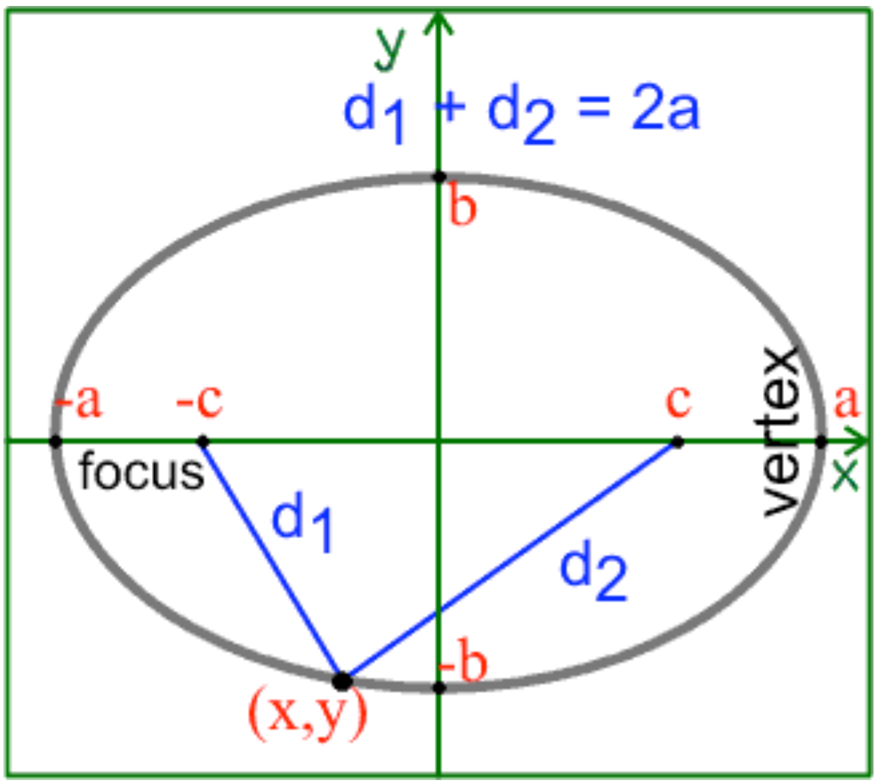
Handout 1: Derivation of the Cartesian Equation for an Ellipse

The purpose of this handout is to illustrate how the usual Cartesian equation for an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is obtained from the Euclidean definition of the ellipse. Consider the ellipse shown in the following diagram[1].



The Euclidean definition of the ellipse is that the total distance of the point (x, y) from the two foci $(-c, 0)$ and $(c, 0)$ is equal to a constant. Following the notation suggested by the diagram we will label these distances d_1 and d_2, and the constant $2a$, where $a > 0$.

Now, using the Theorem of Pythagoras we can calculate that:

$$d_1 = \sqrt{(-c - x)^2 + y^2} = \sqrt{(c + x)^2 + y^2}$$

$$d_2 = \sqrt{(c - x)^2 + y^2},$$

[1] This image was obtained from: http://www.richland.edu

bearing in mind that $x < 0$ in the diagram given above. The Euclidean condition for a point (x, y) to be a point on the ellipse is then:

$$\sqrt{(c+x)^2 + y^2} + \sqrt{(c-x)^2 + y^2} = 2a.$$

Subtracting d_2 from both sides and squaring gives:

$$\sqrt{(c+x)^2 + y^2} = 2a - \sqrt{(c-x)^2 + y^2}$$

$$(c+x)^2 + y^2 = 4a^2 - 4a\sqrt{(c-x)^2 + y^2} + (c-x)^2 + y^2.$$

Simplifying this equation and making the square root the subject gives:

$$4a\sqrt{(c-x)^2 + y^2} = 4a^2 + (c-x)^2 + y^2 - (c+x)^2 - y^2$$

$$4a\sqrt{(c-x)^2 + y^2} = 4a^2 - 4cx$$

$$\sqrt{(c-x)^2 + y^2} = a - \frac{c}{a}x.$$

Squaring both sides of this equation to remove the square root then gives:

$$(c-x)^2 + y^2 = a^2 - \frac{2c}{a}x + \frac{c^2}{a^2}x^2.$$

FOILing and simplifying this expression then gives:

$$c^2 + x^2 + y^2 = a^2 + \frac{c^2}{a^2}x^2.$$

Combining the x^2 terms, subtracting c^2 from both sides and simplifying gives:

$$\frac{a^2 - c^2}{a^2}x^2 + y^2 = a^2 - c^2.$$

Setting $b^2 = a^2 - c^2$ and dividing both sides of this equation by b^2 gives the familiar:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$