Handout 10: The Method of Undetermined Coefficients

The Method of Undetermined Coefficients is a technique for solving non-homogenous second order differential equations with constant coefficients. To solve a differential equation like:

$$y'' + y = 0.001 \cdot x^2$$

to find a formula for $y(x)$, the Method of Undetermined Coefficients involves three steps:

Step 1:	Find the general solution of the related homogeneous differential equation.
Step 2:	Find a particular solution of the non-homogeneous differential equation.
Step 3:	Use any supplied initial values to find the values of unspecified constants.

Step 1: Solving the Homogeneous Differential Equation

Step 1 works in exactly the same way as the homogeneous second order differential equations that you have solved in lecture:

(a) Create a characteristic equation.

(b) Find the roots of the characteristic equation.

(c) Use the roots of the characteristic equation to find a formula for $y(x)$. Table 1 shows the various forms that the function $y(x)$ can take depending on the number and nature of the roots of the characteristic equation.

Note that in Table 1, C_1 and C_2 both represent constants.

Roots	Formula for $y(x)$
Two distinct, real roots r_1 and r_2	$y(x) = C_1 \cdot e^{r_1 \cdot x} + C_2 \cdot e^{r_2 \cdot x}$
One real root, r	$y(x) = (C_1 + C_2 \cdot x) \cdot e^{r \cdot x}$
Complex roots, $\alpha \pm i\beta$	$y(x) = C_1 \cdot e^{\alpha \cdot x} \cdot \cos(\beta \cdot x) + C_2 \cdot e^{\alpha \cdot x} \cdot \sin(\beta \cdot x)$

Table 1: Solutions of the homogeneous differential equation.

Step 2: Finding a Particular Solution of the Non-homogeneous Differential Equation

To find the particular solution, start with the non-homogeneous function from the differential equation. We will refer to this function as $N(x)$.

(a) Calculate derivatives of $N(x)$ (first, second, third, etc.). Keep going until the derivatives start repeating or the derivatives are reduced to zero.

(b) Look through the terms that you have written down and pick out all of the unique functions that have appeared either in the formula for $N(x)$ or in one of its derivatives.

(c) Your particular solution will be the sum of all of these individual functions. Write constants in front of each as you write them down in the sum.

(d) If one of the terms in your also appears in the homogeneous solution that you wrote down in Step 1, multiply that term by x here. (This is sometimes called the "Modification Rule" in differential equations classes.)

(e) To determine the values of the constants in your particular solution, plug the particular solution back into the non-homogeneous differential equation and equate coefficients.

Step 3: Apply Initial Conditions

The most general solution of the non-homogeneous differential equation is the sum of the function that you found in Step 1 and the particular solution that you found in Step 2.

If you did Step 2 correctly, then you should have found values for all of the constants in your particular solution. The idea behind adding the solution of the homogeneous differential equation (which has two unspecified constants in it) to the particular solution is to allow you to take initial conditions into account.

(a) Write out the formula for your most general solution of the non-homogeneous differential equation by adding the solution you obtained in Step 1 to the particular solution that you found in Step 2.

(b) Apply the initial conditions to find the values of C_1 and C_2.

Step 3 can be a computationally challenging step, but remember that you are always using the initial conditions to create two equations that will involve only numbers and C_1 and C_2, and then you are going to solve these equations to find the values of C_1 and C_2.

Example

Solve the following initial value problem:

$$y'' + y = 0.001 \cdot x^2 \qquad y(0) = 0 \qquad y'(0) = 1.5.$$

Solution

Step 1: Solve the homogeneous differential equation.

The homogeneous differential equation is: $y'' + y = 0$. The characteristic equation is $r^2 + 1 = 0$, which has roots of $r = 0 \pm i$. The solution of the homogeneous differential equation is:

$$y_h(x) = C_1 \cdot \cos(x) + C_2 \cdot \sin(x).$$

Step 2: Find a particular solution.

For the non-homogeneous differential equation in this example, $N(x) = 0.001 \cdot x^2$. If we consider $N(x)$ and its non-zero derivatives, the collection of terms that we have is as follows:

$$N(x) = 0.001 \cdot x^2$$
$$N'(x) = 0.002 \cdot x$$
$$N''(x) = 0.002.$$

So, the individual terms that we have here boil down to x^2, x and 1. Putting these together into a sum (with constants F, G and H) gives:

$$y_p(x) = F \cdot x^2 + G \cdot x + H.$$

To determine the values of F, G and H we can plug $y_p(x)$ back into the non-homogeneous differential equation:

$$y'' + y = 0.001 \cdot x^2.$$

Observing that:

$$y'_p(x) = 2 \cdot F \cdot x + G$$

$$y'_p(x) = 2 \cdot F$$

this gives us:

$$y''_p(x) + y_p(x) = 2 \cdot F + F \cdot x^2 + G \cdot x + H = 0.001 \cdot x^2.$$

Equating coefficients of powers of x gives us the set of equations:

Coefficient of x^2: $F = 0.001$

Coefficient of x: $G = 0$

Constant term: $2F + H = 0$.

The solution of this set of equations is: $F = 0.001$, $G = 0$ and $H = -0.002$ so the formula for the particular solution is:

$$y_p(x) = 0.001 \cdot x^2 - 0.002.$$

Step 3: Apply Initial Conditions

The sum of the solution of the homogeneous equation and the particular solution is:

$$y(x) = y_h(x) + y_p(x) = C_1 \cdot \cos(x) + C_2 \cdot \sin(x) + 0.001 \cdot x^2 - 0.002.$$

To determine the values of C_1 and C_2 we will use the supplied initial conditions of $y(0) = 0$ and $y'(0) = 1.5$. Doing this by first plugging in $x = 0$ and $y = 0$ gives:

$$0 = C_1 \cdot \cos(0) + C_2 \cdot \sin(0) + 0.001 \cdot (0)^2 - 0.002,$$

which simplifies to:

$$C_1 - 0.002 = 0,$$

so that $C_1 = 0.002$.

Next, we take the derivative of $y(x)$ and use the initial condition $y'(0) = 1.5$. The derivative of $y(x)$ is given by:

$$y'(x) = -0.002 \cdot \sin(x) + C_2 \cdot \cos(x) + 0.002 \cdot x.$$

Plugging in the initial value and solving for C_2 gives:

$$y'(0) = -0.002 \cdot \sin(0) + C_2 \cdot \cos(0) + 0.002 \cdot (0) = 1.5,$$

so that $C_2 = 1.5$.

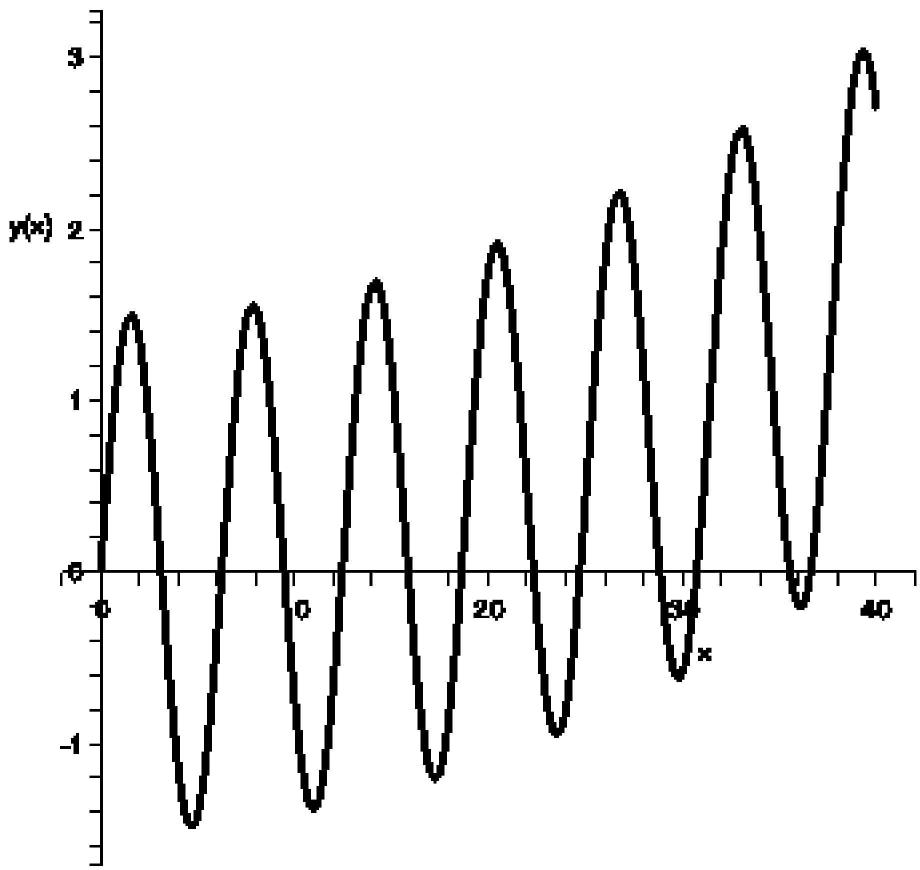
Final Answer:

Putting everything together gives the final answer, which is the formula for $y(x)$ with no unspecified constants remaining. This formula is:

$$y(x) = 0.002 \cdot \cos(x) + 1.5 \cdot \sin(x) + 0.001 \cdot x^2 - 0.002.$$

Check:

The graph of the solution to this initial value problem produced by the mathematical software Maple™ is shown below. Enter the formula for $y(x)$ given above into your graphing calculator to verify that the graph is the same as the one shown below.



Now that you've worked your way through an example, try to apply what you have learned to solve the following three non-homogeneous differential equations.

Your objective in each case is to find a formula for $y(x)$ in which you have found the numerical values of all of the constants.

(a) $y'' - y' - 12y = 144 \cdot x^3 + 12.5$ $y(0) = 5$ $y'(0) = -0.5$

(b) $y'' + 0.2y' + 0.26y = 1.22 \cdot e^{0.5x}$ $\quad y(0) = 3.5$ $\quad y'(0) = 0.35$

(c) $y'' + 4y = 16 \cdot \cos(2x)$ $\quad y(0) = 0$ $\quad y'(0) = 0$

Answers

(a) $y(x) = 2e^{-3x} + 3e^{4x} - 12x^3 + 3x^2 - 6.5x$

(b) $y(x) = 1.5 \cdot e^{-0.1x} \cdot \cos(0.5x) - e^{-0.1x} \cdot \sin(0.5x) + 2 \cdot e^{0.5x}$

(c) $y(x) = 4x \cdot \sin(2x)$