Outline

1. Trapezoid and Midpoint rules.
2. Review of Simpson.
3. Error in Simpson.
4. Type 1 indefinite integrals.

1. Overestimates / Underestimates for Trapezoid and Midpoint

- Concavity is the important property of $f(x)$.

Concavity of $f(x)$	Trapezoid Rule	Midpoint Rule
$f(x)$ is concave down on (a,b)	Under-estimate of  $\int_a^b f(x)\,dx$	Over-estimate of $\int_a^b f(x)\,dx$
$f(x)$ is concave up on (a,b)	Over-estimate of  $\int_a^b f(x)\,dx$	Under-estimate of $\int_a^b f(x)\,dx$

2. Review of Simpson

$$S_{2N} = \frac{1}{3} T_N + \frac{2}{3} M_N$$

Example

Estimate $\int_1^4 e^{x^2} dx$ using Simpson's rule and 100 rectangles.

Solution

$\boxed{N=50}$

Midpoint = 1138813.643

Trapezoid = 1170636.855

Simpson = 1149421.38.

$\boxed{N = 100}$

3. Error Bound for Simpson's Method

$$|\text{Error}| \leq \frac{M \cdot (b-a)^5}{180 \cdot N^4}$$

(a,b) interval integral is over.

$N = \#$ of rectangles.

$M =$ maximum of $|f^{(4)}(x)|$ over interval $[a,b]$.

Example

How many rectangles do you need to estimate $\int_1^4 e^{x^2}\,dx$ with an error less than 0.00001?

Solution

$$\text{Error} \le \frac{M\cdot(b-a)^5}{180\cdot N^4} < 0.00001$$

$a = 1 \quad b = 4 \quad f(x) = e^{x^2}$

Find M.

Calculate N.

To find M:

$$f^{(4)}(x) = 16x^4 e^{x^2} + 48x^2 e^{x^2} + 8e^{x^2}$$

Use $M = 4.333 \times 10^{10}$

$$\frac{(4.333 \times 10^{10})(4-1)^5}{180 \cdot N^4} < 0.00001$$

$$N > \left(\frac{(4.333 \times 10^{10})(4-1)^5}{(180)(0.00001)}\right)^{1/4}$$

$N > 8746.$ (Round up.)

Use at least 8746 rectangles.

Note: ① Simpson's rule **always** uses an **even** number of rectangles.

② When rounding an error calculation for Simpson's rule, round up to the nearest **even** whole number.

4. Type 1 Improper Integrals.

- Integrals that have $+\infty$ or $-\infty$ as one of the limits of integration

e.g. $\int_1^\infty \frac{1}{x^2}\,dx$.

- Is the area <u>finite</u> (integral <u>converges</u>) or <u>infinite</u> (integral <u>diverges</u>)?

Example

Does $\int_1^\infty \frac{1}{x^2}\,dx$ converge or diverge?

Solution

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x^2}\,dx$$ Express integral using limits.

$$= \lim_{a\to\infty} \left[-\frac{1}{x}\right]_1^a$$ Calculate anti-derivative

$$= \lim_{a\to\infty} \frac{-1}{a} - \frac{-1}{1}$$ Plug in limits of integration

$$= 1$$ Take limit as $a \to \infty$.

Integral converges.

Example

Does $\int_2^\infty \frac{1}{x^3}\,dx$ converge or diverge?

Solution

Express integral using limit notation.

$$\int_2^\infty \frac{1}{x^3}\,dx = \lim_{a\to\infty} \int_2^a \frac{1}{x^3}\,dx$$

Find anti-derivative

$$= \lim_{a\to\infty} \left[\frac{-x^{-2}}{2} \right]_2^a$$

Evaluate at limits of integration

$$= \lim_{a\to\infty} \underbrace{-\frac{a^{-2}}{2}}_{\to 0} - \underbrace{\frac{2^{-2}}{2}}_{\to 1/8}$$

$= 1/8$ Integral converges.

Useful Integrals

p- integrals:

$$\int_c^{\infty} \frac{k}{x^p}\,dx \rightarrow \boxed{\text{Converges for } p > 1}$$

$$\rightarrow \boxed{\text{Diverges for } p \leq 1}$$

c positive #

Exponential functions.

$$\int_c^{\infty} e^{kx}\,dx \rightarrow \boxed{\text{Converges } k < 0}$$

$$\rightarrow \boxed{\text{Diverges for } k \geq 0}$$

c positive #