MATH 122 – SECOND UNIT TEST

Friday, October 24, 2008.

NAME: SOLUTIONS.

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A	B	D	F	H	J
	C	E	G	I	K

Instructions:

1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. If you evaluate an improper integral, be sure to use appropriate algebraic and limit notation.
7. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	20	
2	20	
3	20	
4	15	
5	15	
6	10	
Total	100	

SOLUTIONS.

1. 20 Points. SHOW ALL WORK. NO WORK = NO CREDIT.

Solve each of the following initial value problems. Your final answers should not include any unspecified constants. Clearly indicate your final answer, for example by circling it.

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

(a) **(10 points)** $\frac{dP}{dt} = 10 - 0.5 \cdot P$ $\qquad P(0) = 15$

We will use the technique of Separation of Variables:

$$\frac{dP}{dt} = -0.5(P - 20)$$

$$\int \frac{1}{P-20}\, dP = \int -0.5\, dt$$

$$\ln(|P-20|) = -0.5t + C$$

$$P - 20 = Ae^{-0.5t} \qquad A = \pm e^{C}$$

$$P = 20 + Ae^{-0.5t}$$

To find the value of A use $P(0) = 15$.

$$15 = 20 + Ae^{0}$$

$$-5 = A.$$

Putting all of this together gives the final answer:

$$P = 20 - 5e^{-0.5t}.$$

Continued on the next page.

Solve each of the following initial value problems. Your final answers should not include any unspecified constants. Clearly indicate your final answer, for example by circling it.

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(b) **(10 points)** $\frac{dz}{dt} + \cos(t) \cdot z = \cos(t)$ $\quad z(0) = 3$

We will use the technique of Integrating Factors.

$$p(t) = \cos(t)$$

$$\int p(t)\,dt = \sin(t)$$

Integrating factor $= I = e^{\int p(t)dt} = e^{\sin(t)}$.

Multiply each term in the differential equation by the integrating factor.

$$e^{\sin(t)} \cdot \frac{dz}{dt} + e^{\sin(t)} \cdot \cos(t) \cdot z = \cos(t) e^{\sin(t)}$$

so:

$$\frac{d}{dt}\left(e^{\sin(t)} \cdot z\right) = \cos(t) \cdot e^{\sin(t)}$$

and:

$$\int \frac{d}{dt}\left(e^{\sin(t)} \cdot z\right) dt = \int \cos(t) \cdot e^{\sin(t)}\,dt$$

$$e^{\sin(t)} \cdot z = e^{\sin(t)} + C$$

$$z = \frac{e^{\sin(t)} + C}{e^{\sin(t)}}$$

To find C, use $z(0) = 3$. This gives $C = 2$ and the final answer:

$$z = \frac{e^{\sin(t)} + 2}{e^{\sin(t)}}.$$

SOLUTIONS.

2. **15 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.**

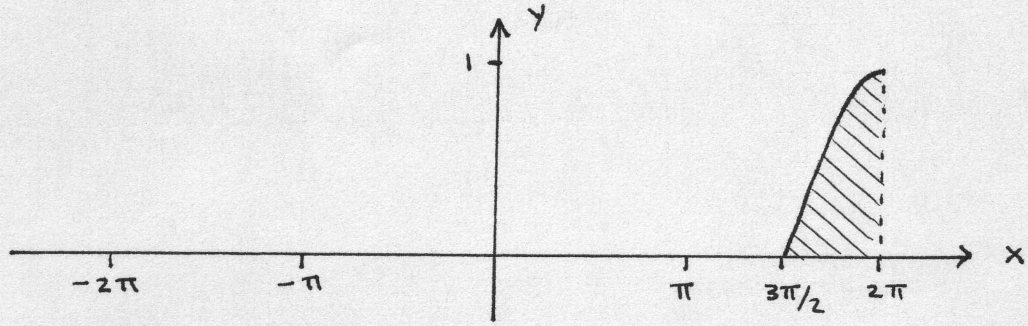
You should not use your calculator on this problem for anything besides arithmetic. In particular, finding antiderivatives or evaluating improper integrals on your calculator is not acceptable.

Calculate the **exact** volume of the solid created by revolving the region of the *xy*-plane bounded by the curves:

$x = \frac{3\pi}{2}$ $\quad x = 2\pi$ $\quad y = 0$ $\quad y = \cos(x)$

around the *y*-axis. Clearly indicate your final answer.

The region that will be revolved around the y-axis is shown in the diagram below.



We will use the Shell method to set up the integral for this volume.

Integration by parts with

$u = x \qquad v' = \cos(x)$

$u' = 1 \qquad v = \sin(x)$

$$\text{Volume} = \int_{3\pi/2}^{2\pi} 2\pi \cdot x \cdot \cos(x) \cdot dx$$

$$= \Big[2\pi x \cdot \sin(x) \Big]_{3\pi/2}^{2\pi} - \int_{3\pi/2}^{2\pi} 2\pi \cdot \sin(x)\, dx$$

$$= \Big[2\pi x \cdot \sin(x) + 2\pi \cdot \cos(x) \Big]_{3\pi/2}^{2\pi}$$

$$= 3\pi^2 + 2\pi$$

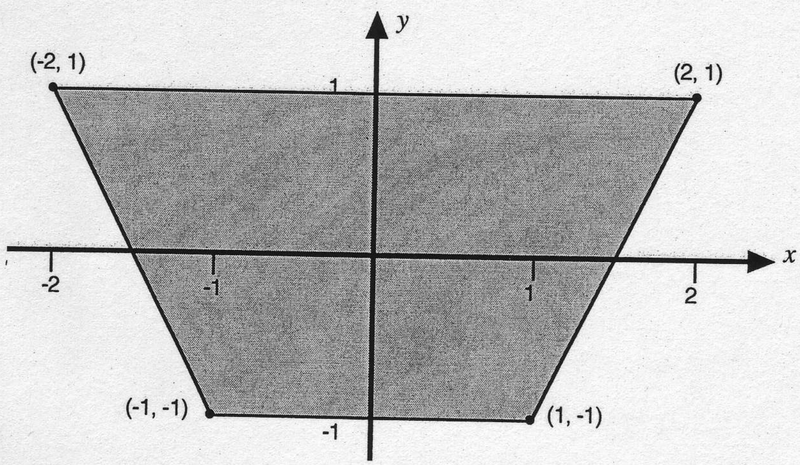
$$\approx 35.89199851$$

SOLUTIONS.

3. **20 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.**

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

The object that you will be studying in this problem is a trapezoidal plate with the dimensions shown in the diagram given below. All dimensions and measurements are in units of meters.



(a) **(10 points)** The density of the trapezoidal plate varies from place to place. The density of the plate at any point is given by the function:

$$\delta(y) = 2 + y \ \text{kg/m}^2.$$

Find the **exact** total mass of the trapezoidal plate. Clearly indicate your final answer.

Let w = width of trapezoid as a function of y.

y	-1	1
w	2	4

so $w = y + 3$.

The mass of the trapezoidal plate will be:

$$\text{Mass} = \int_{-1}^{1} (2+y)(y+3)\,dy$$

$$= \int_{-1}^{1} (y^2 + 5y + 6)\,dy$$

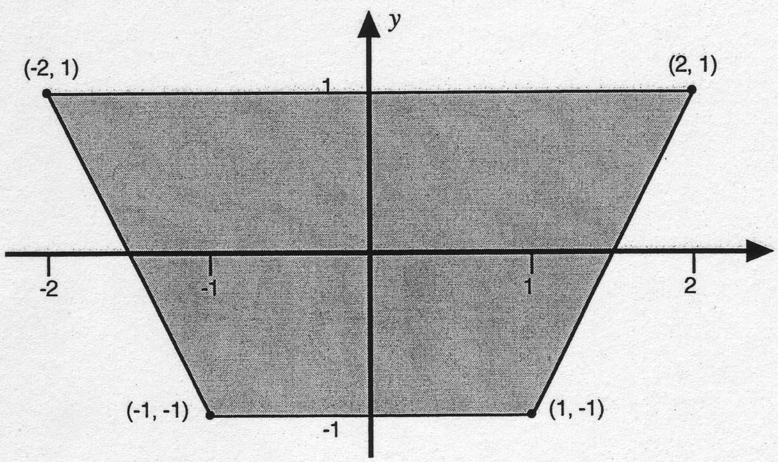
$$= \left[\frac{1}{3}y^3 + \frac{5}{2}y^2 + 6y\right]_{-1}^{1}$$

$$= 12\,{}^2/_3.$$

Continued on the next page.

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

The object that you will be studying in this problem is a trapezoidal plate with the dimensions shown in the diagram given below. All dimensions and measurements are in units of meters.



(b) **(10 points)** The x-coordinate of the trapezoidal plate's center of mass is $x = 0$. (You do not have to show this.) Find the **exact** y-coordinate of the trapezoidal plate's center of mass. Clearly indicate your final answer.

The y-coordinate of the center of mass is given by:

$$\bar{y} = \frac{\int_{-1}^{1} y \cdot (y+3)(2+y)\,dy}{\int_{-1}^{1} (y+3)(2+y)\,dy.}$$

The numerator is equal to:

$$\int_{-1}^{1} y(y+3)(2+y)\,dy = \int_{-1}^{1} y^3 + 5y^2 + 6y\,dy$$

$$= \left[\tfrac{1}{4}y^4 + \tfrac{5}{3}y^3 + \tfrac{6}{2}y^2\right]_{-1}^{1}$$

$$= 3\tfrac{1}{3}$$

So:

$$\bar{y} = \frac{\frac{10}{3}}{\frac{38}{3}} = \frac{5}{19}\ \text{m}.$$

4. **15 Points. SHOW YOUR WORK. EXPLAIN IN PART (b).**

The function $y = f(x)$ is defined by the following differential equation and initial condition:

Differential equation: $f'(x) = x + f(x)$ or $y' = x + y$

Initial value: $f(0) = 1$ or $y(0) = 1$.

(a) **(10 points)** Use Euler's Method and $\Delta x = 0.5$ to estimate $f(2.5)$. Clearly indicate your final answer.

Current x	Current y	Derivative y'	Rise $y' \cdot \Delta x$	New y
0	1	1	0.5	1.5
0.5	1.5	2	1.0	2.5
1.0	2.5	3.5	1.75	4.25
1.5	4.25	5.75	2.875	7.125
2.0	7.125	9.125	4.5625	11.6875

$f(2.5) \approx 11.6875$

(b) **(5 points)** Is the estimate of $f(2.5)$ that you calculated in Part (a) an over-estimate or an under-estimate of the actual value of $f(2.5)$? Be careful to show your work and explain how you know.

$f(2.5) \approx 11.6875$ is an underestimate.

The derivative $f'(x)$ is increasing (see the derivative column above) so the function is concave up.

When the function being estimated is concave up, Euler's method underestimates the actual function values.

SOLUTIONS.

5. 15 Points. SHOW ALL WORK. NO WORK = NO CREDIT.

Find the solution to the following initial value problem. Note that your final answer should not contain any unspecified constants. Clearly indicate your final answer.

$$y'' + 9 \cdot y = e^x \qquad y(0) = 1.1 \qquad y'(0) = 6.1$$

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

Step 1: Homogeneous Equation.

$$y'' + 9y = 0$$

characteristic equation: $r^2 + 9 = 0$

Roots : $r = \pm 3i$

Homogeneous solution: $y_h(x) = C_1 \cos(3x) + C_2 \sin(3x)$

Step 2: Particular Solution

$$\left.\begin{aligned} N(x) &= e^x \\ N'(x) &= e^x \end{aligned}\right\} \quad e^x \text{ is important.}$$

Particular Solution: $y_p(x) = F \cdot e^x$. To find F:

$$y_p''(x) + 9\, y_p(x) = e^x$$

$$10 F e^x = e^x$$

$$F = 1/10$$

Particular solution: $y_p(x) = \frac{1}{10} e^x$.

Step 3: Initial Values.

$y(0) = 1.1$: $C_1 + \frac{1}{10} = 1.1 \Rightarrow C_1 = 1$

$y'(0) = 6.1$: $3C_2 + \frac{1}{10} = 6.1 \Rightarrow C_2 = 2$

Additional space is provided on the following page if you need it.

FINAL ANSWER: $y(x) = \cos(3x) + 2\sin(3x) + \frac{1}{10} e^x$

SOLUTIONS.

6. 15 Points. CLEARLY INDICATE YOUR FINAL ANSWER.

A cylindrical urn, standing upright, contains hot chocolate. The top of the urn has a *diameter* of 0.8 m. The urn has a *height* of 1.1 m and is filled to a *depth* of 0.9 m. The density of the hot chocolate at a depth of h meters below the surface is given by the function:

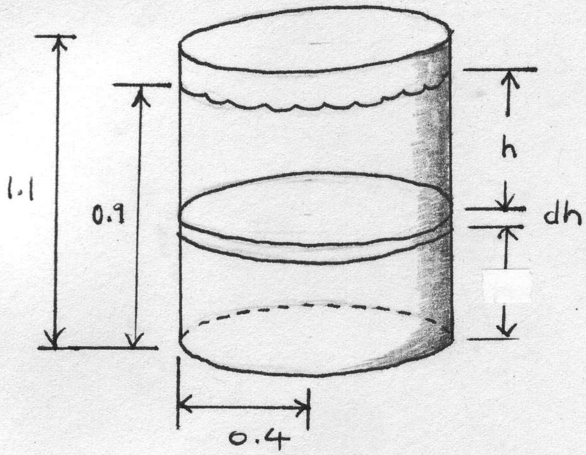
$$\delta(h) = 1 + A \cdot h \quad \text{kg/m}^3,$$

where A is a positive constant.

Set up an integral that will give the total work (in joules) that must be done in order to pump the hot chocolate to the top rim of the urn. Remember to include appropriate units in your answer.

NOTE:
(1) Your final answer may include the constant A.
(2) In metric (or SI) units, the constant for gravity is $g = 9.8$ m/s^2.
(3) You do not have to evaluate the integral.

The situation is illustrated in the diagram shown below:



The distance that the horizontal slice has to move is:

$$\text{Distance} = h + 0.2$$

The force of gravity on the slice is:

$$\text{Force} = (9.8)(1 + A \cdot h)\pi(0.4)^2 \cdot dh$$

So the work to lift this slice will be:

$$\text{Work} = (h + 0.2)(9.8)(1 + A \cdot h)\pi(0.4)^2 \cdot dh$$

The total work to empty the urn is:

$$\text{Work} = \int_0^{0.9} (h + 0.2)(9.8)(1 + Ah)\pi(0.4)^2\, dh.$$