Handout 15: Review Problems for the Cumulative Final Exam

The topics that will be covered on Final Exam are as follows.

- Integration formulas.
- U-substitution.
- Integration by parts.
- Integration using trigonometric formulas and identities.
- Trigonometric substitution.
- Partial fractions.
- Integration tricks such as polynomial long division and completing the square.
- Approximating integrals (Riemann sums, trapezoid and midpoint rules, Simpson's rule).
- Conditions under which different integration methods produce over and under estimates.
- Error estimates for trapezoid, midpoint and Simpson's rules.
- Improper integrals.
- Areas between curves.
- Calculating volumes (non-rotation) using integrals.
- Volumes of revolution – disk method.
- Volumes of revolution – washer method.
- Volumes of revolution – shell method.
- Arc length.
- Using integrals to calculate masses.
- Center of mass.
- Using integrals to calculate work in physics.
- Using integrals to calculate hydrostatic force.
- Euler's method.
- Slope fields.
- Equilibrium solutions.
- Separation of variables.
- Integrating factors.
- Second order, homogeneous differential equations with constant coefficients.
- Method of Undetermined Coefficients.
- Calculating formulas for partial sums (e.g. for telescoping series).
- Convergence of infinite series by definition (limit of partial sums).
- Geometric series (finite) and their applications.
- Geometric series (infinite) and their applications.
- n^{th} Term test for divergence.
- Integral test.
- Ratio test.
- Comparison test (compare with p-series or infinite geometric series).
- Alternating series test.
- Absolute versus conditional convergence for alternating series.
- Estimating the sum of an alternating series to a given level of accuracy.
- Summing a finite series with a calculator.
- Finding a formula for the Taylor series of $f(x)$ with center a from the definition.
- Finding a formula for the Taylor series of $f(x)$ with center a by modifying an existing series.
- Radius of convergence of a power series or Taylor series.
- Interval of convergence of a power series or Taylor series.
- Accuracy of Taylor polynomial approximations for functions.
- Parametric equations for circles, line segments, ellipses and other curves in the plane.

- Calculating tangent lines for curves defined by parametric equations.
- Finding arc lengths for curves defined by parametric equations.
- Sketching curves defined by polar equations.
- Finding equations for tangent lines when curves are defined by polar equations.

This (roughly) covers the end of Chapter 5, all of Chapters 6-8 and the first half of Chapter 9 of the textbook, together with additional topics (such as differential equations).

1. Find the Taylor series of the function $f(x) = \dfrac{x^2}{\sqrt{2+x}}$ centered at $a = 0$.

2. Consider the function $f(x) = x^{\frac{3}{4}}$.

(a) Find the Taylor polynomial of degree 3 for $f(x)$ centered at $a = 16$.

(b) Use your answer from Part (a) to estimate the value of $17^{\frac{3}{4}}$.

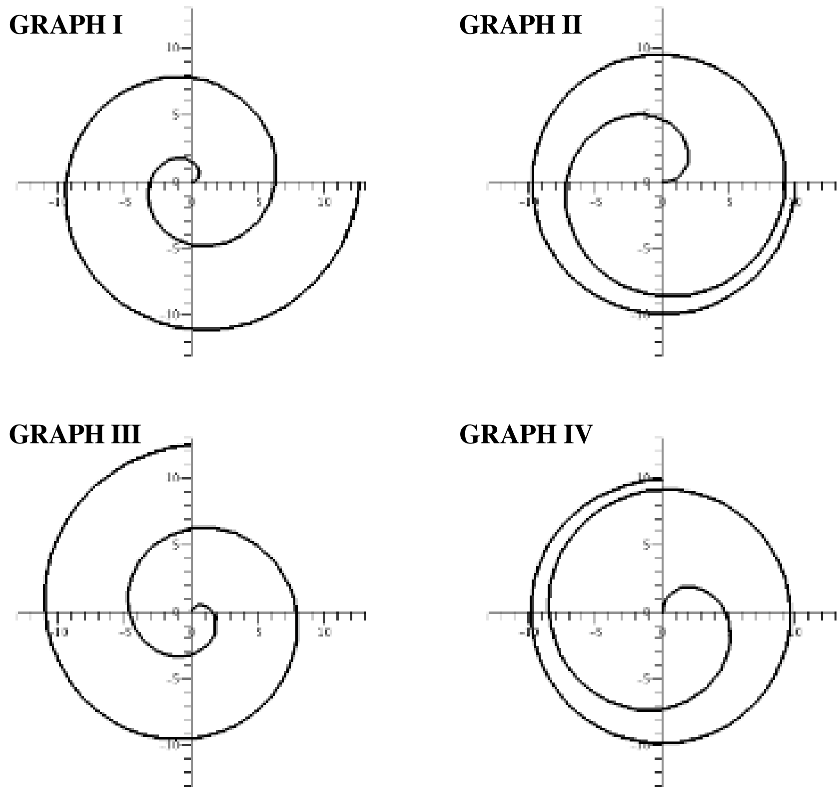
(c) Find a reasonable estimate for the error in your approximation from Part (b).

3. A flying ladybug lands in a spot of wet paint at the origin on the *xy*-plane. The ladybug then walks away from the origin tracing out a path described by the equations

$$x(t) = t \cdot \cos(t) \qquad y(t) = t \cdot \sin(t) \qquad t \geq 0.$$

t is measured in seconds and x, y are measured in millimeters.

(a) Circle the diagram that does the best job of representing the path traced out by the ladybug over the interval $0 \leq t \leq 4\pi$.



(b) Circle the integral that gives the *exact* length of the path that the ladybug travels during the first 2 seconds.

$$\int_0^2 \sqrt{\left(t \cdot \sin(t)\right)^2 + \left(t \cdot \cos(t)\right)^2}\, dt \qquad \int_0^2 \sqrt{\left(-t \cdot \sin(t)\right)^2 + \left(t \cdot \cos(t)\right)^2}\, dt$$

$$\int_0^2 \sqrt{\left(\cos(t) - t \cdot \sin(t)\right)^2 + \left(\sin(t) + t \cdot \cos(t)\right)^2}\, dt \qquad \int_0^2 \sqrt{1 + 2\sin(t)\cos(t) + t^2}\, dt$$

$$\int_0^2 \sqrt{1 + t^2 \cdot \left(\sin^2(t) - \cos^2(t)\right)}\, dt \qquad \int_0^2 \sqrt{1 + \left(\cos(t) - \sin(t)\right)^2}\, dt$$

Continued on the next page.

(c) Calculate the speed of the ladybug when $t = 2\pi$. Give at least three decimal places in your answer. Remember to include appropriate units with your answer.

(d) Find the equation of the tangent line to the ladybug's path at $t = \frac{7\pi}{2}$.

4. Determine the convergence or divergence of each of the following series. In each case, CIRCLE either CONVERGES or DIVERGES.

In each case, *demonstrate that your answer is correct* step-by-step using *an appropriate convergence test*. Be sure to explicitly state which convergence test you have used. Be careful to show how the convergence test justifies your answer. If you do not justify your answer, you will get zero credit, even if you circle the correct response.

(a) $$\sum_{n=1}^{\infty} \frac{5+(-1)^n}{n \cdot \sqrt{n}}$$ CONVERGES DIVERGES

JUSTIFICATION:

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \cdot (\ln(n))^4}$ CONVERGES DIVERGES

JUSTIFICATION:

(c) $$\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}{n!}$$ CONVERGES DIVERGES

JUSTIFICATION:

5. Consider the alternating series: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n^2}{10^n}$.

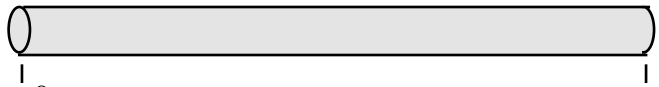
(a) Does this series converge absolutely, converge conditionally or diverge?

Continued on the next page.

(b) What is the smallest value of N needed to ensure that $\sum_{n=1}^{N} \frac{(-1)^{n-1} \cdot n^2}{10^n}$ approximates the value of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n^2}{10^n}$ with an error of less than 0.0001?

(c) Approximate the value of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n^2}{10^n}$ with an error of less than 0.0001.

6. A carbon rod of length 10 cm is placed on the x-axis with its left endpoint at the origin.



$x = 0$ $x = 10$

The density of the carbon rod is given by the function: $\delta(x) = a + bx$ in units of grams per centimeter (g/cm). In this formula, a and b are both positive constants.

(a) Calculate the mass of the carbon rod. Your answer may include the constants a and b. Include appropriate units with your answer.

(b) Calculate the x-coordinate of the *center of mass* of the carbon rod. Your answer may include the constants a and b.

7. A cylindrical urn, standing upright, contains hot chocolate. The top of the urn has a *diameter* of 0.8 m. The urn has a *height* of 1.1 m and is filled to a *depth* of 0.9 m. The density of the hot chocolate at a depth of h meters below the surface is given by the function:

$$\delta(h) = 1 + A \cdot h \qquad \text{kg/m}^3,$$

where A is a positive constant. Find the total work required to pump the hot chocolate to the top rim of the urn. Remember to include appropriate units in your answer. Note that in SI units, the constant for gravity is $g = 9.8$ m/s^2.

8. A new toy is being developed called the “Infini-bunny.” The Infini-bunny jumps up and down on the same spot. To activate the Infini-bunny, a person places the toy on the ground and presses the start button. The Infini-bunny then starts jumping up and down. The first jump is to a height of 10 feet. The second jump is to a height of $10\left(\frac{5}{6}\right)$. The third jump is to a height of $10\left(\frac{5}{6}\right)^2$, and so on. Each jump is $\frac{5}{6}$ of the height of the previous jump. The Infini-bunny will keep jumping up and down until the person catches it and turns it off.

(a) Write down an expression (or formula) for the height (in feet) of the Infini-bunny's n^{th} jump.

(b) Write down an expression (or formula) for the total vertical distance the Infini-bunny has traveled when it lands at the end of its n^{th} jump. Express your answer in closed form.

NOTE: The old saying, “What goes up must come down” may be useful here and in Part (c).

Continued on the next page.

(c) Find the total vertical distance (in feet) the Infini-bunny will travel if a person activates the Infini-bunny and just leaves it jumping (i.e. they never catch the Infini-bunny and turn it off).

(d) To complete a jump of height h feet takes the Infini-bunny $\sqrt{\frac{h}{8}}$ seconds. If a person activates the Infini-bunny and then leaves it alone, will the bunny keep jumping forever or does it eventually stop jumping? Briefly justify your answer. Ignore practical considerations like friction, air-resistance, battery life, etc.

9. An object is moving along a curve in the x-y plane. The position of the object at time t is given by the parametric equations $x(t)$ and $y(t)$. All that you can assume about $x(t)$ and $y(t)$ is that their derivatives are given by:

$$\frac{dx}{dt} = t \cdot \cos\left(t^2 + 1\right) \qquad \text{and} \qquad \frac{dy}{dt} = -t \cdot \sin\left(t^2 + 1\right)$$

for $0 \leq t \leq 3$, and that at time $t = 2$ their values are:

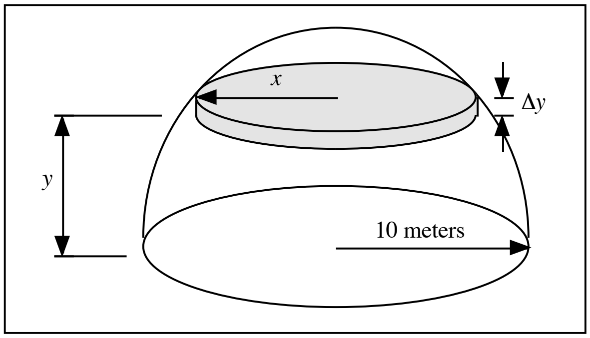
- $x(2) = 10$, and,
- $y(2) = -3$.

(a) Write down an equation for the tangent line to the curve at the point $(10, -3)$.

(b) Find the speed of the object at time $t = 2$.

(c) Find the **exact** distance that the object travels in the time interval $0 \leq t \leq 1$.

10.



Consider the shape shown in the diagram. This shape is the top half of a sphere with a radius of 10 meters. If you look at the shape "side on" then it looks like the top half of the circle described by the equation:

$$x^2 + y^2 = 100.$$

The shape does not have the same density throughout. The density is high near the base of the shape and lower near the top of the shape. In fact, the mass density at a height of y meters is given by the function:

$$p(y) = 2200 - 20y^2 \text{ kilograms per cubic meter.}$$

(a) If you were asked to work out the *mass* of the shape, briefly explain (in a sentence or two why it would be best to slice up the shape into horizontal slices.

(b) Set up an integral that will give the *mass* of the shape.

(c) Calculate the mass of the object.

11. Find solutions to the differential equations below, subject to the given initial conditions. In each case, provide step-by-step work to show how your answer was obtained. If you do not provide any work to justify your answer, you will get zero credit.

(a) $\dfrac{dP}{dt} = 3P - 6$ $\qquad$ $P(0) = 20$.

(b) $\dfrac{dy}{dx} = \dfrac{5y}{x}$ $\qquad$ $y(2) = 64$.

12. A perfectly spherical pearl has a radius of b mm, where b is a positive number. To prepare it for sale, a circular hole is bored through the center of the pearl. The radius of the hole is a mm, where a is a positive number and $0 < a < b$. The circular hole goes completely through the pearl, from one side of the pearl to the opposite side of the pearl.

Before the circular hole was bored, the volume of the pearl was $\frac{4}{3}\pi b^3$. Find the volume of the pearl that remains after the hole has been bored. Remember to include appropriate units with your answer.

13. Suppose that $g(0) = 2$, $g(3) = 5$ and $\int_0^3 g(x)dx = 7$. Calculate the exact numerical value of each of the following definite integrals.

(i) $\int_0^3 g(3-x)dx$

(ii) $\int_0^9 g\left(\frac{x}{3}\right)dx$

(iii) $\int_0^3 x \cdot g'(x)dx$

14. Suppose that *f* is a twice differentiable function with $f(0)=6$, $f(1)=5$ and $f'(1)=2$. Calculate the exact value of $\int_0^1 x \cdot f''(x)dx$. Show your work.

15. Use integrals to calculate the volume described in each part of this problem.

(a) The region R is bounded by $y = x^2$, $y = 1$ and the y-axis. Find the volume of the solid obtained by rotating the region R around the y-axis.

(b) A region R of the xy-plane is enclosed by the curves: $y = 0, x = 0,\ y = \sqrt{x}$ and $x = 4$. Find the volume generated when the region R is revolved around the y-axis.

Continued on the next page

(c) Calculate the volume of the solid whose base is the region bounded by:

- $y = e^x$,
- the x-axis, and,
- the lines $x = 0$ and $x = 1$

and whose cross-sections are equilateral triangles that are perpendicular to the x-axis.

(d) The region R is bounded by:

- The line $y = 0$,
- The line $x = 1$,
- The line $x = 4$, and by
- The curve $y = e^{-x}$.

Consider the solid whose base is the region R, and whose cross-sections perpendicular to the x-axis are squares. Calculate the **exact** volume of this solid.

16. **(a)** Using the techniques of integration that you have learned (and not calculator integration), find the *exact* value of the definite integral:

$$\int_{-b}^{b}\left(\frac{1}{1+x^2}+\frac{x}{1+x^2}\right)dx.$$

Your final answer may contain the symbol b but should contain no other unspecified constants.

Note that: $$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C.$$

Continued on the next page.

(b) Use your answer to Part (a) to calculate the limit: $\displaystyle \lim_{b \to \infty} \int_{-b}^{b} \left(\frac{1}{1+x^2} + \frac{x}{1+x^2} \right) dx$.

Note that $\lim_{x \to \infty} \arctan(x) = \frac{\pi}{2}$.

(c) Does the improper integral $\displaystyle \int_{-\infty}^{\infty} \frac{1+x}{1+x^2}\, dx$ converge or diverge? Briefly explain how you know.

17. On afternoon in January, my furnace broke at 1:00pm. At the time this happened, the temperature in the house was 68ºF. When I got home at 7:00pm, the temperature in the house was 48ºF while the outside temperature was 10ºF. You can assume that this was an unusual day during which the outside temperature stayed at exactly 10ºF all day.

(a) Assuming that the temperature T in my house obeys Newton's Law of Cooling, write down the differential equation for T. (This will involve one unknown constant that you will have to find later.)

(b) Solve the differential equation from Part (a).

(c) Assuming that the furnace is not repaired, at what time will the temperature in the house reach the freezing point of 32ºF?

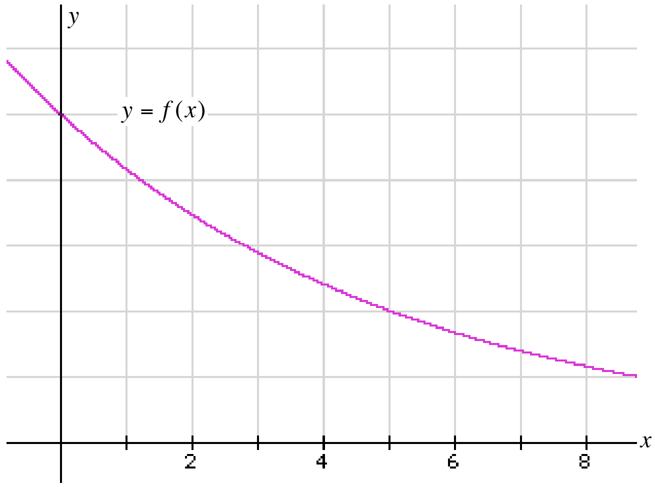
18. On Staten Island, garbage is dumped into a pyramid-shaped hole with a square base. (The base is at ground level; the hole gets narrower as you go down into the earth.) The length of each side of the square base is 100 yards. For each one yard you go down vertically in the hole, the length of the sides decreases by a yard. For example, if you go down one yard vertically, then the length of each side of the pyramid-shaped hole is 99 yards. Initially the hole is 20 feet deep.

(a) If 65 cubic yards of garbage arrive at the dump each day, how long will it be (in days) before the dump is full?

Continued on the next page.

(b) Suppose garbage weighs 800 pounds per cubic yard. A few years after the hole is completely filled with garbage, environmentalists force the city to dig up all the garbage and remove it from the hole. How much work must be done to remove the garbage from the hole?

19. The graph of a function $f(x)$ is shown below.



(a) Use the graph given above to sketch the rectangles that you would use if you were asked to estimate the value of the definite integral $\int_1^7 f(x)dx$ using the *midpoint rule* and a total of three (3) rectangles.

(b) The table (below) gives selected values of $f(x)$. Use the *trapezoid rule*, the values given in the table and a total of three (3) rectangles to approximate the value of the definite integral $\int_1^7 f(x)dx$.

x	1	2	3	4	5	6	7
$f(x)$	4.2	3.5	2.9	2.4	2.0	1.7	1.4

(c) Is the value that you calculated in Part (b) an over- or an under-estimate of the definite integral? Briefly explain how you can tell.

(d) Give an example of a function for which the trapezoid and midpoint rules will give exactly the same value.

20. **(a)** Find the approximate value of $\int_1^4 \sin\left(\sqrt{x}\right)dx$ obtained when the integral is approximated using the Midpoint Rule and 25 rectangles.

(b) Find the approximate value of $\int_1^4 \sin\left(\sqrt{x}\right)dx$ obtained when the integral is approximated using the Trapezoid Rule and 25 trapezoids.

(c) Find the approximate value of $\int_1^4 \sin\left(\sqrt{x}\right)dx$ obtained when the integral is approximated using Simpson's Rule and 50 rectangles.

(d) How many rectangles should you use if you wanted to approximate the value of $\int_1^4 \sin\left(\sqrt{x}\right)dx$ using the Midpoint Rule and with an error of less than 0.005? Show your work.

21. Use the technique of Trigonometric substitution to evaluate the definite integrals shown below.

(a) $$\int_1^2 \frac{1}{y^2 \cdot \sqrt{5 - y^2}} \cdot dy$$

Continued on the next page.

(b) $$\int_0^{\pi/2} \frac{\cos(z)}{\sqrt{1+\sin^2(z)}}\,dz$$

HINT: **(a)** Try a u-substitution to begin with; $u = 1+\sin^2(z)$ is probably not going to be very helpful.

(b) You may use the integration formula: $\int \sec(x)dx = \ln(|\sec(x)+\tan(x)|)+C$.

22. In this problem, all that you may assume that $f(x)$ is an **odd** function, that $g(x)$ is an **even** function, that the domains of both functions include $-4 \le x \le 4$ and that they have the values shown below.

$$f(-2) = 5 \qquad f(0) = 0 \qquad g(2) = -7 \qquad g'(2) = -1.$$

(a) Find the exact value of the definite integral: $\int_{-3}^{3} f(x)^3 \cdot [1 + g(x)] \cdot dx$.

(b) Find the exact value of the definite integral: $\int_{-2}^{2} x \cdot f'(x) dx$.

(c) Find the exact value of the definite integral: $\int_{0}^{2} f'(x) \cdot \sqrt{10 + f(x)} \cdot dx$.

(d) Find the exact value of the definite integral: $\int_{0}^{\sqrt{2}} x \cdot f'(x^2) dx$.

23. **(a)** Find a formula for the most general antiderivative of: $f(x) = \dfrac{e^x}{(e^x - M)(e^x + 3N)}$ where M and N are constants and $3N + M \neq 0$.

Continued on the next page.

(b) Find a formula for the most general antiderivative of: $g(x) = \dfrac{1}{x^2 - 6Bx + 10B^2}$ where B is a positive constant.

Continued on the next page.

(c) Find a formula for the most general antiderivative of: $j(x) = \dfrac{x^3 + Rx^2 - R^2x - R^3}{x - R}$ where R is a positive constant.

24. Evaluate each of the indefinite integrals to find the most general antiderivative.

(a) $\int \cos^4(x) \cdot \tan^3(x) \cdot dx$

(b) $\int \sec^6(\theta) \cdot d\theta$

25. Find the solution to the following initial value problem. Note that your final answer should not contain any unspecified constants. Clearly indicate your final answer.

$$y'' + 16 \cdot y = \sin(x) \qquad y(0) = 1 \qquad y'(0) = 0$$

26. The function $y = f(x)$ is defined by the following differential equation and initial condition:

Differential equation: $f'(x) = x^2 + f(x)$ or $y' = x^2 + y$

Initial value: $f(0) = 1$ or $y(0) = 1$.

(a) Use Euler's Method and $\Delta x = 0.25$ to estimate $f(2)$. Clearly indicate your final answer.

(b) Is the estimate of $f(2)$ that you calculated in Part (a) an over-estimate or an under-estimate of the actual value of $f(2)$? Be careful to show your work and explain how you know.

27. A tank initially contains 120 liters of pure water. A saline solution (with a concentration of γ grams per liter of salt) is pumped into the tank at a rate of 2 liters per minute. The well-stirred mixture leaves the tank at the same rate. In this problem, t is the time (in minutes) after the saline solution started pumping.

(a) Write down an initial value problem (i.e. a differential equation and a function value) for $u(t)$, the **mass** of salt in the tank at time t.

(b) Find an explicit formula for $u(t)$.

(c) Find an expression for the limit of $u(t)$ as $t \to \infty$.

28. In this problem you will be concerned with the differential equation: $y'' - 3y' - 4y = e^t$.

(a) Find the general solution of the homogeneous equation:

$$y'' - 3y' - 4y = 0.$$

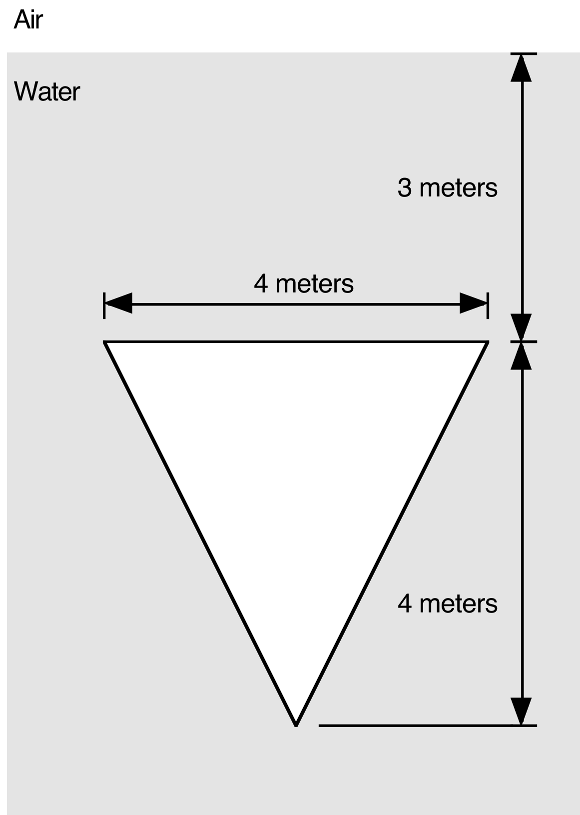
(b) Find a particular solution of the non-homogeneous equation:

$$y'' - 3y' - 4y = e^t.$$

(c) Find a solution of the initial value problem:

$$y'' - 3y' - 4y = e^t \qquad y(0) = 1 \qquad y'(0) = 0.$$

29. A triangular plate is submerged in a tank of fresh water (density = 1000 kg/m^3) as shown in the diagram below. The top of the triangular plate is three (3) meters below the surface of the water and the triangular plate is vertical in the water. Calculate the hydrostatic force exerted by the water on one side of the triangular plate.



30. Find the solution of the following initial value problem:

$$y'' + 8y' - 9y = 0 \qquad y(1) = 1 \qquad y'(1) = 0.$$