*Math 122, Fall 2008. Answers to Unit Test 3 Review Problems – Set B.*

**Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)**

**1.(a)** $1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$

**1.(b)** $1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3$

**1.(c)** $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$

**1.(d)** $1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6$

**2.(a)** Radius = 0.2.

**2.(b)** Radius = 2.

**2.(c)** Radius = 1.

**2.(d)** The radius of convergence is infinite.

**2.(e)** Radius = 1.

**3.(a)** True.

**3.(b)** False.

**3.(c)** True.

**3.(d)** False.

**3.(e)** False.

**3.(f)** False.

**3.(g)** False.

**4.(a)** The first two non-zero terms of the Taylor series are: $2x + \frac{2}{3}x^3$.

**4.(b)** A highly educated guess for this extrapolation is: $\sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}$. If you calculate the next non-zero term in the Taylor series from Part (a) you get $\frac{2}{5}x^5$, which might help to guide your extrapolation efforts.

**4.(c)** Note that (not worrying too much about the $+C$):

$$\ln(1-x) = -\int \frac{1}{1-x}\,dx = -\sum_{n=0}^{\infty}\int x^n \cdot dx = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}.$$

**4.(d)** Note that (not worrying too much about the +$C$):

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty}\int (-1)^n \cdot x^n \cdot dx = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{n+1}}{n+1}.$$

**4.(e)** $$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{n+1}}{n+1} + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{2}{2n+1}x^{2n+1}.$$

**4.(f)** Use the Ratio Test to determine the radius of convergence. For the series in 4.(e) it is $r = 1$.

**5.(a)** The coefficient of $(x-2)^n$ will be $\frac{n+1}{3^n}$.

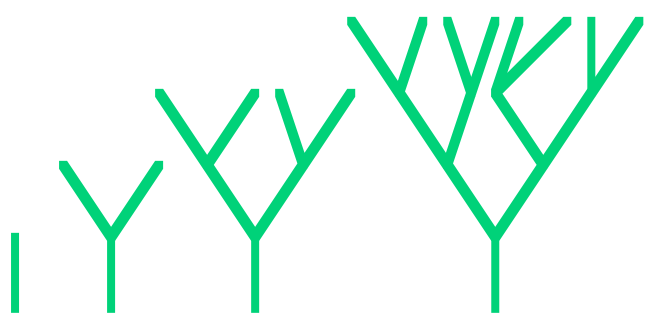
**5.(b)** The Taylor series is: $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{3^n}(x-2)^n$

**5.(c)** The radius of convergence of the Taylor series is 3.

**5.(d)** $f(2.5) = 1 + \frac{2}{3}(2.5-2) + \frac{3}{3^2}(2.5-2)^2 + \frac{4}{3^3}(2.5-2)^3 = 1.435185185.$

**5.(e)** The maximum error is $\displaystyle\frac{f^{(iv)}(2)}{4!}(2.5-2)^4 = \frac{\frac{4!}{3^4}}{4!}(2.5-2)^4 = 0.000771.$

**6.(a)** Rough sketches of a *Codium tomentosum* alga just before each of the first four "splits" are shown below.



| Just before first split | Just before second split | Just before third split | Just before fourth split |
|---|---|---|---|

**6.(b)** The completed table is shown below. You may not have written out the entries of the table in quite the same way as shown below. That's fine so long as the entries you have written down are equivalent ot the entries shown below. I have written the entries in this way to try to make the pattern in the table a little more evident.

| Just before *Codium tomentosum* has undergone this many splits ... | The total volume of the alga is ... (cubic centimeters) |
|---|---|
| 1 | 24 |
| 2 | $24 + 24\cdot 2$ |
| 3 | $24 + 24\cdot 2 + 24\cdot 2^2$ |
| 4 | $24 + 24\cdot 2 + 24\cdot 2^2 + 24\cdot 2^3$ |
| $N$ | $24 + 24\cdot 2 + 24\cdot 2^2 + \ldots + 24\cdot 2^{N-1}$ |

**6.(c)** Using the summation formula for a geometric series with the last entry in the table above gives:

$$\textit{Volume before } N^{th} \textit{ Split} = \frac{24\cdot\left[1-2^N\right]}{1-2} = 24\cdot\left[2^N - 1\right].$$

**6.(d)** The amount of glucuronic acid in one package of tablets is: $600\times 120 = 72{,}000$ mg. As each cubic centimeter of *Codium tomentosum* contains 0.001 mg of glucuronic acid the total volume of *Codium tomentosum* will be: $72{,}000/0.001 = 72{,}000{,}000$ cubic centimeters of *Codium tomentosum*.

**6.(e)** To determine the number of splits that the alga had undergone, we can set the equation from Part (c) equal to 72,000,000 and solve for $N$. Performing this calculation:

$$24\cdot\left[2^N - 1\right] = 72{,}000{,}000$$

$$2^N - 1 = \frac{72{,}000{,}000}{24} = 3{,}000{,}000$$

$$N = \frac{\log(3{,}000{,}000+1)}{\log(2)} = 21.517$$

So, $N = 21.517$. All of that glucuronic acid could be obtained from an alga that was just about to split for the 21st or 22nd time.

**7.(a)** Polynomial: 1
Maximum error: 0.8243606354

Note that the point of creating such a Taylor polynomial would be to approximate the value of quantities such as $e^{0.5}$, for example if you didn't have ready access to a high precision calculator. In such a case, including $e^{0.5}$ in your error calculation doesn't make much sense because you don't know what the precise value of this quantity is. Instead, you would note that $e < 4$ and use $4^{0.4} = 2$ as a bound for $e^{0.5}$ in the error calculation. This gives a slightly higher value for the error than using $e^{0.5}$.

**7.(b)** Polynomial: $x$
Maximum error: 0.3916634548

Note that the actual error here is considerably less. This is because the Taylor series for $\sin(x)$ expanded around $a = 0$ does not have any terms with even powers of $x$. As such, the Taylor polynomial given above is not only the degree 1 Taylor polynomial, it is also the degree 2 Taylor polynomial for sin($x$) around $a = 0$. If you carry out the error calculation using $N = 2$ and $M = 1$ (a better upper bound is possible but only if you have a means for evaluating sine and cosine with great precision) gives the smaller error of $0.16666666 \approx 0.17$).

**7.(c)** Polynomial: $1 - \frac{1}{2}x^2$
Maximum error: 0.0003375. (Note that $M = 1$ was used here. See comments following 7.(b).)

**7.(d)** Polynomial: $x + \frac{1}{3}x^3$
Maximum error: 14.31635

The maximum error given above was calculated using $M = 343.5924$. This value was obtained by finding a formula for the fourth derivative of tan($x$) and then graphing the absolute value of the fourth derivative over the interval [-1, 1]. The maximum occurred at $x = 1$, and it was equal to 343.5924. If you didn't have a sophisticated graphing calculator to find such a precise value of $M$, then 400 would do as a reasonable upper bound for the absolute value of the fourth derivative of tan($x$) on the interval [-1, 1]. Using $M = 400$ gives a maximum error of about 16.5.

**7.(e)** Polynomial: $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$
Maximum error: 0.0004293544976

If you use the value of 2 for $M$ (which is slightly larger than $e^{0.5\text{-}}$ - see comment following 7.(a) above) then you get an error estimate of 0.0005208333 which can be rounded up to 0.006.

**8.(a)** The center of the power series is $a = 1$. So, $t = 1$ is when a finite number of terms will give the value of $v(t)$ exactly.

**8.(b)** The radius of convergence is 2.

**8.(c)** The power series for the particle's acceleration is: $\sum_{n=1}^{\infty} \frac{(t-1)^n}{n \cdot 2^n}$.

**8.(d)** The acceleration is zero at $t = 1$.

**8.(e)** There is a local minimum of velocity at $t = 1$.

**9.(a)** The interval is [3, 5).

**9.(b)** The interval is [4, 8].

**9.(c)** The interval is [−4, 2).

**9.(d)** The interval is (−3, −1].

**10.(a)** Define $S_N$ to be the $N^{th}$ partial sum of the series $\sum_{n=1}^{\infty} b_n$. Then:

$$S_N = a_{N+1} - a_N + a_N - a_{N-1} + \ldots + a_3 - a_2 + a_2 - a_1 = a_{N+1} - a_1.$$

Since $\sum_{n=1}^{\infty} a_n$ is a convergent series, the limit as $N \to \infty$ of $a_{N+1}$ will be zero. This means that the limit of $S_n$ as $N \to \infty$ will be $-a_1$. As the limit of the partial sums exists and is finite, the series $\sum_{n=1}^{\infty} b_n$ converges.

**10.(b)** Depending on the specific series that we choose for $\sum_{n=1}^{\infty} a_n$, the series $\sum_{n=1}^{\infty} b_n$ could converge or diverge. For example, choosing $a_n = \frac{1}{n}$ leads to a convergent series for $b_n$:

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{-1}{n^2+n}.$$

On the other hand, some choices for $a_n$ will lead to divergent series. For example, choosing $a_n = 2^n$ leads to the following divergent series for $b_n$:

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\left(2^{n+1} - 2^n\right) = \sum_{n=1}^{\infty} 2^n .$$