1. Outline

1. Ratio test.
2. p-series.
3. Comparison test.

1. Ratio Test

Example

$$n! = \underbrace{(1)(2)(3)(4)\ldots(n-2)(n-1)\cdot n}_{\text{product of first } n \text{ whole numbers.}}$$

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$

converge or diverge?

Solution

Ratio test.

$$a_n = \frac{10^n}{n!} \qquad a_{n+1} = \frac{10^{n+1}}{\underbrace{(n+1)!}_{\text{NOT } n!+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}}$$

$10^n \cdot 10^1$

$$= \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n}$$

$(n+1) \cdot n!$

$$= \frac{10}{n+1}$$

Take limit:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{10}{n+1} = 0.$$

Conclusion:

$\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converges absolutely.

Absolute and Conditional Convergence

- Infinite series: $\sum_{n=1}^{\infty} a_n$.

- Absolute convergence: $\sum_{n=1}^{\infty} |a_n|$ converges.

- Conditional convergence: $\sum_{n=1}^{\infty} a_n$ converges.

- For series with positive and negative terms, conditional and absolute convergence are different.

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ "alternating harmonic series"

converges conditionally, but since $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| =$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, the alternating harmonic series does not converge absolutely because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Example

Does : $$\sum_{n=1}^{\infty} \frac{n^2 \cdot 2^{n+3}}{(4n)!}$$

converge or diverge?

Solution

$$a_n = \frac{n^2 \cdot 2^{n+3}}{(4n)!}$$

NOT n^2+1

$$a_{n+1} = \frac{(n+1)^2 \cdot 2^{n+4}}{(4n+4)!}$$

NOT $(4n+1)!$ NOT $(4n)!+1$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 \cdot 2^{n+4}}{(4n+4)!} \cdot \frac{(4n)!}{n^2 \cdot 2^{n+3}}$$

$$= \frac{2 \cdot}{(4n+4)(4n+3)(4n+2)(4n+1)} \cdot \frac{(n+1)^2}{n^2}$$

$$= \frac{2 \cdot n^2 + \overset{\cancel{O(n)}}{\text{peanuts}}}{256\, n^6 + \overset{\cancel{O(n^5)}}{\text{peanuts}}}.$$

Take limit:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2n^2}{256n^6} = 0.$$

Conclusion:

$$\sum_{n=1}^{\infty} \frac{n^2 \cdot 2^{n+3}}{(4n)!}$$ converges absolutely.

2. p-series

- A series : $\displaystyle\sum_{n=1}^{\infty} \frac{C}{n^p}$ is called a p-series. ← constant

- If $p > 1$: p-series converges.

- If $p \leq 1$: p-series diverges.

(Proved using integral test.)

Example

Does $$\sum_{n=1}^{\infty} \frac{n^2 \cdot (2n)!}{3^{n+3}}$$

converge or diverge?

Solution

$$a_n = \frac{n^2 \, (2n)!}{3^{n+3}}$$

$$a_{n+1} = \frac{(n+1)^2 \cdot \big(2 \cdot (n+1)\big)!}{3^{n+1+3}}$$

$$= \frac{(n+1)^2 \cdot (2n+2)!}{3^{n+4}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 \, (2n+2)!}{3^{n+4}} \cdot \frac{3^{n+3}}{n^2 \cdot (2n)!}$$

$$= \frac{(2n+2)(2n+1)}{3} \; \frac{(n+1)^2}{n^2}$$

Take limit:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{4n^4 + \text{peanuts}}{3n^2}\right|$$

$$= +\infty .$$

Conclusion:

$\displaystyle\sum_{n=1}^{\infty} \frac{n^2 \cdot (2n)!}{3^n}$ diverges.

3. Comparison Tests

If $0 \le a_n < b_n$, then:

(a) If $\sum_{n=1}^{\infty} b_n$ converges,

then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ diverges then

$\sum_{n=1}^{\infty} b_n$ diverges.

Example

Does $\sum_{n=1}^{\infty} \frac{n-1}{n^3+4}$ converge or diverge?

Solution

$\frac{n-1}{n^3+4} \approx \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so guess $\sum_{n=1}^{\infty} \frac{n-1}{n^3+4}$ converges.

Objective: Create a convergent series that is larger than $\sum_{n=1}^{\infty} \frac{n-1}{n^3+4}$.

To do is:

$$n^3 < n^3 + 4$$

$$\frac{1}{n^3} > \frac{1}{n^3+4}$$

$$\frac{n}{n^3} > \frac{n-1}{n^3} \geqslant \frac{n-1}{n^3+4}$$

<u>So:</u> $$\frac{1}{n^2} > \frac{n-1}{n^3+4}.$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series $p=2$)

so by the Comparison Test

$\sum_{n=1}^{\infty} \frac{n-1}{n^3+4}$ also converges.