Recitation Handout 9: Optimizing the Design for Bottled Water

Today's recitation will focus on optimization problems. These are problems where you are try to find the maximum or minimum value of a quantity (usually subject to a constraint).

For each of the problems you will attempt on today's recitation, you will be trying to minimize the surface area of a water bottle with the constraint that the volume of the water bottle has to equal a certain fixed value.

The specific goals of today's recitation are for you to:

1. Use geometrical formulas for perimeters, areas, volumes and surface areas to create functions for surface areas and volumes.

2. Locate the critical points of a function by calculating the first derivative of a function f and then solving the equation:

$$f'(x) = 0.$$

3. Recall the relationship between the behavior of a derivative and the concavity of the original function (see table below).

Second Derivative is ...	Derivative is ...	Original function is ...
Positive	Increasing	Concave up
Negative	Decreasing	Concave down
Zero	Neither increasing nor decreasing	Possibly at a point of inflection

4. Use the change in sign of the first derivative evaluated near a critical point to classify the critical point as a local maximum or local minimum.

5. Use the sign of the second derivative evaluated at a critical point to classify the critical point as a local maximum or a local minimum.

The aquapod may be the most environmentally friendly water bottle as its shape is close to a sphere.

The overall goal in each of the optimization problems you will complete during today's recitation will be to find the most environmentally friendly dimensions for each common way of designing a plastic water bottle. The amount of plastic used to manufacture the bottle is proportional to the surface area of the bottle, which is why we try to minimize the surface area each time.

However, the surface area formulas that you set up will normally involve at least two variables. You will also typically set up an equation for the volume of the bottle. This equation (sometimes called the *constraint*) will allow you to eliminate one of the variables from your surface area formula.

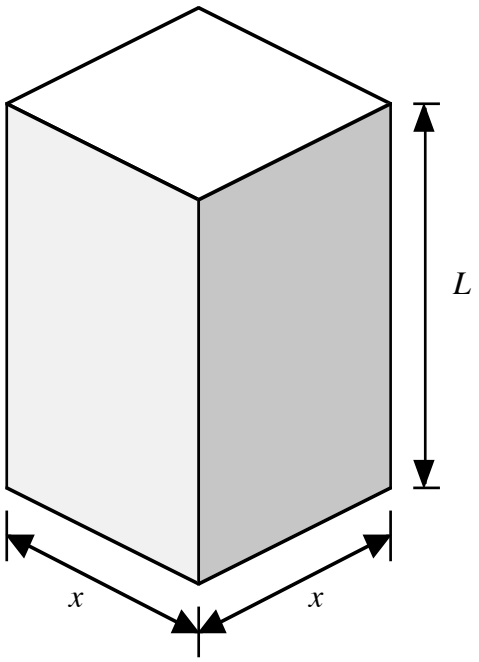
How environmentally friendly is a FIJI brand plastic water bottle?

Fiji brand bottled water comes in a bottle that closely resembles a rectangular prism with a square base (see the picture). These bottles contain one point five liters of water (which is the same as 1500 cubic centimeters).

1. The amount of plastic that is used to create the water bottle will be most closely related to the surface area of the bottle.

Because the actual bottle slopes at the top, this is actually pretty hard to calculate. As a first attempt at this problem, we will use the surface area of the simplified version of the water bottle that is shown below.

Find a formula (involving x, L and numbers) for the surface area of the water bottle.



2. The total volume of a bottle of FIJI brand water is 1500 cubic centimeters. Express this fact as an equation involving x, L and numbers.

3. Use your answers to Questions 1 and 2 to create a formula for the surface area of the bottle that involves only the variable x and numbers.

4. Find the value of x that makes the surface area as small as possible.

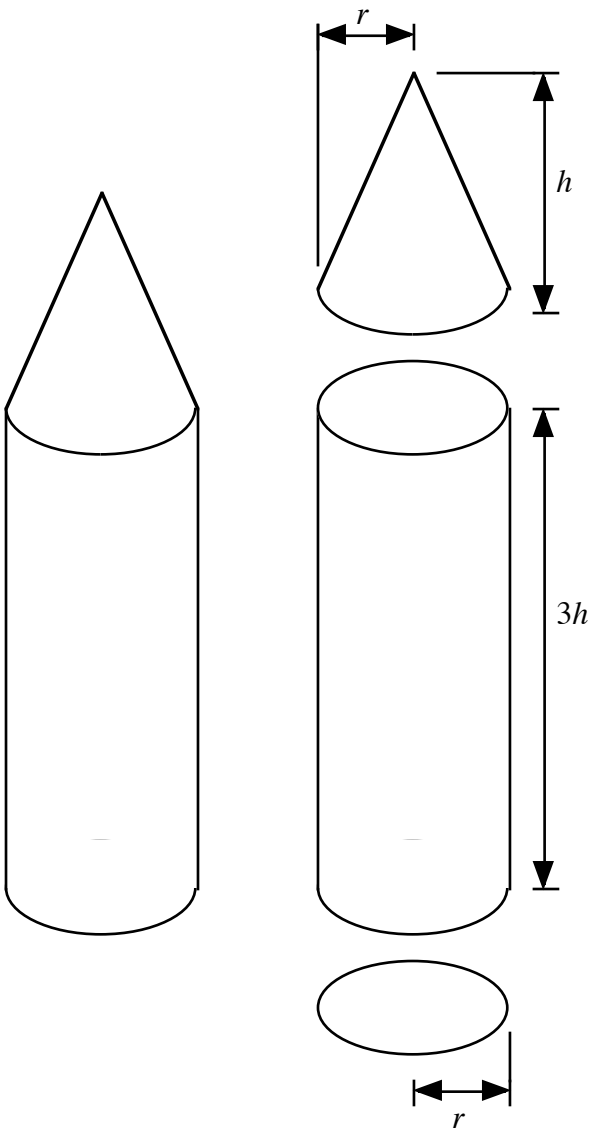
5. Find the value of L that makes the surface area as small as possible.

6. How do you know that the values of x and L that you found actually give the *smallest* surface area possible? Use the first derivative of surface area to confirm that this is the case.

x			
First derivative			

How environmentally friendly is an Evian brand water bottle?

One point five liter (1500 cubic centimeters) bottles of Evian mineral water come in a bottle that consists of a cone and a cylinder.



7. Find a formula (in terms of r and h) for the volume of the Evian water bottle.

8. Find a formula (in terms of r and h) for the surface area of the Evian water bottle. Remember (as we learned in class) that the surface area of a cone with height h and radius r is given by:

$$S = \pi r\sqrt{r^2 + h^2}\,.$$

9. Use the volume formula that you found to eliminate h from the surface area formula.

10. What is the value of r that gives the smallest possible value for the surface area of the bottle?

11. What is the value of h that gives the smallest possible value for the surface area of the bottle?

12. How do you know that the values of r and h that you found actually give the *smallest* surface area possible? Use the second derivative of surface area to confirm that this is the case.

Summary

13. Examine a FIJI and an Evian water bottle. Which seems like the most wasteful?

14. According to the values of x and L that you calculated, what is the smallest possible surface area for a FIJI brand water bottle?

15. Use a ruler to measure the actual dimensions of a FIJI brand water bottle. What is the actual surface area of the bottle?

16. By what percentage does the amount of plastic in the FIJI bottle exceed the minimum?

17. According to the values of r and h that you calculated, what is the smallest possible surface area for an Evian water bottle?

18. Use a ruler to measure the actual dimensions of an Evian water bottle. What is the actual surface area of the bottle?

19. By what percentage does the amount of plastic in the Evian bottle exceed the minimum?

20. Which of the two bottles (Evian or FIJI) is actually the mast wasteful?