21123 (Calculus of Approximation) Lecture 7 - Alternating Series

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So far, we have acquired some powerful tools for the analysis of series and the question of their convergence. Today, we step back a bit and look at series whose terms oscillate. It turns out that in the case of terms that steadily decrease to 0, this is enough for convergence of our series.

1 Recall.....

From last class, we saw that

$$\sum_{n=1}^{\infty} \frac{1}{n} \tag{1}$$

diverged. So, what about

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \tag{2}$$

Let's get our hands dirty! Take into account that we have to take partial sums, so let's look at the even partial sums first:

$$\begin{aligned} S_2 &= 1 - \tfrac{1}{2} \\ S_4 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} \\ &\cdots\cdots \\ &\cdots\cdots \\ S_{2n} &= S_{2n-2} + \tfrac{1}{2n-1} - \tfrac{1}{2n} > S_{2n-2} \\ S_{2n} &\leq 1 \end{aligned} \tag{3}$$

and so our new sequence of even partial sums is increasing, yet bounded by 1. By our theorem from Lecture 5, we know that this sequence must converge, i.e. $S_{2n} \to S$. Now, we also know that

$$S_{2n+1} = S_{2n} + \frac{1}{2n+1} \tag{4}$$

which implies that

$$S_{2n+1} \to S + 0 \tag{5}$$

and so in fact $S_n \to S$. Now, we can generalize this result: all we used was that $b_n > b_{n+1} > 0$ and that $b_n \to \infty$ (to show that S_{2n+1} converges as well.) Formally stated:

Lemma 1. *Let* $b_n > b_{n+1} > 0$ *and* $b_n \to \infty$*. Then*

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n < \infty \tag{6}$$

Proof. See discussion above. □

1.1 Examples

Let's talk about the convergence properties of the following:

$$\begin{aligned} S_\infty &= \textstyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{0.9}} \\ S_\infty &= \textstyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n \\ S_\infty &= \textstyle\sum_{n=2}^{\infty} (-1)^n \ln(1 - \frac{1}{n}) \\ S_\infty &= \textstyle\sum_{n=1}^{\infty} (-1)^n \sin(\frac{1}{n}) \end{aligned} \tag{7}$$

2 Remainder Theorem

In the first example of $b_n = \frac{1}{n}$ we used above, we could see that no matter what we added, we were always < 1. So, what if we wanted to find $S_n - S$, i.e.

$$S - S_n = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... - \frac{1}{n} + \frac{1}{n+1} - ...\right) - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... - \frac{1}{n}\right) \tag{8}$$

and so we have that our error $\mid S - S_n \mid$ is no bigger than $b_{n+1} = \frac{1}{n+1}$.

In general

Theorem 1. *For a series satisfying the conditions of a convergent alternating series, we have that* $\mid S - S_n \mid \leq b_{n+1}$

Proof. (Idea) Once again, see the preceeding discussion □

2.1 Examples

Find the largest possible error associated to approximating S by S_n:

$$\begin{aligned}
S_\infty &= \textstyle\sum_{k=1}^{\infty}(-1)^{k+1}\sin(\frac{1}{k}) \\
S_\infty &= \textstyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2+k} \\
S_\infty &= \textstyle\sum_{k=1}^{\infty}(-1)^{k+1}\ln(1+\frac{1}{k}) \\
S_\infty &= \textstyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^{1+\frac{1}{k}}}
\end{aligned} \tag{9}$$

3 Homework

Section 11.5 - pp 739-740, Ex. 7,19,24,27