Math 122, Fall 2008. Answers to Unit Test 2 Review Problems – Set B.

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1.(a) Volume = $\int_0^1 \pi \cdot \left[x^2 - \sin^2(x)\right] \cdot dx$.

1.(b) Volume = $\int_0^1 2\pi \cdot x \cdot \left[x - \sin(x)\right] \cdot dx$.

1.(c) Volume = $\int_0^1 \left[x - \sin(x)\right]^2 \cdot dx$.

1.(d) Volume = $\int_0^{\sin(1)} \frac{1}{2} \cdot \pi \cdot \left[\frac{\sin^{-1}(y) - y}{2}\right]^2 \cdot dy + \int_{\sin(1)}^1 \frac{1}{2} \cdot \pi \cdot \left[\frac{1-y}{2}\right]^2 \cdot dy$.

1.(e) Perimeter = $\sqrt{2} + \left(1 - \sin(1)\right) + \int_0^1 \sqrt{1 + \cos^2(x)} \cdot dx$.

2.(a) Volume = $\int_0^3 \pi \cdot \frac{69}{3} \cdot \left(y + \frac{100}{23}\right) \cdot dy = 72.2566 \cdot \left[\frac{1}{2}y^2 + \frac{100}{23} \cdot y\right]_0^3 = 1267.632$ m^3.

2.(b) Force = $\int_0^3 (9.8) \cdot (1000) \cdot (3 - y) \cdot dy = 9800 \cdot \left[3y - \frac{1}{2}y^2\right]_0^3 = 44100$ N.

2.(c) Work =

$\int_0^3 (3-y) \cdot (9.8) \cdot (1100) \cdot \pi \cdot \frac{69}{3} \cdot \left(y + \frac{100}{23}\right) \cdot dy = 778926.4825 \cdot \left[\frac{300}{23}y - \frac{31}{46}y^2 - \frac{1}{3}y^3\right]_0^3 = 18745035.13$ Joules.

3.(a) We will start out by finding a differential equation for the amount of carbon dioxide, $A(t)$, measured in cubic meters. When we have a differential equation for $A(t)$ we can convert it to a differential equation for $P(t)$ by observing that:

$$P(t) = \frac{A(t)}{350} \times \frac{100}{1} = \frac{A(t)}{3.5} \qquad \text{and} \qquad \frac{dP}{dt} = \frac{1}{3.5} \cdot \frac{dA}{dt}.$$

To set up the differential equation for $A(t)$ we will use the familiar pattern:

Derivative = (Rate in) – (Rate out).

Here the derivative is $\frac{dA}{dt}$ and the rate in is the product of the flow rate (0.005 m^3 per minute) multiplied by 0.1. The rate out will be the proportion of the atmosphere of the greenhouse that is carbon dioxide, namely $\frac{A(t)}{350}$ multiplied by the rate at which the well-mixed atmosphere leaves the greenhouse, 0.005 m^3 per minute. Putting all of this together gives the differential equation:

$$\frac{dA}{dt} = (0.005) \cdot (0.1) - (0.005) \cdot \frac{A}{350} = -0.0000142857 \cdot (A - 35).$$

Converting this to a differential equation for the percentage $P(t)$ gives:

$$3.5 \cdot \frac{dP}{dt} = -0.0000142857 \cdot (3.5 \cdot P - 35)$$

$$\text{or: } \frac{dP}{dt} = -0.0000142857 \cdot (P - 10).$$

3.(b) Using the initial value of $P(0) = 0.05$, we can solve for $P(t)$ using the technique of Separation of Variables. Doing this:

$$\frac{dP}{dt} = -0.0000142857 \cdot (P - 10)$$

$$\int \frac{1}{P-10} \cdot dP = \int -0.0000142857 \cdot dt$$

$$\ln(|P - 10|) = -0.0000142857 \cdot t + C$$

$$P - 10 = A \cdot e^{-0.0000142857 \cdot t} \qquad \text{where } A = \pm e^{C}.$$

Substituting $P(0) = 0.05$ to solve for A and rearranging to make $P = P(t)$ the subject of the equation:

$$P = P(t) = 10 - 9.95 \cdot e^{-0.0000142857 \cdot t}.$$

3.(c) To answer this question we set $P = 4$ and solve for t. Doing this:

$$4 = 10 - 9.95 \cdot e^{-0.0000142857 \cdot t}$$

$$\tfrac{6}{9.95} = e^{-0.0000142857 \cdot t}$$

$$t = \tfrac{1}{-0.0000142857} \cdot \ln\left(\tfrac{6}{9.95}\right) \approx 35406.95 \text{ minutes. (About 25 days.)}$$

3.(d) To answer this question we set $P = 0.1$ and solve for t. Doing this:

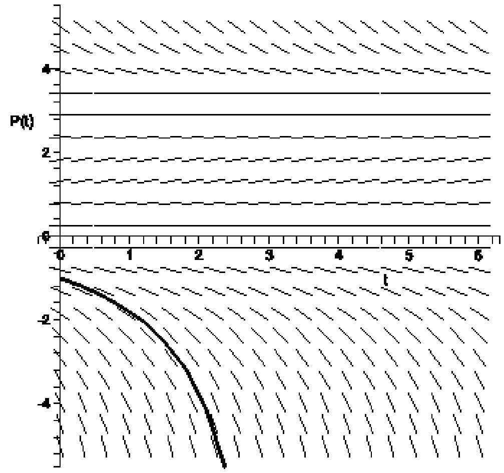
$$0.1 = 10 - 9.95 \cdot e^{-0.0000142857 \cdot t}$$

$$\tfrac{9.9}{9.95} = e^{-0.0000142857 \cdot t}$$

$$t = \tfrac{1}{-0.0000142857} \cdot \ln\left(\tfrac{9.9}{9.95}\right) \approx 352.65 \text{ minutes. (About 6 hours.)}$$

4.(a) There are two equilibrium solutions. One is the horizontal line $P = 0$ (it is unstable) and the other is the horizontal line $P = 3$ (it is stable).

4.(b) A picture of the slope field showing the graph that begins at the point (0, –1) is shown on the next page.



4.(c) The table showing the work for Euler's method is given below. $P(2) \approx -2.8567$.

Current t	Current P	Derivative	Rise	New P
0	–1	–0.533	–0.266	–1.266
0.5	–1.266	–0.7206	–0.3603	–1.6269
1.0	–1.6269	–1.0037	–0.5018	–2.1288
1.5	–2.1288	–1.4557	–0.7278	–2.8567

4.(d) The estimate will be an overestimate. This is because in the region of interest, the function $P(t)$ is concave down.

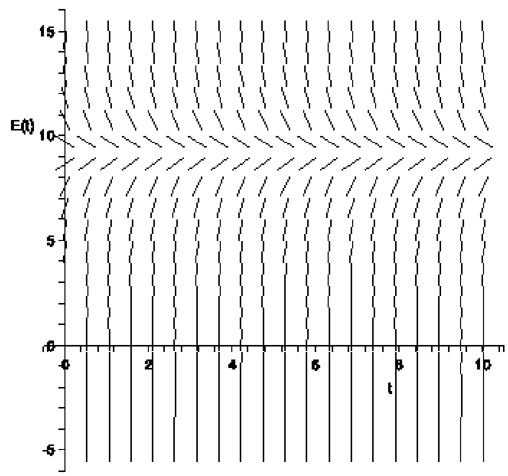
5.(a) The patient begins with no endostatin in their body so the initial value will be $E(0) = 0$. The differential equation will follow the familiar pattern for a differential equation:

Derivative = (Rate in) – (Rate out).

Here the derivative is $\frac{dE}{dt}$, the rate in is 25 mg per hour and the rate out is the constant of proportionality (2.7) times the amount of endostatin in the patient, $E(t)$. Putting all of this together gives the differential equation:

$$\frac{dE}{dt} = 25 - 2.7 \cdot E = -2.7 \cdot (E - 9.259).$$

5.(b) The differential equation has exactly one equilibrium solution, the horizontal line $E = 9.259$. This is a stable equilibrium as the slope field of the differential equation (see next page) shows. The practical significance of this equilibrium solution is that as the treatment goes on for a long time, the amount of endostatin in the patient will approach 9.259 mg.



5.(c) To find a formula for $E(t)$ we will use the technique of Separation of Variables:

$$\frac{dE}{dt} = -2.7 \cdot (E - 9.259)$$

$$\int \frac{1}{E - 9.259} \cdot dE = \int -2.7 \cdot dt$$

$$\ln(|E - 9.259|) = -2.7 \cdot t + C$$

$$E - 9.259 = A \cdot e^{-2.7 \cdot t} \quad \text{where } A = \pm e^{C}.$$

Using the initial value $E(0) = 0$ to determine the value of A and rearranging to make $E = E(t)$ the subject of the equation, we get:

$$E = E(t) = 9.259 - 9.259 \cdot e^{-2.7 \cdot t}.$$

5.(d) To determine when the level of endostatin reaches 9 mg, we will substitute $E = 9$ into the above equation and solve for t. Doing this gives:

$$9 = 9.259 - 9.259 \cdot e^{-2.7 \cdot t}$$

$$t = \tfrac{-1}{2.7} \cdot \ln\left(\tfrac{0.259}{9.259}\right) \approx 1.33 \text{ hours.}$$

6.(a) To locate the equilibrium solutions of a differential equation, you set the derivative equal to zero, and then (in this case) solve for $P(T)$. Setting the derivative equal to zero gives the following equation.

$$0 = 0.25 \cdot P(T) \cdot [P(T) + 1] \cdot [P(T) - 1] \cdot [2 - P(T)]^2.$$

The solutions of this equation are: $P(T) = 0$, $P(T) = -1$, $P(T) = 1$ and $P(T) = 2$. These are the equilibrium solutions of the differential equation.

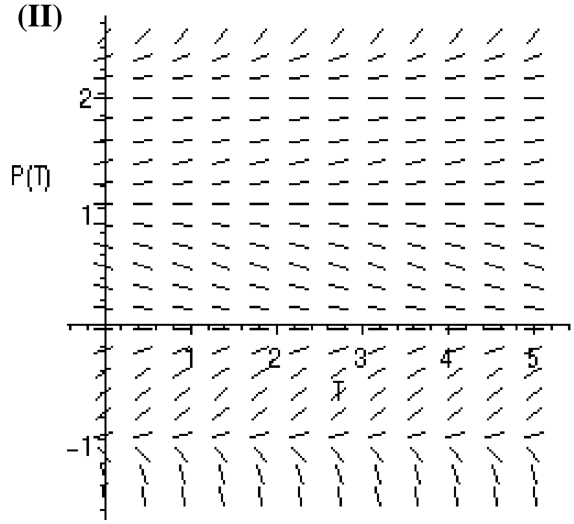
6.(b) The values of the derivative can be calculated by plugging the given values of $P(T)$ into the differential equation. The supplied values of T could also be plugged into the differential equation, but as T

does not appear directly anywhere in the formula for the differential equation, the value of the derivative $P'(T)$ will not be affected by the specific value of *T*. The completed table is shown below.

T	*P*(*T*)	$P'(T)$
0	2.5	0.8203
1	1.5	0.1172
2	0.5	−0.2109
3	−0.5	0.5859
4	−1.5	−5.7422

6.(c) The slope field for the differential equation must be compatible with the information that you have calculated in Parts (a) and (b) of this question. The only one of the four slope fields given that actually does this is slope field **(II)** pictured below. Brief reasons for rejecting the other three slope fields are given below the picture.

(II)



Slope field (I): At *P*(*T*) = −1.5 the little line segments are pointing upwards indicating a positive slope. However, according to the differential equation $P'(T)$ = −5.7422.

Slope field (III): There is no equilibrium solution at height *P*(*T*) = −1.

Slope field (IV): At *P*(*T*) = 2.5 the little line segments are pointing downwards indicating a negative slope. However, according to the differential equation $P'(T)$ = +0.8203.

6.(d) The appearance of slope field **(II)** near each of the equilibrium solutions is what allows you to classify the solutions as stable, unstable or semi-stable. Recall that:

Type of equilibrium solution	Appearance of little line segments near the equilibrium solution (when read from **left to right**)
Stable	Little line segments are attracted towards the equilibrium solution
Unstable	Little line segments are repelled away from the equilibrium solution
Semi-stable	On one side (above or below) the little line segments are attracted towards the equilibrium solution. On the other side, (below or above) the little line segments are repelled away from the equilibrium solution.

The equilibrium solutions and their classifications are listed in the table below.

Equilibrium solution	Classification (stable, unstable, semi-stable)
$P(T) = -1$	Unstable
$P(T) = 0$	Stable
$P(T) = 1$	Unstable
$P(T) = 2$	Semi-stable

7.(a) $y(t) = \frac{1}{5}e^{4t} + \frac{4}{5}e^{-t}$.

7.(b) $y(t) = 2e^{-3t} \cdot \sin(t)$.

7.(c) $y(x) = \sec(x)$.

7.(d) $y(t) = -2e^{-3t} + 3e^{-2t}$.

8.(a) Let y be the vertical distance up from the bottom of the tank. Then the hydrostatic force exerted against each of the triangular ends of the tank is given by:

$$\text{Force} = \int_0^3 (9.8)\cdot(1000)\cdot(3-y)\cdot(y)\cdot dy = 44100 \text{ N}.$$

8.(b) Approximately 1.06×10^6 J.

8.(c) The water will be 2 meters deep.

9.(a) The differential equation will be: $\frac{df}{dt} = k \cdot f \cdot (1-f) = -k \cdot f \cdot (f-1)$.

9.(b) The differential equation from Part (a) can be solved using the techniques of Separation of Variables and partial fractions. Doing this:

$$\frac{df}{dt} = -k \cdot f \cdot (f-1)$$

$$\int \frac{1}{f\cdot(f-1)}\cdot df = \int -k\cdot dt$$

$$\int \left(\frac{-1}{f}+\frac{1}{f-1}\right)\cdot df = \int -k\cdot dt$$

$$-\ln(|f|)+\ln(|f-1|) = -k\cdot t + C$$

$$\ln\left(\left|\frac{f-1}{f}\right|\right) = -k\cdot t + C$$

$$\frac{f-1}{f} = A\cdot e^{-k\cdot t} \qquad \text{where } A = \pm e^{C}.$$

$$f-1 = A\cdot e^{-k\cdot t}\cdot f$$

$$f - A\cdot e^{-k\cdot t}\cdot f = 1$$

$$f = f(t) = \frac{1}{1-A\cdot e^{-k\cdot t}}.$$

9.(c) Between 3:30pm and 3:45pm, 900 of the 1000 town residents will have heard the rumor. You can calculate this by using the two function values: $f(0) = 0.08$ and $f(4) = 0.5$ to determine the values of A and k in the function from Part (b). Once A and k have been identified, set $f = 0.9$ and solve for t.

10.(a) Area = 0.5.

10.(b) Area = 71/6.

10.(c) Area = 32/3.