Recitation Handout 17: Radius and Interval of Convergence

Interval of Convergence

The interval of convergence of a power series: \( \sum_{n=0}^{\infty} c_n \cdot (x - a)^n \) is the interval of x-values that can be plugged into the power series to give a convergent series.

The center of the interval of convergence is always the anchor point of the power series, \( a \).

Radius of Convergence

The radius of convergence is half of the length of the interval of convergence. If the radius of convergence is \( R \) then the interval of convergence will include the open interval: \( (a - R, a + R) \).

Finding the Radius of Convergence

To find the radius of convergence, \( R \), you use the Ratio Test.

Step 1: Let \( a_n = c_n \cdot (x - a)^n \) and \( a_{n+1} = c_{n+1} \cdot (x - a)^{n+1} \).

Step 2: Simplify the ratio \( \frac{a_{n+1}}{a_n} = \frac{c_{n+1} \cdot (x - a)^{n+1}}{c_n \cdot (x - a)^n} = \frac{c_{n+1}}{c_n} \cdot (x - a) \).

Step 3: Compute the limit of the absolute value of this ratio as \( n \to \infty \).

Step 4: Interpret the result using the table below.

<table>
<thead>
<tr>
<th>Limit of absolute value of ratio as ( n \to \infty ).</th>
<th>Radius of convergence, ( R ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero.</td>
<td>( R = \frac{1}{N} ). The interval of convergence includes ( (a - \frac{1}{N}, a + \frac{1}{N}) ) and possibly the end-points ( x = a - \frac{1}{N} ) and ( x = a + \frac{1}{N} ).</td>
</tr>
<tr>
<td>( N \cdot</td>
<td>x - a</td>
</tr>
</tbody>
</table>

Are the end-points in the Interval of Convergence?

Each of the two end-points (\( x = a - R \) and \( x = a + R \)) may or may not be part of the interval of convergence. To determine whether the end-points are in the interval of convergence, you have to plug them into the power series (one at a time) to get an infinite series. You then use a convergence test to determine whether or not the infinite series converges or diverges. If the infinite series converges, then the end-point that you plugged into the power series is in the interval of convergence. Otherwise, the end-point is not in the interval of convergence.

If you use the ratio test at each end-point you usually get an inconclusive test so it is best to try a different convergence test when investigating the end-points of the interval of convergence.
(a) Find the radius of convergence and interval of convergence for: 

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (x - 1)^n \]

\[ a_n = \]

\[ \frac{a_{n+1}}{a_n} = \]

Limit of absolute value of ratio =

**RADIUS OF CONVERGENCE:** ________________________________
Find the radius of convergence and interval of convergence for:

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (x - 1)^n \]

End points: and

Convergence or divergence at first end-point:

Convergence or divergence at second end-point:

INTERVAL OF CONVERGENCE:
(b) Find the radius of convergence and interval of convergence for: \[ \sum_{n=0}^{\infty} 2^n \cdot x^{2n} \]

\[ a_n = \] 

\[ \frac{a_{n+1}}{a_n} = \] 

Limit of absolute value of ratio = 

RADIUS OF CONVERGENCE: ________________________________
Find the radius of convergence and interval of convergence for:

\[
\sum_{n=0}^{\infty} 2^n \cdot x^{2n}
\]

End points: and

Convergence or divergence at first end-point:

Convergence or divergence at second end-point:

INTERVAL OF CONVERGENCE: ____________________________
(c) Find the radius of convergence and interval of convergence for:

\[ \sum_{n=1}^{\infty} \frac{4^n}{n} \cdot (x - 3)^n \]

\[ a_n = \]

\[ \frac{a_{n+1}}{a_n} = \]

Limit of absolute value of ratio =

RADIUS OF CONVERGENCE: ________________________________
Find the radius of convergence and interval of convergence for:

\[ \sum_{n=1}^{\infty} \frac{4^n}{n} \cdot (x - 3)^n \]

End points: and

Convergence or divergence at first end-point:

Convergence or divergence at second end-point:

INTERVAL OF CONVERGENCE: ____________________________
Find the radius of convergence and interval of convergence for:

\[ 1 + 2 \cdot (x + 5) + \frac{4!}{(2!)^2} \cdot (x + 5)^2 + \frac{6!}{(3!)^2} \cdot (x + 5)^3 + \frac{8!}{(4!)^2} \cdot (x + 5)^4 + \ldots \]

\[ a_n = \]

\[ \frac{a_{n+1}}{a_n} = \]

Limit of absolute value of ratio =

RADIUS OF CONVERGENCE: ________________________________
Find the radius of convergence and interval of convergence for:

\[
1 + 2 \cdot (x + 5) + \frac{4!}{(2!)^2} \cdot (x + 5)^2 + \frac{6!}{(3!)^2} \cdot (x + 5)^3 + \frac{8!}{(4!)^2} \cdot (x + 5)^4 + \ldots
\]

End points: \[ \] and \[ \]

Convergence or divergence at first end-point: \[ \]

Convergence or divergence at second end-point: \[ \]

INTERVAL OF CONVERGENCE: \[ \]
ANSWERS:

(a) Radius of convergence = 1. Interval of convergence is (0, 2].

(b) Radius of convergence = 0.5. Interval of convergence is (−0.5, 0.5).

(c) Radius of convergence = 0.25. Interval of convergence is [2.75, 3.25).

(d) Radius of convergence = 0.25. Interval of convergence is (−5.25, −4.75).