MATH 122 – FINAL EXAM

Friday, December 12, 2008.

NAME: SOLUTIONS

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A	B	D	F	H	J
	C	E	G	I	K

Instructions:

1. Do not separate the pages of the exam.
2. Please read the instructions for each individual question carefully.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. If you evaluate an improper integral, be sure to use appropriate algebraic and limit notation.
7. **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	7	
2	5	
3	10	
4	11	
5	12	
6	10	
7	11	
8	6	
9	9	
10	10	
11	9	
Total	**100**	

SOLUTIONS

1. 7 Points. CIRCLE TRUE OR FALSE.

A power series $\sum_{n=0}^{\infty} C_n \cdot (x+7)^n$ converges at $x = 0$ and diverges at $x = -17$. This is all that you may assume about the power series. For each of the following statements, determine whether they are true or false. Indicate your answer by circling **TRUE** or **FALSE**.

(a) The radius of convergence must be at least 7. **TRUE** **FALSE**

(b) The interval of convergence must be $(-17, 0)$. **TRUE** **FALSE**

(c) The interval $(-14, 0)$ must be part of the interval of convergence. **TRUE** **FALSE**

(d) The radius of convergence could be greater than or equal to 17. **TRUE** **FALSE**

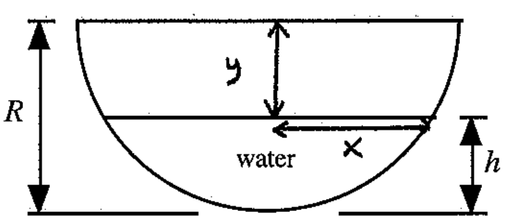
(e) The interval of convergence includes all real numbers except $x = 17$. **TRUE** **FALSE**

(f) The radius of convergence could be equal to 24. **TRUE** **FALSE**

(g) The only point where the power series is guaranteed to converge is $x = 0$. **TRUE** **FALSE**

2. 5 Points. CLEARLY INDICATE YOUR FINAL ANSWER.

A **hemispherical** pond has a radius of R and is filled with water to a level $h < R$. Write down an integral that gives the total amount of water in the pond. You do not have to evaluate the integral.



$$x^2 + y^2 = R^2$$

Volume of slice

$$= \pi x^2 \cdot dy$$

$$= \pi (R^2 - y^2)\, dy$$

Volume of water $= \int_{R-h}^{R} \pi (R^2 - y^2)\, dy.$

SOLUTIONS

3. **10 Points. MULTIPLE CHOICE. CIRCLE ONE ANSWER IN EACH PART.**

(a) **(2 points)** The region R is bounded by $y = \sqrt{x}$, the x-axis and the lines $x = 0$ and $x = 4$. The volume generated by revolving the region R around the horizontal line $y = 3$ is:

(i) $\pi\int_0^4 x\,dx$

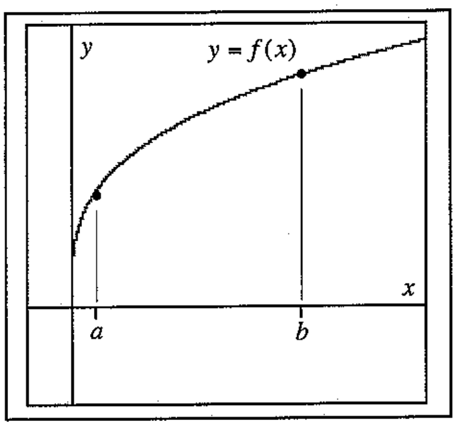
(ii) $\pi\int_0^4 \left(3-\sqrt{x}\right)^2 dx$

(iii) $\pi\int_0^4 \left(9-\left(3-\sqrt{x}\right)^2\right)dx$

(iv) $\pi\int_0^2 \left(3-y^2\right)dy$

(v) $\pi\int_0^2 \left(\sqrt{3}-y^2\right)^2 dy$

(b) **(2 points)** The following graph shows the function $f(x)$ and the points $x = a$ and $x = b$ ($a < b$). Three quantities are defined using this graph as follows:



Quantity I: $\mathrm{QI} = b - a$

Quantity II: $\mathrm{QII} = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$

Quantity III: $\mathrm{QIII} = \sqrt{(b-a)^2 + \left(f(b)-f(a)\right)^2}$

Which ONE of the following MUST be true?

(i) QI < QII < QIII

(ii) QIII < QI < QII

(iii) QII < QI < QIII

(iv) QI < QIII < QII

(v) QII < QIII < QI

Continued on the next page

(c) **(2 points)** Let $g(x)$ be defined by: $g(x) = \sum_{n=0}^{\infty} x^{2n}$. Which ONE of the following MUST be true?

(i) $g(x)$ is increasing on the interval $(-1, 0)$.

(ii) $g(x)$ is increasing on the interval $(0, 1)$.

(iii) $g(x) < 0$ on the interval $(-1, 0)$.

(iv) $g(x)$ is increasing for every real number $x > 0$.

(v) $g(x)$ has a local maximum at $x = 0.5$.

(d) **(2 points)** If $\int x^2 \cos(x)dx = f(x) - \int 2x\sin(x)dx$, then $f(x)$ is equal to:

(i) $2\sin(x) + 2x\cos(x)$

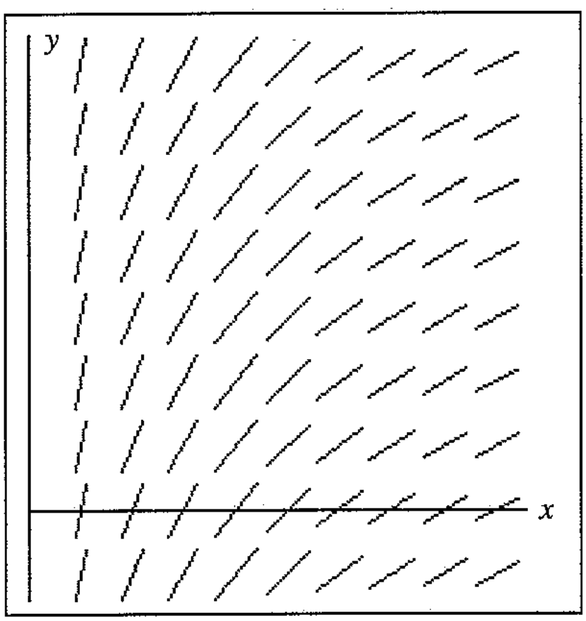
(ii) $x^2 \sin(x)$

(iii) $2x\cos(x) - x^2\sin(x)$

(iv) $4\cos(x) - 2x\sin(x)$

(v) $(2 - x^2)\cos(x) - 4\sin(x)$

(e) **(2 points)** The slope field shown below corresponds to a certain differential equation. Which of the following functions could be a solution of that differential equation?



(i) $y = x^2$

(ii) $y = e^x$

(iii) $y = e^{-x}$

(iv) $y = \cos(x)$

(v) $y = \ln(x)$

4. 11 Points. SHOW YOUR WORK. CLEARLY INDICATE YOUR FINAL ANSWERS.

(a) **(2 points)** The function $f(x)=\dfrac{x-5}{x^2-1}$ can be written in the form $\dfrac{A}{x-1}+\dfrac{B}{x+1}$ where A and B are numbers. Calculate the values of A and B. Circle your final answers.

$$\frac{x-5}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$A(x+1) + B(x-1) = x-5$$

$$A + B = 1$$

$$A - B = -5$$

So: $2A = -4$

$2B = 6$

$$\boxed{A = -2 \qquad B = 3}$$

(b) **(6 points)** Suppose that $g(0) = 2$, $g(3) = 5$ and $\int_0^3 g(x)dx = 7$. Calculate the exact numerical value of each of the following definite integrals. Circle your final answer in each case.

(i) $$\int_0^3 g(3-x)dx = -\int_3^0 g(u)\,du = \int_0^3 g(u)\,du = 7.$$

$u = 3-x$

$-du = dx$

(ii) $$\int_0^9 g(\tfrac{x}{3})dx = 3\int_0^3 g(u)\,du = (3)(7) = 21.$$

$u = x/3$

$3\,du = dx$

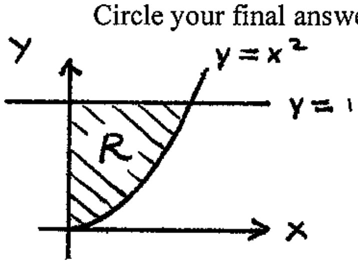
(iii) $$\int_0^3 x\cdot g'(x)dx = \left[x\cdot g(x)\right]_0^3 - \int_0^3 g(x)\,dx = (3)(5) - (0)(2) - 7$$

$$= 8.$$

$u = x \qquad u' = 1$

$v' = g' \qquad v = g$

(c) **(3 points)** The region R is bounded by $y = x^2$, $y = 1$ and the y-axis. Find the volume of the solid obtained by rotating the region R around the y-axis. Do not use your calculator, except for arithmetic. Circle your final answer.



$$\text{Volume} = \int_0^1 \pi\cdot x^2\cdot dy$$

$$= \int_0^1 \pi y\,dy$$

$$= \left[\tfrac{1}{2}\pi y^2\right]_0^1$$

$$= \pi/2.$$

5. 12 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

Determine the convergence or divergence of each of the following series. If you do not justify your answer, you will get zero credit, even if you circle the correct final answer.

In each case, demonstrate that your answer is correct in a step-by-step fashion using *an appropriate convergence test*. Be sure to explicitly state which convergence test you have used and show that it can be used with the series you are working on. Be careful to show all of your work. As the final part of your answer in each part, CIRCLE either CONVERGES or DIVERGES.

(a) **(6 points)** $$\sum_{n=1}^{\infty} \frac{3^n \cdot (n!)^2}{(2n)!}$$

JUSTIFICATION: We will use the Ratio Test.

$$a_n = \frac{3^n \cdot (n!)^2}{(2n)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1} \cdot ((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{3^n \cdot (n!)^2}$$

$$= \frac{3 \cdot (n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \frac{3(n+1)}{2(2n+1)}$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{3(n+1)}{2(2n+1)} \right| = \frac{3}{4} < 1$$

By the Ratio Test, $\sum_{n=1}^{\infty} \frac{3^n \cdot (n!)^2}{(2n)!}$ converges.

FINAL ANSWER (CIRCLE ONE): CONVERGES DIVERGES

Demonstrate that your answer is correct in a step-by-step fashion using *an appropriate convergence test*. Be sure to explicitly state which convergence test you have used and show that it can be used with the series you are working on. Be careful to show all of your work. As the final part of your answer in each part, CIRCLE either CONVERGES or DIVERGES.

(b) **(6 points)** $\displaystyle\sum_{n=1}^{\infty}\frac{3+\sin(n)}{n^4}$

JUSTIFICATION: We will use the Comparison test.

Initial Guess: Note that $-1 \leq \sin(n) \leq 1$

So that $2 \leq 3+\sin(n) \leq 4$.

This means that $\dfrac{3+\sin(n)}{n^4} \leq \dfrac{4}{n^4}$. The series $\displaystyle\sum_{n=1}^{\infty}\frac{4}{n^4}$ is a p-series with $p=4>1$ so it converges. I guess that the given series does something similar.

Formal Comparison: $0 \leq \sin(n)+3 \leq 4$

$$0 \leq \frac{\sin(n)+3}{n^4} \leq \frac{4}{n^4}$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{4}{n^4}$ is a p-series with $p=4>1$ it converges. Hence by the Comparison test, $\displaystyle\sum_{n=1}^{\infty}\frac{3+\sin(n)}{n^4}$ converges.

FINAL ANSWER (CIRCLE ONE): CONVERGES DIVERGES

6. 10 Points. SHOW YOUR WORK.

Consider the alternating series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

(a) **(5 points)** Does the series converge absolutely, converge conditionally or diverge? Clearly state your answer and use a convergence test to demonstrate that your answer is correct.

Be sure to explicitly state which convergence test you have used and show that it can be used with the series you are working on. Be careful to show all of your work and clearly state your final conclusion.

The series converges conditionally but not absolutely.

Absolute Convergence:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$ is a p-series with $p = 1/2 < 1$

so it diverges.

Conditional Convergence:

We will use the Alternating Series Test.

$$a_n = \frac{1}{\sqrt{n}}$$

Condition I: $0 \le \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ so $0 \le a_{n+1} < a_n$

Condition II: $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$ so $\lim_{n \to \infty} a_n = 0$.

By the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

converges conditionally.

Continued on the next page.

The alternating series from the previous page is:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}.$$

(b) **(3 points)** Let S represent the sum of the series. Suppose that S, the sum of the series, is approximated by the N^{th} partial sum:

$$S_N = \sum_{n=1}^{N}\frac{(-1)^n}{\sqrt{n}}.$$

What is the smallest value of N that could be used to approximate S by S_N and an error of less than 0.1?

Let $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Then according to the Alternating Series Estimation Theorem:

$$|S - S_N| < a_{N+1} = \frac{1}{\sqrt{N+1}} < 0.1$$

Solving for N:

$$N+1 > \frac{1}{(0.1)^2}$$

$$N > \frac{1}{(0.1)^2} - 1$$

$$N > 99$$

Smallest value of N is $N = 100$.

(c) **(2 points)** Approximate the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ with an error of less than 0.1. Include at least **8 decimal places** in your answer.

y1 = (-1) ^ X / √(X)

sum(seq(y1(K), K, 1, 100)) = − 0.5550236395

7. **11 Points. SHOW YOUR WORK. IN (b) EXPLAIN YOUR REASONING.**

The function $y = f(x)$ is defined by the following differential equation and initial condition:

Differential equation: $f'(x) = x^2 + [f(x)]^2$ or $y' = x^2 + y^2$

Initial value: $f(0) = 1$ or $y(0) = 1$.

(a) **(4 points)** Use Euler's Method and $\Delta x = 0.25$ to estimate $f(1)$.

Current x	Current f(x)	f'(x)	Rise = f'(x) · Δx	New f(x)
0	1	1	0.25	1.25
0.25	1.25	1.625	0.40625	1.65625
0.50	1.65625	2.993164063	0.7482910156	2.404541016
0.75	2.404541016	6.344317496	1.586079374	3.99062039

$$f(1) \approx 3.99062039$$

(b) **(3 points)** Is the estimate of $f(1)$ that you calculated in Question (a) an over-estimate or an under-estimate of the actual value of $f(1)$? Be careful to show your work and explain how you know.

The estimate $f(1) \approx 3.99062039$ is an underestimate. The increasing values of $f'(x)$ from the table shown above show that $f(x)$ is concave up. When used to estimate the value of a concave up function, Euler's method gives underestimates.

(d) **(4 points)** How many rectangles should you use if you wanted to approximate the value of $\int_1^4 \sin(\sqrt{x})\,dx$ using the Midpoint Rule and with an error of less than 0.005? Show your work.

$f(x) = \sin(\sqrt{x})$

$f'(x) = \cos(\sqrt{x}) \cdot \frac{1}{2} x^{-1/2}$

$f''(x) = -\sin(\sqrt{x}) \cdot \frac{1}{4} \cdot x^{-1}$

$\qquad - \cos(\sqrt{x}) \cdot \frac{1}{4} \cdot x^{-3/2}$

From calculator,

$M = \max\limits_{1 \le x \le 4} |f''(x)| = 0.34544332$

$$\frac{M \cdot (4-1)^3}{24 \cdot N^2} < 0.005$$

Solve for N:

$$N > \sqrt{\frac{M(4-1)^3}{(24)(0.005)}}$$

$N > 8.816$

Use at least 9 rectangles.

8. 6 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

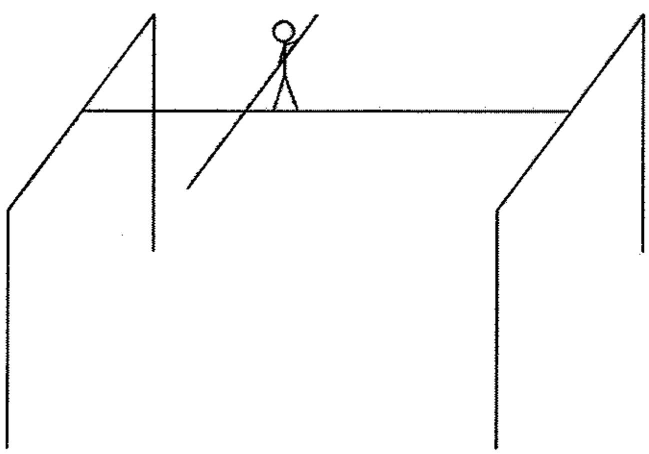
The Great Brunetti is a tightrope walker. The Great Brunetti captivates his audiences by using a specially constructed balancing pole that makes it appear to audience members that he is off balance and might fall. The Great Brunetti's balancing pole is 5 meters long. If x is the distance from the left end of the pole (in meters) then the density of the pole (in kilograms per meter) is given by the function:

$$\delta(x) = 3 \cdot e^{-x^2} \qquad 0 \le x \le 5.$$

The TOTAL MASS of the balancing pole is 2.66 kilograms. You can use this figure when solving this problem – there is no need for you to calculate the total mass of the balancing pole yourself.

Where (measured from the left end of the pole) should the Great Brunetti hold the pole so as to be perfectly balanced? Include units with your answer.

NOTE: You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.



We need to find the location of the center of mass.

$$\text{Center of mass} = \frac{\int_0^5 x \cdot \delta(x) \cdot dx}{2.66}$$

$$\int_0^5 x \cdot \delta(x) \cdot dx = \int_0^5 3x e^{-x^2} \cdot dx$$

$$= \left[-\frac{3}{2} e^{-x^2} \right]_0^5$$

$$\doteq 1.5$$

$$\text{Center of mass} = \frac{1.5}{2.66} = 0.5639 \text{ m.}$$

The Great Brunetti should hold the pole 0.5639 m from the left end of the pole.

9. 9 Points. SHOW ALL WORK. NO WORK = NO CREDIT.

Find the solution to the following initial value problem. Note that your final answer should not contain any unspecified constants. Clearly indicate your final answer.

$$y'' + 25 \cdot y = e^{2x} \qquad y(0) = \tfrac{59}{29} \qquad y'(0) = \tfrac{2}{29}$$

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

Homogeneous Equation.

$$y'' + 25y = 0$$

Char. Eqn. $r^2 + 25 = 0$

Roots : $r = \pm 5i$

Homogeneous solution:

$$y_h(x) = C_1 \cos(5x) + C_2 \sin(5x).$$

Particular Solution.

$y_p(x) = F \cdot e^{2x}$, F = constant.

$$y_p''(x) + 25 y_p(x) = e^{2x}$$

$$4Fe^{2x} + 25Fe^{2x} = e^{2x}$$

$$F = 1/29$$

Particular solution:

$$y_p(x) = \frac{1}{29} e^{2x}$$

Determine Constants.

$$y(x) = c_1 \cdot \cos(5x) + c_2 \cdot \sin(5x) + \frac{1}{29} e^{2x}$$

$y(0) = 59/29$: $\quad c_1 + \frac{1}{29} = \frac{59}{29}$ so that $c_1 = 2$

$y'(0) = 2/29$ $\quad 5c_2 + \frac{2}{29} = \frac{2}{29}$ so that $c_2 = 0$

FINAL ANSWER: $y(x) = 2\cos(5x) + \frac{1}{29} e^{2x}$

10. **10 Points. SHOW YOUR WORK. CLEARLY INDICATE YOUR FINAL ANSWERS.**

Solve each of the following initial value problems. Your final answers should not include any unspecified constants. Clearly indicate your final answer, for example by circling it.

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

(a) **(5 points)** $\frac{dy}{dx} - x \cdot y = e^{\frac{1}{2}x^2}$ $\quad\quad y(0) = -1$

We will use the technique of Integrating Factors:

$p(x) = -x$. $\quad \int p(x)\,dx = -\frac{1}{2}x^2$. $\quad I = e^{\int p(x)dx} = e^{-\frac{1}{2}x^2}$.

$$e^{-1/2x^2} \cdot \frac{dy}{dx} - x \cdot e^{-1/2x^2} \cdot y = e^{-1/2x^2} \cdot e^{1/2x^2}$$

$$\frac{d}{dx}\left(e^{-1/2x^2} \cdot y\right) = 1$$

$$y = \frac{x + C}{e^{-1/2x^2}}$$

To find C, use $y(0) = -1$ to get $C = -1$ and: $y = \frac{x-1}{e^{-1/2x^2}}$

(b) **(5 points)** $\frac{dM}{dx} = 6 - 2 \cdot M$ $\quad\quad M(0) = 7$

We will use the Technique of Separation of Variables.

$$\frac{dM}{dx} = -2(M-3)$$

$$\int \frac{1}{M-3}\,dM = \int -2\,dx$$

$$\ln(|M-3|) = -2x + C$$

$$M = 3 + Ae^{-2x}$$

To find A, use $M(0) = 7$ to get $A = 4$ and:

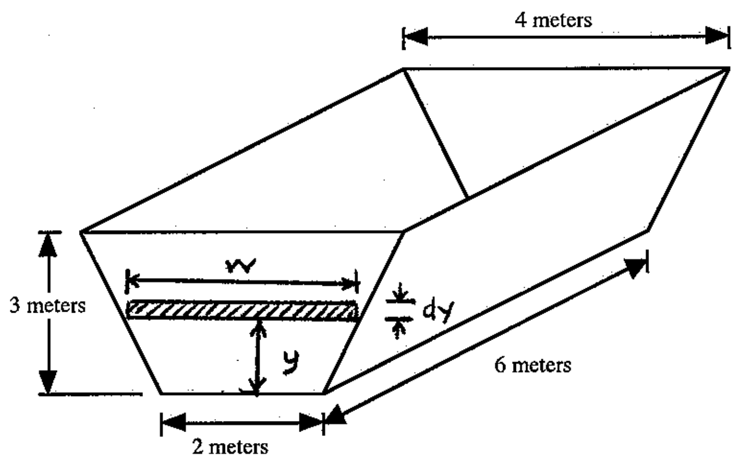
$$M(x) = 3 + 4e^{-2x}$$

SOLUTIONS

11. **9 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.**

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

The object that you will be studying in this problem is a water trough with a trapezoidal end plate as shown in the diagram given below. All dimensions and measurements are in units of meters.



(a) **(4 points)** Suppose that the trough is completely filled with pure water (density = 1000 kg/m^3). Calculate the exact total hydrostatic force exerted on one of the trapezoidal end plates of the trough. Include appropriate units with your answer.

First we will calculate the hydrostatic force exerted on the shaded slice of area shown above.

Let y = distance (in m) from the bottom of the trough.

Force on slice = (pressure)(area)

$$= (1000)(9.8)(3-y)\cdot w \cdot dy$$

The relationship between w and y is a linear function.

y	0	3
w	2	4

so: $w = \frac{2}{3}y + 2$

The total force on the trapezoidal end plate is:

$$\text{Total force} = \int_0^3 (1000)(9.8)(3-y)\left(\tfrac{2}{3}y+2\right)dy$$

$$= 9800\int_0^3 \left(-\tfrac{2}{3}y^2+6\right)dy$$

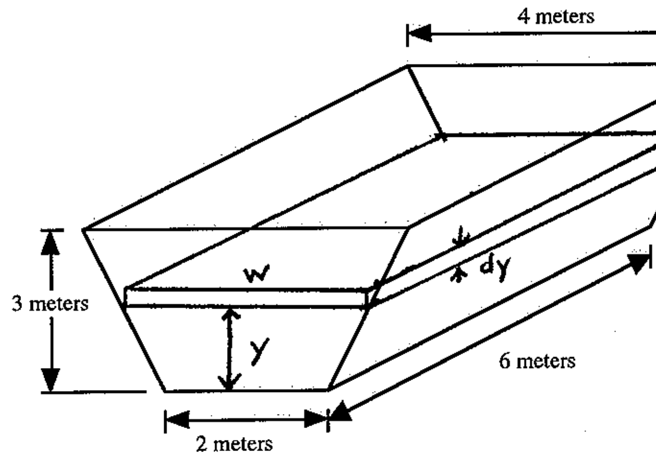
Continued on the next page.

$$= 9800\left[-\tfrac{2}{9}y^3+6y\right]_0^3$$

$$= 117600 \text{ N}$$

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

The object that you will be studying in this problem is a water trough with a trapezoidal end plate as shown in the diagram given below. All dimensions and measurements are in units of meters.



(b) **(5 points)** Suppose that the trough is completely filled with pure water (density = 1000 kg/m^3). Calculate the exact total amount of work that must be done to pump all of the water out of the trough. Include appropriate units with your answer.

First we will calculate the work needed to pump the given slice of water out of the top of the trough.

$$\text{Work} = (\text{Force})(\text{Distance})$$
$$= (9.8)(1000)(6)(w)(dy) \cdot (3-y)$$
$$= (9.8)(1000)(6)\left(\tfrac{2}{3}y+2\right)(3-y)\cdot dy$$

The total work to remove all water from the trough is given by:

$$\text{Total work} = \int_0^3 (9.8)(1000)(6)\left(\tfrac{2}{3}y+2\right)(3-y)\,dy$$
$$= 58\,800 \cdot \int_0^3 \left(-\tfrac{2}{3}y^2+6\right)dy$$
$$= 58\,800\left[-\tfrac{2}{9}y^3+6y\right]_0^3$$
$$= 705\,600 \quad \text{N}\cdot\text{m}$$