Solutions to Homework Assignment #7

1. To verify that the given function is a solution of the given differential equation, it is sufficient to plug the given function into the left hand side of the differential equation and show that the expression obtained simplifies to the right hand side of the differential equation.

(a) $t \cdot y' - y = t^2$ $\qquad y(t) = 3t + t^2$

$$\textbf{Left hand side} = t \cdot y' - y$$
$$= t \cdot (3 + 2t) - (3t + t^2)$$
$$= 3t + 2t^2 - 3t - t^2$$
$$= t^2 = \textbf{Right hand side.}$$

(b) $y' - 2t \cdot y = 1$ $\qquad y(t) = e^{t^2} \cdot \int_0^t e^{-s^2}\,ds + e^{t^2}$

$$\textbf{Left hand side} = y' - 2t \cdot y$$
$$= 2t \cdot e^{t^2} \cdot \int_0^t e^{-s^2}\,ds + e^{t^2} \cdot \left(e^{-t^2} - 0\right) - 2t \cdot e^{t^2} \cdot \int_0^t e^{-s^2}\,ds + e^{t^2}$$
$$= e^{t^2} \cdot e^{-t^2}$$
$$= 1 = \textbf{Right hand side.}$$

2. We will solve the following initial value problem using the technique of Separation of Variables:

$$\frac{dy}{dt} = ry + k \qquad y(0) = y_0.$$

Doing this:

$$\frac{dy}{dt} = r\left(y + \frac{k}{r}\right)$$

$$\int \frac{1}{y+\frac{k}{r}} \cdot dy = \int r \cdot dt$$

$$\ln\left(\left|y+\tfrac{k}{r}\right|\right) = r \cdot t + C$$

$$y + \tfrac{k}{r} = A \cdot e^{r \cdot t} \qquad \text{where } A = \pm e^{C}.$$

To evaluate the constant A, use the initial condition $y(0) = y_0$:

$$y_0 + \tfrac{k}{r} = A.$$

Putting all of this together gives the final answer to this problem:

$$y = \left(y_0 + \tfrac{k}{r}\right) \cdot e^{r \cdot t} - \tfrac{k}{r}.$$

3. We will use the method of **Integrating Factors** to solve the following initial value problem:

$$y' - y = 2te^{2t} \qquad\qquad y(0) = 1.$$

From the differential equation, $p(t) = -1$ so the integrating factor I is given by:

$$I = e^{\int p(t)dt} = e^{-t}.$$

Multiplying each term in the differential equation by this integrating factor gives:

$$e^{-t} \cdot y' - e^{-t} \cdot y = 2t \cdot e^{-t} \cdot e^{2t}.$$

Simplifying using the Law of Exponents and the product rule gives the following equation:

$$\frac{d}{dt}\left(e^{-t} \cdot y\right) = 2t \cdot e^{t}.$$

Integrating both sides of this equation with respect to t (using integration by parts to integrate the right hand side of the equation with $u = 2t$ and $v' = e^{t}$) we obtain:

$$e^{-t} \cdot y = 2t \cdot e^{t} - 2e^{t} + C.$$

To evaluate the constant C we can use the initial condition $y(0) = 1$. Doing this gives:

$$1 = -2 + C,$$

so that $C = 3$ and the final answer to this problem is:

$$y = 2t \cdot e^{2t} - 2e^{2t} + 3e^{t}.$$

4. We will use the method of **Integrating Factors** to solve the following initial value problem:

$$y' + \tfrac{2}{t} y = \tfrac{\cos(t)}{t^2} \qquad\qquad y(\pi) = 0,\ t > 0$$

From the differential equation, $p(t) = \tfrac{2}{t}$ so the integrating factor I is given by:

$$I = e^{\int p(t)dt} = e^{\int \frac{2}{t} \cdot dt} = e^{2 \cdot \ln(t)} = e^{\ln(t^2)} = t^2.$$

Multiplying each term in the differential equation by this integrating factor gives:

$$t^2 \cdot y' + 2t \cdot y = \cos(t).$$

Recognizing the left hand side of the equation as the derivative of the product $t^2 \cdot y$ and integrating both sides of the equation gives:

$$t^2 \cdot y = \sin(t) + C.$$

To determine the value of the constant C, we can substitute the initial condition $y(\pi) = 0$ into the above equation. Doing this gives $C = 0$. Putting everything together gives the final answer to this problem:

$$y = \frac{\sin(t)}{t^2}.$$

5. We will use the method of **Integrating Factors** to solve the following initial value problem:

$$t^3 \cdot y' + 4t^2 \cdot y = e^{-t} \qquad\qquad y(-1) = 0$$

Before identifying $p(t)$ and calculating the integrating factor, we will divide each of the terms in the differential equation by the coefficient of y'. The equation then resembles:

$$y' + \frac{4}{t} \cdot y = \frac{e^{-t}}{t^3}.$$

Here, $p(t) = \tfrac{4}{t}$ so that the integrating factor, I, is calculated to be:

$$I = e^{\int p(t)dt} = e^{\int \frac{4}{t} \cdot dt} = e^{4 \cdot \ln(t)} = e^{\ln(t^4)} = t^4.$$

Multiplying each term in the differential equation by this integrating factor gives:

$$t^4 \cdot y' + 4t^3 \cdot y = t \cdot e^{-t}.$$

Recognizing the left hand side of the equation as the derivative of the product $t^4 \cdot y$ and integrating both sides of the equation (using integration by parts to integrate the right hand side of the equation with $u = t$ and $v' = e^{-t}$) gives:

$$t^4 \cdot y = -t \cdot e^{-t} - e^{-t} + C.$$

To evaluate the constant C we can use the initial condition $y(-1) = 0$. Doing this gives $C = 0$ so that the final answer to this problem can be written as:

$$y = \frac{-t \cdot e^{-t} + e^{-t}}{t^4}.$$

6. To find the solution, $y(x)$, of the following initial value problem:

$$y'' + y' - 2y = 0 \qquad y(0) = 1 \qquad y'(0) = 1,$$

we will begin by setting up a characteristic equation and finding its roots.

Characteristic Equation: $r^2 + r - 2 = 0$
$(r+2)(r-1) = 0$

so that the roots of the characteristic equation are $r = -2$ and $r = 1$. The form of the solution to the differential equation is then:

$$y(x) = C_1 \cdot e^{x} + C_2 \cdot e^{-2x}.$$

To determine the values of the two constants C_1 and C_2 we will use the two initial conditions to set up a pair of equations for C_1 and C_2, and then solve these.

$$y(0) = 1: \qquad C_1 + C_2 = 1$$

$$y'(0) = 1: \qquad C_1 - 2 \cdot C_2 = 1$$

These equations give $C_1 = 1$ and $C_2 = 0$ so that the final answer to this problem is:

$$y(x) = e^{x}$$

7. To find the solution, $y(x)$, of the following initial value problem:

$$2y'' + y' - 4y = 0 \qquad y(0) = 0 \qquad y'(0) = 1,$$

we will begin by setting up a characteristic equation and finding its roots.

Characteristic Equation: $2r^2 + r - 4 = 0$.

We can find the roots of this equation using the quadratic formula:

$$r = \frac{-1 \pm \sqrt{1 - 4(2)(-4)}}{2(2)} = \frac{-1+\sqrt{33}}{4}, \frac{-1-\sqrt{33}}{4}.$$

The form of the solution to the differential equation is:

$$y(x) = C_1 \cdot e^{\frac{-1+\sqrt{33}}{4}\cdot x} + C_2 \cdot e^{\frac{-1-\sqrt{33}}{4}\cdot x}.$$

To determine the values of the two constants C_1 and C_2 we will use the two initial conditions to set up a pair of equations for C_1 and C_2, and then solve these.

$$y(0) = 0: \qquad C_1 + C_2 = 0$$

$$y'(0) = 1: \qquad \frac{-1+\sqrt{33}}{4} \cdot C_1 + \frac{-1-\sqrt{33}}{4} \cdot C_2 = 1.$$

Solving these two equations for C_1 and C_2, we get:

$$C_1 = \frac{2}{\sqrt{33}} \qquad \text{and} \qquad C_2 = \frac{-2}{\sqrt{33}},$$

so that the final answer for this problem can be written as:

$$y(x) = \tfrac{2}{\sqrt{33}} \cdot e^{\frac{-1+\sqrt{33}}{4}\cdot x} - \tfrac{2}{\sqrt{33}} \cdot e^{\frac{-1-\sqrt{33}}{4}\cdot x}.$$

8. A particular solution is a solution of the nonhomogeneous differential equation that does not have any arbitrary, unspecified constants in it.

(a) To find a particular solution of the nonhomogeneous differential equation $y'' - 3y' - 4y = 3e^{2t}$ we begin with the function $N(t) = 3e^{2t}$.

$$N(t) = 3e^{2t}$$
$$N'(t) = 6e^{2t}$$

What is important here (e^{2t}) has already repeated so we will use $y_p(t) = F \cdot e^{2t}$ as our particular solution. To find the value of the constant F, substitute $y_p(t)$ into the nonhomogeneous differential equation. Doing this gives:

$$4F \cdot e^{2t} - 6F \cdot e^{2t} - 4F \cdot e^{2t} = 3e^{2t}.$$

Simplifying and solving for F gives $F = \frac{-1}{2}$ so that the particular solution here is given by:

$$y_p(t) = \tfrac{-1}{2} \cdot e^{2t}.$$

(b) To find a particular solution of the nonhomogeneous differential equation $y'' - 3y' - 4y = 2 \cdot \sin(t)$ we will begin with the function $N(t) = \sin(t)$.

$$N(t) = \sin(t)$$
$$N'(t) = \cos(t)$$
$$N''(t) = -\sin(t)$$

What's important here has started to repeat, so we will use:

$$y_p(t) = G \cdot \sin(t) + H \cdot \cos(t)$$

as our particular solution. To determine the values of the constants G and H we will substitute $y_p(t)$ into the nonhomogeneous differential equation and then equate coefficients of $\sin(t)$ and $\cos(t)$ to create two equations that we can solve to find G and H.

$$(-G \cdot \sin(t) - H \cdot \cos(t)) - 3(G \cdot \cos(t) - H \cdot \sin(t)) - 4(G \cdot \sin(t) + H \cdot \cos(t)) = -2 \cdot \sin(t).$$

Equating coefficients of $\sin(t)$ and $\cos(t)$ gives the following two equations:

$\sin(t)$: $\quad -5G + 3H = -2$

$\cos(t)$: $\quad -3G - 5H = 0$

Solving these two equations gives $G = \frac{5}{17}$ and $H = \frac{-3}{17}$. Putting everything together gives the following particular solution:

$$y_p(t) = \tfrac{5}{17} \cdot \sin(t) - \tfrac{3}{17} \cdot \cos(t).$$

9. Find the solution, $y(x)$, of the following initial value problem.

$$y'' + 4y = t^2 + 3e^t \qquad y(0) = 0 \qquad y'(0) = 2.$$

To solve this initial value problem we will follow the three steps that we normally follow when solving an initial value problem based on a nonhomogeneous differential equation.

Step 1: Solve the Homogeneous Equation

The homogeneous equation is $y'' + 4y = 0$. The corresponding characteristic equation is $r^2 + 4 = 0$ which has the roots $r = \pm 2i$. The homogeneous solution is then:

$$y_h(t) = C_1 \cdot \cos(2t) + C_2 \cdot \sin(2t).$$

Step 2: Create the Particular Solution

We will create the particular solution in two parts, the first being based on the nonhomogeneous function $N(t) = t^2$. Writing down this function and its derivatives gives the following list:

$$\begin{aligned} N(t) &= t^2. \\ N'(t) &= 2t. \\ N''(t) &= 2. \\ N'''(t) &= 0. \end{aligned}$$

Taking what is important from this list gives the following particular solution:

$$y_{p1}(t) = F \cdot t^2 + G \cdot t + H.$$

To determine the values of the constants F, G and H we will substitute this particular solution into the nonhomogeneous differential equation that has the nonhomogeneous part $N(t) = t^2$ and then equate coefficients of powers of t.

$$2F + 4\left(F \cdot t^2 + G \cdot t + H\right) = t^2.$$

This gives $F = 0.25$, $G = 0$ and $H = -0.125$ so that this part of the particular solution is given by:

$$y_{p1}(t) = 0.25 \cdot t^2 - 0.125.$$

We will now repeat this process using the nonhomogeneous function $N(t) = 3e^t$.

$$\begin{aligned} N(t) &= 3e^t. \\ N'(t) &= 3e^t. \end{aligned}$$

What is important (e^t) is already repeating so we will use $y_{p2}(t) = J \cdot e^t$. Substituting this into the nonhomogeneous differential equation with nonhomogeneous part $N(t) = 3e^t$ will enable use to determine the value of the constant J.

$$J \cdot e^t + 4 \cdot J \cdot e^t = 3 \cdot e^t$$

so that $J = 0.6$ and $y_{p2}(t) = 0.6 \cdot e^t$.

Putting the two parts of the particular solution together gives the particular solution we will use in Step 3 of this process:

$$y_p(t) = 0.25 \cdot t^2 - 0.125 + 0.6 \cdot e^t.$$

Step 3: Use the Initial Values

Before we use the initial values to find the constants C_1 and C_2, we will add the homogeneous and particular solutions.

$$y(t) = y_h(t) + y_p(t) = C_1 \cdot \cos(2t) + C_2 \cdot \sin(2t) + 0.25 \cdot t^2 - 0.125 + 0.6 \cdot e^t.$$

$$y(0) = 0: \qquad C_1 - 0.125 + 0.6 = 0$$

$$y'(0) = 2: \qquad 2C_2 + 0.6 = 2.$$

Solving these equations and substituting the values into $y(t)$ gives the final answer for this problem:

$$y(t) = -0.475 \cdot \cos(2t) + 0.7 \cdot \sin(2t) + 0.25 \cdot t^2 - 0.125 + 0.6 \cdot e^t.$$

10. The statement of the problem in #10 was as follows:

"Suppose that a room containing 1200 cubic feet of air is initially free of carbon monoxide. Beginning at time $t = 0$ cigarette smoke (containing 4% carbon monoxide) is introduced to the room at a rate of 0.1 cubic feet per minute. The well-circulated mixture is allowed to leave the room at the same rate."

(a) Find and expression for the concentration $x(t)$ of carbon monoxide in the room at any time $t > 0$.

To do this we will find and solve an initial value problem (a differential equation together with an initial value) for $x(t)$.

Setting up a differential equation for $x(t)$ from scratch is not that easy. We will begin by setting up a differential equation for $a(t)$, the amount (in cubic feet) of carbon monoxide in the room after t minutes and then adapt this to become a differential equation for $x(t)$.

Before we set up the differential equation for $a(t)$, note that:

$$x(t) = \frac{a(t)}{1200} \qquad \text{and that} \qquad \frac{da}{dt} = 1200 \cdot \frac{dx}{dt}.$$

To set up the differential equation for $a(t)$ note that the rate in will be given by the rate of flow into the room (0.1 cubic feet per minute) multiplied by the proportion of this flow that is carbon monoxide (0.04). The rate at which carbon monoxide leaves the room will be the rate of flow out (0.1 cubic feet per minute) multiplied by the proportion of the atmosphere of the room that is carbon monoxide. This proportion is $a(t)$ divided by 1200 cubic feet. Putting all of this into the template that we have for a differential equation gives:

$$\frac{da}{dt} = (0.1) \cdot (0.04) - (0.1) \cdot \frac{a(t)}{1200}.$$

Using the above equations to replace the derivative of $a(t)$ and the function $a(t)$ gives the differential equation:

$$1200 \cdot \frac{dx}{dt} = 0.004 - (0.1) \cdot x(t),$$

which can be rearranged to the more conventional forms:

$$\frac{dx}{dt} = \frac{0.004}{1200} - \frac{0.1}{1200} \cdot x(t) = \frac{-0.1}{1200} \cdot \left(x(t) - 0.04\right).$$

As the room is initially free of carbon dioxide, the initial condition will be:

$$x(0) = 0.$$

We will solve the differential equation for $x(t)$ using the technique of Separation of Variables.

$$\int \frac{1}{x - 0.04} \cdot dx = \int \frac{-0.1}{1200} \cdot dt$$

$$\ln\left(|x - 0.04|\right) = \frac{-0.1}{1200} \cdot t + C$$

$$x = 0.04 + A \cdot e^{\frac{-0.1}{1200} \cdot t} \qquad \text{where } A = \pm e^{C}.$$

Using $x(0) = 0$ to solve for A gives $A = -0.04$ so that the formula for $x(t)$ is:

$$x = 0.04 - 0.04 \cdot e^{\frac{-0.1}{1200} \cdot t}.$$

(b) Extended exposure to carbon monoxide in concentrations as low as 0.00012 is harmful to the human body. Find the amount of time that elapses before this concentration is reached in the room.

To solve this part of the problem we will set $x(t) = 0.00012$ and solve for t.

$$0.00012 = 0.04 - 0.04 \cdot e^{\frac{-0.1}{1200} \cdot t}$$

$$t = \frac{-1200}{0.1} \cdot \ln\left(\frac{0.04 - 0.00012}{0.04}\right) \approx 36.05 \text{ minutes.}$$