21123 (Calculus of Approximation) Lecture 6 - Comparison Tests

Albert Cohen

May 25 2004

Last class, we compared our series sum with an analogous (improper) integral. Today, we shall further this notion of proof of convergence or divergence via comparison.

1 Recall.....

From last class, we saw that

$$\sum_{n=1}^{\infty} \frac{1}{n} \tag{1}$$

diverged. So, what about

$$\sum_{n=1}^{\infty} \frac{1}{n - \frac{1}{n+1}} \tag{2}$$

We can see that for

$$\begin{aligned} a_n &= \tfrac{1}{n} \\ b_n &= \tfrac{1}{n-\frac{1}{n+1}} \\ b_n &\geq a_n \\ S_\infty &= \textstyle\sum_{n=1}^{\infty} \tfrac{1}{n-\frac{1}{n+1}} \\ &\geq \textstyle\sum_{n=1}^{\infty} \tfrac{1}{n} \end{aligned} \tag{3}$$

and so our new sum diverges as it's partial sums are always bigger than those of the Harmonic series, which diverges. Let's formalize this:

Lemma 1. *If we have two sequences of terms, a_n and b_n, to be summed in an infinite series, then if $a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n < \infty$,then $\sum_{n=1}^{\infty} a_n < \infty$. Otherwise, if $a_n \geq b_n$ and $\sum_{n=1}^{\infty} b_n = \infty$,then $\sum_{n=1}^{\infty} a_n = \infty$. (By $< \infty$, we mean converge, and by $= \infty$, we mean diverge.)*

Proof. Look at the Partial sums S_n and T_n for $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$ respectively. $\square$

1.1 Examples

Let's talk about the convergence properties of the following:

$$\begin{aligned} S_\infty &= \textstyle\sum_{n=1}^{\infty} \frac{e^{-n}}{n} \\ S_\infty &= \textstyle\sum_{n=1}^{\infty} \frac{1}{n} + \frac{1}{n^2} \\ S_\infty &= \textstyle\sum_{n=1}^{\infty} \frac{n^2+3n}{n^3-\frac{1}{2}} \\ S_\infty &= \textstyle\sum_{k=0}^{\infty} \frac{\cos^2(n)}{n^2+1} \end{aligned} \tag{4}$$

2 Limit Comparison Test

Direct comparison can also be used in a limiting sense:

Theorem 1. *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$*, then* $\sum_{n=1}^{\infty} a_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$ *both converge or both diverge.*

Proof. (Idea) Since $\frac{a_n}{b_n} \to L$, there exists an N such that $n \geq N$ implies that $\frac{L}{2} \leq \frac{a_n}{b_n} \leq 2L$, which allows us to write $\frac{L}{2}b_n \leq a_n \leq 2Lb_n$ and so we use the comparison idea from the section above. Similarily, $\frac{b_n}{a_n} \to \frac{1}{L}$, and so $\frac{1}{2L}b_n \leq a_n \leq \frac{2}{L}b_n$. □

Once again, let's test this theorem out!

2.1 Examples

Let's talk about the convergence properties of the following:

$$\begin{aligned} S_\infty &= \textstyle\sum_{k=1}^{\infty} \sin(\frac{1}{k}) \\ S_\infty &= \textstyle\sum_{k=1}^{\infty} \frac{1}{k^2+k} \\ S_\infty &= \textstyle\sum_{k=1}^{\infty} \ln(1+\frac{1}{k}) \\ S_\infty &= \textstyle\sum_{k=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} \end{aligned} \tag{5}$$

3 Homework

Section 11.4 - pp 734-735, Ex. 7,10,19,30