Math 122, Fall 2008. Answers to Unit Test 3 Review Problems – Set A.

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1.(a) The series diverges by the n^{th} term test. (The limit of a_n is 1.)

1.(b) The series diverges by the n^{th} term test. (The limit of a_n is 1.)

1.(c) The n^{th} term test does not reveal anything about this series. The series converges (try comparing to the series $\sum_{n=1}^{\infty} \frac{n^2}{n!}$; the convergence of this series can be demonstrated using the Ratio Test).

1.(d) The n^{th} term test does not reveal anything about this series. The series diverges (try simplifying and then comparing to the p-series with $p = 0.5$).

2.(a) The series converges.

2.(b) The series diverges.

2.(c) The series converges.

2.(d) The series converges.

2.(e) The series converges.

2.(f) The series converges.

2.(g) The series diverges.

3.(a) The completed table is shown below.

Months since loan obtained	Amount still owed
1	$200000 \cdot \left(1+\frac{0.07}{12}\right) - M$
2	$200000 \cdot \left(1+\frac{0.07}{12}\right)^2 - M \cdot \left(1+\frac{0.07}{12}\right) - M$
3	$200000 \cdot \left(1+\frac{0.07}{12}\right)^3 - M \cdot \left(1+\frac{0.07}{12}\right)^2 - M \cdot \left(1+\frac{0.07}{12}\right) - M$
N	$200000 \cdot \left(1+\frac{0.07}{12}\right)^N - M \cdot \left(1+\frac{0.07}{12}\right)^{N-1} - \ldots - M \cdot \left(1+\frac{0.07}{12}\right) - M$

3.(b) Thirty years is equivalent to 360 months. Assuming that the individual has been responsible and has actually paid off the mortgage in those 360 months, he or she would owe nothing after 360 months. Using the last entry from the table in Part (a) with $N = 360$, and setting the resulting expression equal to zero gives:

$$200000 \cdot \left(1+\frac{0.07}{12}\right)^{360} - M \cdot \left(1+\frac{0.07}{12}\right)^{359} - \ldots - M \cdot \left(1+\frac{0.07}{12}\right) - M = 0$$

The idea is to now solve this expression to find the value of M. Moving all of the terms that involve M to the right hand side of the equation gives:

$$200000\cdot\left(1+\tfrac{0.07}{12}\right)^{360} = M + M\cdot\left(1+\tfrac{0.07}{12}\right) + \ldots + M\cdot\left(1+\tfrac{0.07}{12}\right)^{359}$$

Using the summation formula for a geometric series on the right hand side of this equation gives:

$$200000\cdot\left(1+\tfrac{0.07}{12}\right)^{360} = \frac{M\cdot\left[1-\left(1+\tfrac{0.07}{12}\right)^{360}\right]}{1-\left(1+\tfrac{0.07}{12}\right)}.$$

Re-arranging to make M the subject of the equation and evaluating all of the numerical quantities gives:

M = \$1330.60 per month.

3.(c) If the mortgage is a 15 year mortgage instead of a 30 year mortgage then it must be paid off after 180 months. Therefore,

$$200000\cdot\left(1+\tfrac{0.07}{12}\right)^{180} - M\cdot\left(1+\tfrac{0.07}{12}\right)^{179} - \ldots - M\cdot\left(1+\tfrac{0.07}{12}\right) - M = 0.$$

Again, the idea is to solve this equation to find the value of M. Moving all of the terms that involve M to the right side of the equation gives:

$$200000\cdot\left(1+\tfrac{0.07}{12}\right)^{180} = M + M\cdot\left(1+\tfrac{0.07}{12}\right) + \ldots + M\cdot\left(1+\tfrac{0.07}{12}\right)^{179}$$

Using the geometric summation formula with the geometric series on the right hand side of the equation gives:

$$200000\cdot\left(1+\tfrac{0.07}{12}\right)^{180} = \frac{M\cdot\left[1-\left(1+\tfrac{0.07}{12}\right)^{180}\right]}{1-\left(1+\tfrac{0.07}{12}\right)}.$$

Re-arranging this equation to make M the subject and evaluating the numerical quantities gives:

M = \$1797.66 per month.

3.(d) The total amounts paid to the lender in each case are shown the table below.

Number of Months	Monthly payment (\$)	Total paid to lender (\$)
180	1797.66	323578.50
360	1330.60	479016.00

The table shows that although the monthly payment is lower for the 30 year mortgage, the total amount that is paid to the lender is considerably higher than the total amount paid to the lender with the 15 year mortgage.

3.(e) I would prefer to have a low interest rate at the beginning of the mortgage and a high interest rate at the end of the mortgage. This is because at the beginning of the mortgage almost all of your monthly mortgage payment goes to offsetting the interest that the outstanding balance incurs. If the interest rate is low at this point of the mortgage then more of your monthly payment will go towards reducing the outstanding balance rather than just offsetting the interest. At the end of the mortgage the situation is reversed and most of the money that you send in each month is going towards reducing the size of the outstanding balance with very little offsetting interest. Therefore, if you have to have a high interest rate during some point of your mortgage, the high interest rate would exact the least amount of money from you if it came along towards the end of the mortgage.

4.(a) The series converges. $\displaystyle\int_1^{\infty}\frac{3}{(2x-1)^2}dx = \frac{3}{2}.$

4.(b) The series diverges. $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\cdot\ln\left(1+x^2\right)+C.$

4.(c) The series converges. $\int_1^{\infty} \frac{1}{x^2+1}\,dx = \frac{\pi}{4}.$

4.(d) The series converges. $\int_1^{\infty} x\cdot e^{-x}\cdot dx = \frac{2}{e}.$

4.(e) The series diverges. $\int \frac{\ln(x)}{x}\,dx = \frac{1}{2}\cdot\left(\ln(x)\right)^2+C.$

4.(f) The series converges. $\int_2^{\infty} \frac{1}{x\cdot\left(\ln(x)\right)^2}\,dx = \frac{1}{\ln(2)}.$

5.(a) The series diverges. Note that $n > n-3$ so that $\frac{1}{n} < \frac{1}{n-3}$. Compare the given series to the series to the Harmonic series.

5.(b) The series converges. When $n > 0$, $0 < e^{-n} < 1$ so that $\frac{e^{-n}}{n^2} < \frac{1}{n^2}$. Compare the given series to the p-series with $p = 2$.

5.(c) The series converges. You can compare to either the p-series with $p = 4$ or to the geometric series with $r =$ 0.25 as both of the following inequalities are true: $\frac{1}{n^4+4^n} < \frac{1}{n^4}$ and $\frac{1}{n^4+4^n} < \left(\frac{1}{4}\right)^n$. Both of these series are convergent (the geometric series because $-1 < r < 1$).

5.(d) The series is divergent. As $-1 \le \sin(n) \le 1$, $1 \le 2-\sin(n)$ and so $\frac{1}{n} < \frac{2-\sin(n)}{n}$. Compare the given series with the harmonic series.

5.(e) The series is divergent. For $n > 0$, $\ln(n) < n$ so that when $n > 1$, $\frac{1}{n} < \frac{1}{\ln(n)}$. Compare the given series to the harmonic series.

5.(f) The series is convergent. Note that $\frac{3}{2^n+1} < \frac{3}{2^n} = 3\cdot\left(\frac{1}{2}\right)^n$. Compare to the geometric series with $a = 3$ and $r = 0.5$. This series converges because $-1 < r < 1$.

6.(a) The completed table is shown below.

Number of hours that student has been taking Vivarin®	Number of Vivarin® tablets taken in addition to initial dose	Total amount of caffeine in the student's body is … (mg)
0	0	200
4	1	$200 + 200\cdot(0.8908987181)^4$

8	2	$200 + 200\cdot(0.8908987181)^4 + 200\cdot(0.8908987181)^8$
12	3	$200 + 200\cdot(0.8908987181)^4 + + 200\cdot(0.8908987181)^8 + 200\cdot(0.8908987181)^{12}$
$4\cdot N$	N	$200 + 200\cdot(0.8908987181)^4 + + 200\cdot(0.8908987181)^8 + 200\cdot(0.8908987181)^{12} + \ldots + 200\cdot(0.8908987181)^{4\cdot N}$

6.(b) The series in the last entry of the table from Part (a) is a geometric series with initial value $a = 200$, $r = (0.8908987181)^4$ and a total of $N + 1$ individual terms added together. The geometric summation formula gives that the total amount of caffeine (in mg) in the person's body after they have taken the initial dose and N additional doses of Vivarin® will be:

$$\frac{200\cdot[1-((0.8908987181)^4)^{N+1}]}{1-(0.8908987181)^4}.$$

6.(c) To determine how long a person has to take Vivarin® to accumulate 500 mg of caffeine and begin to experience adverse reactions, we will set the summation formula from Question (c) equal to 500 and solve for N, which will give the number of additional doses. As the person is taking one dose every 4 hours, we will deduce the length of time that the person has been taking Vivarin® by multiplying N by 4 hours.

$$\frac{200\cdot[1-((0.8908987181)^4)^{N+1}]}{1-(0.8908987181)^4} = 500$$

$$1-((0.8908987181)^4)^{N+1} = \frac{500\cdot[1-(0.8908987181)^4]}{200}$$

$$((0.8908987181)^4)^{N+1} = 1-\frac{500\cdot[1-(0.8908987181)^4]}{200}$$

$$(N+1)\cdot\ln[(0.8908987181)^4] = \ln\left(1-\frac{500\cdot[1-(0.8908987181)^4]}{200}\right)$$

$$N = \frac{\ln\left(1-\frac{500\cdot[1-(0.8908987181)^4]}{200}\right)}{\ln[(0.8908987181)^4]} - 1 \approx 4.6.$$

Rounding this up to the nearest whole number gives that $N = 5$. Therefore, after about 20 hours of taking Vivarin® the person will accumulate 500 mg of caffeine in their system and probably begin to experience some adverse reactions.

6.(d) The summation formula that gives the amount of caffeine that has accumulated in the person's body is:

$$\frac{200\cdot[1-((0.8908987181)^4)^{N+1}]}{1-(0.8908987181)^4} = \frac{200}{1-(0.8908987181)^4} - \frac{200\cdot((0.8908987181)^4)^{N+1}}{1-(0.8908987181)^4}.$$

As $N \to \infty$, this summation formula will approach a limit of:

$$\frac{200}{1-(0.8908987181)^4} \approx 540 \text{ mg}.$$

Therefore, if the person continues to take Vivarin® indefinitely, he or she will eventually accumulate about 540 mg of caffeine in their system.

7.(a) The Comparison test can be used here. Noting that for $n > 0$, $n^2 + 2n > n^2$ gives that $\frac{2}{n^2+2n} < \frac{2}{n^2}$. Comparison to a constant multiple of the p-series that has $p = 2$ (which is convergent) gives that the series in this problem also converges.

7.(b) The partial fractions will be:
$$\frac{2}{n^2+2n} = \frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}.$$

Cross multiplying and equating coefficients of powers of n gives the two equations:

$$2A = 2 \qquad\qquad A + B = 0.$$

These equations have the solution: $A = 1$ and $B = -1$.

7.(c) $S_N = \frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}$.

7.(d) The sum of the series is the limit of S_N as $N \to \infty$. Taking the limit of the expression from 7.(d) gives that the sum of the series is 1.5.

8.(a) The series is divergent. (Use the n^{th} term test for divergence.)

8.(b) If you use the ratio test, you can establish that this series converges absolutely (and therefore also converges conditionally). To determine the value of N to use we must solve the equation:

$$\left(\tfrac{2}{3}\right)^{N+1} < 0.01$$

which can be solved using logarithms and gives $N > 10$. For this series, $S_{11} = -0.404624408$.

8.(c) This series does not converge absolutely. (If you take absolute values of terms you get the p-series with $p = 0.5 < 1$.) The series does converge conditionally as it satisfies the requirements of the Alternating Series test. To find the value of N we need to solve the equation:

$$\frac{1}{\sqrt{N+1}} < 0.1.$$

This can be done with algebra to give $N > 99$. For this series, $S_{100} = 0.5550236395$.

8.(d) The series is divergent. (Use the n^{th} term test for divergence.)

9.(a) Diverges.

9.(b) Converges.

9.(c) Converges.

9.(d) Converges.

10.(a) We will use the Squeeze Lemma. Note that if $S = \sum_{k=1}^{\infty} a_k$ then S is a finite number. As each a_k is positive, then for each value of $n > 1$, $0 \le \sum_{k=1}^{n} a_k \le S$. Dividing this inequality by $n > 1$ gives:

$$0 \le b_n = \frac{1}{n} \cdot \sum_{k=1}^{n} a_k \le \frac{S}{n}.$$

Now, the limit of zero as $n \to \infty$ is zero and the limit of $\frac{S}{n}$ as $n \to \infty$ is also zero. The Squeeze Lemma then gives that the limit of b_n (as $n \to \infty$) is similarly zero.

10.(b) As the series $\sum_{n=1}^{\infty} b_n$ is **not** an alternating series, this limit tells you nothing about the convergence of this series.

10.(c) The infinite series $\sum_{n=1}^{\infty} b_n$ diverges. This can be demonstrated using the Comparison test. To make the comparison, note that $a_1 > 0$ and that for all values of n:

$$0 < a_1 \le \sum_{k=1}^{n} a_k.$$

Dividing this inequality by $n > 0$ gives:

$$0 < \frac{a_1}{n} \le \frac{1}{n} \cdot \sum_{k=1}^{n} a_k = b_n.$$

Now, the infinite series $\sum_{n=1}^{\infty} \frac{a_1}{n}$ is a constant multiple of the harmonic series and so it diverges. The Comparison test then gives that the series $\sum_{n=1}^{\infty} b_n$ must also diverge.