Unit Test 3 Review Problems – Set B

We have chosen these problems because we think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the exam will resemble these problems in any way whatsoever. Remember that on exams you will have to supply evidence for your conclusions and may have to explain why your answers are reasonable and appropriate.

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1. Find the first four non-zero terms of the Taylor series for each of the functions given below. In each case, you can use $a = 0$ as the center of your Taylor series.

(a) $f(x) = e^{-x}$ **(b)** $g(x) = \sqrt{1-2x}$

(c) $h(x) = \arcsin(x)$ **(d)** $j(x) = \dfrac{1}{\sqrt{1-x^2}}$

Note: $\dfrac{d}{dx}(\arcsin(x)) = \dfrac{1}{\sqrt{1-x^2}}$.

2. Find the radius of convergence for each of the power series listed below.

(a) $\displaystyle\sum_{n=1}^{\infty} 5^n \cdot x^n$ **(b)** $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)x^n}{2^n + n}$

(c) $\displaystyle\sum_{n=1}^{\infty} n^2 \cdot x^n$ **(d)** $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$

(e) $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$

3. A power series $\displaystyle\sum_{n=0}^{\infty} C_n \cdot (x+7)^n$ converges at $x = 0$ and diverges at $x = -17$. This is all that you may assume about the power series. For each of the following statements, determine whether they are true or false.

(a) The radius of convergence must be at least 7.

(b) The interval of convergence must be $(-17, 0)$.

(c) The interval $(-14, 0)$ must be part of the interval of convergence.

(d) The radius of convergence could be greater than or equal to 17.

(e) The interval of convergence includes all real numbers except $x = 17$.

(f) The radius of convergence could be equal to 24.

(g) The only point where the power series is guaranteed to converge is $x = 0$.

4. In this problem, the function f will always refer to the function defined by the formula:

$$f(x) = \ln\left(\frac{1+x}{1-x}\right).$$

(a) Find the first two non-zero terms of the Taylor series for f centered at $a = 0$.

(b) Based on your answer to Part (a), try to extrapolate the full Taylor series for $f(x)$ centered at $a = 0$. Express your answer in Σ notation.

(c) As you probably know, $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$. Use this fact to find the Taylor series for the function $g(x) = \ln(1-x)$.

(d) As you probably know, $\dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty} (-1)^n \cdot x^n$. Use this fact to find the Taylor series for the function $h(x) = \ln(1+x)$.

(e) Based on your answers to Parts (c) and (d), find the full Taylor series for $f(x)$ centered at $a = 0$. Express your answer in Σ notation.

(f) Find the radius of convergence for the Taylor series of $f(x)$ centered at $a = 0$.

5. In this problem you will be dealing with a function $f(x)$. All that you know about this function is that the n^{th} derivative of $f(x)$ evaluated at $a = 2$ is:

$$f^{(n)}(a) = f^{(n)}(2) = \frac{(n+1)!}{3^n} \qquad \text{for all } n \geq 1.$$

and that the value of the function at the point $a = 2$ is: $f(a) = f(2) = 1$.

(a) Find a formula for the coefficient of $(x-2)^n$ in the Taylor series of the function $f(x)$ about $a = 2$.

(b) Write down the Taylor series for $f(x)$ with center $a = 2$. Express your answer using Σ notation.

(c) Find the interval of convergence for the Taylor series of $f(x)$ about $a = 2$.

(d) Use the Taylor polynomial of degree 3 to approximate the value of $f(2.5)$.

(e) Suppose that $f^{(iv)}(2)$ is the maximum value of $f^{(iv)}(x)$ on the interval [2, 2.5]. What is the maximum error in your answer to Part (d)?

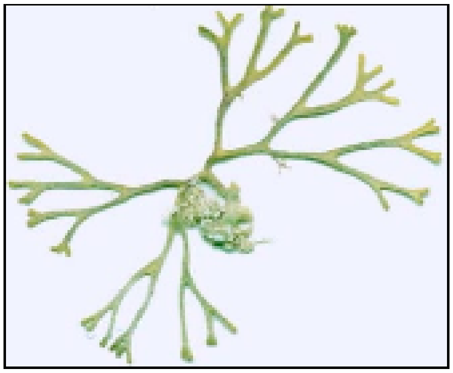

Figure 7: A small specimen of *Codium tomentosum*. The "blob" in the center of the picture is a piece of the rock that the alga was anchored to.

6. The marine alga (i.e. seaweed) *Codium tomentosum* (see Figure 7[1]) is also known by the common names of "velvet horn," "spongy weed" and "green sea finger." Extracts from the alga are used in anti-aging skin creams[2] and as an experimental drug in AIDS research[3]. There is some concern over the fate of this plant in the UK as there is evidence to suggest that *Codium tomentosum* is being replaced by the less valuable introduced species *Codium fragile*[4].

This alga grows by dichotomous brachiation - i.e. each branch of the alga grows to a certain size and then it splits into two new branches. Each branch of the alga grows to a volume of about 24 cubic centimeters before splitting into two new branches[5].

(a) Draw some diagrams that show the appearance of a very young *Codium tomentosum* just before the first, second, third and fourth "splits" that the alga undergoes.

(b) Use the drawings that you made in Part (a) to help you fill in the first few entries of the table given on the next page. When you have filled in the first few entries, look for a pattern in the table and generalize to write down an expression for the volume of the alga just before the N^{th} split.

Just before *Codium tomentosum* has undergone this many splits ...	The total volume of the alga is ... (cubic centimeters)
1	24
2	
3	
4	
N	

(c) Write down a convenient formula (or closed form) for the total volume of the alga just before the N^{th} split.

(d) Chrondoritin sulfate is an experimental AIDS medication[6]. It is not clear whether this drug has any significant clinical effects in the case of AIDS patients. The active ingredients in this compound are

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[1] Image source: http://www.horta.uac.pt/species/Algae/Codium_tometosum/Codium_tomentosum.htm
[2] Source: http://www.scotland.gov.uk/
[3] Source: http://www.spirulinasource.com/library-antiviral.html
[4] Source: Farnham, W.F. 1980. Studies on aliens in the marine flora of southern England. in J.H. Price, D.E.G. Irvine and W.F. Farnham. Eds. *The Shore Environment. Volume 2: Ecosystems*. London, England: Academic Press.
[5] Source: P.E. Dixon and L.M. Irvine. 1977. *Seaweed of the British Isles*. London, England: The British Museum.
[6] The details given here were obtained from: http://www.aidstreatment.org/ProdList.html

glucuronic acid and N-Acetyl galactosomine. This compound is available in packages of 120 tablets. Each tablet contains 600 mg of glucuronic acid and the entire packet of 120 tablets costs \$22.19. One cubic centimeter of *Codium tomentosum* contains about 0.001 mg of glucuronic acid[7]. How many cubic centimeters of *Codium tomentosum* would be required to supply all of the glucuronic acid on one package of Chrondoritin sulfate tablets?

(e) Remember that each branch of *Codium tomentosum* has a volume of about 24 cubic centimeters. If all of the glucuronic acid for one package of Chrondoritin sulfate came from a single specimen of *Codium tomentosum*, how many splits would the alga have undergone?

7. For each of the functions given below,

(i) Find a formula for the Taylor polynomial of the function using $a = 0$ and the degree specified.

(ii) Calculate the maximum error incurred if the Taylor polynomial is used to approximate the function over the interval indicated.

(a) Function: $f(x) = e^x$.
Polynomial: Degree zero.
Interval: $[0, 0.5]$.

(b) Function: $f(x) = \sin(x)$.
Polynomial: Degree 1.
Interval: $[-1, 1]$.

(c) Function: $f(x) = \cos(x)$
Polynomial: Degree 3.
Interval: $[-0.3, 0.3]$

(d) Function: $f(x) = \tan(x)$
Polynomial: Degree 3.
Interval: $[-1, 1]$.

(e) Function: $f(x) = e^x$.
Polynomial: Degree 4.
Interval: $[-0.5, 0.5]$.

8. The speed of a particle (in units of meters per second) as it moves along the y-axis is given by a function $v(t)$. No convenient formula for $v(t)$ exists. However, the power series representing $v(t)$ is:

$$v(t) = \sum_{n=1}^{\infty} \frac{(t-1)^{n+1}}{n \cdot (n+1) \cdot 2^n},$$

where t is the time (in seconds) since the particle started moving.

[7] Source: M.E. Lai, V. Scotto and A. Bergel. 1999. Analytical characterization of natural marine biofilms. *Proceedings of the 10th Annual Conference on Marine Corrosion and Fouling*. (Melbourne, Australia. February 7-12, 1999.)

(a) If you used a finite number of terms of the power series to approximate the value of $v(t)$, what is the only time at which your approximation would be guaranteed to be completely accurate?

(b) Find the radius of convergence of the power series for $v(t)$.

(c) Find a power series representation for the acceleration of the particle as a function of t.

(d) At what time(s) is the acceleration of the particle equal to zero?

(e) At each of the times that you found in Part (a), determine whether the particle is experiencing a (local) maximum or a (local) minimum of velocity at each of those times?

9. Find the interval of convergence for each of the power series listed below.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(x-4)^n}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(x-6)^n}{n^2\cdot 2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)\cdot 3^n}(x+1)^n$

(d) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+3}(x+2)^n$

10. **(a)** Consider an infinite series $\displaystyle\sum_{n=1}^{\infty}a_n$. Suppose that each $a_n > 0$ and that the infinite series converges. Define $b_n = a_{n+1} - a_n$. Does the series $\displaystyle\sum_{n=1}^{\infty}b_n$ converge or diverge?

(b) Consider an infinite series $\displaystyle\sum_{n=1}^{\infty}a_n$. Suppose that each $a_n > 0$ and that the infinite series diverges. Define $b_n = a_{n+1} - a_n$. Does the series $\displaystyle\sum_{n=1}^{\infty}b_n$ converge or diverge?