21123 (Calculus of Approximation) Lecture 4 - Integral Test for Sequences

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Today, we begin the study of the Integral Test Theorem for sequences. Previously, the Test for Divergence told us if a series *diverged.* However, it gave us no information on whether a sequence converged if it passed the Divegence test. The Integral Test, however, can tell us so. The idea is that a series can be over or under-estimated by an analogous integral. In fact, the error associated with calculating only the first N terms of a series can also be over or under-estimated in the same way.

1 Computing the series $\sum_{n=1}^{\infty} \frac{1}{n}$

Let's visualize the series for $\sum_{n=1}^{\infty} \frac{1}{n}$ as the sum of rectangles:

Now, let's draw the curve $f(x) = \frac{1}{x}$ in two different ways, one under, and one over-estimating the sum:

Using our imagination, and some chalk, we can see that **for any positive, increasing function $f : [1, \infty) \to \Re$, and for $a_n = f(n)$**, we have that

$$\int_{n+1}^{\infty} f(x)dx \leq \sum_{k=n+1}^{\infty} a_n \leq \int_{n+1}^{\infty} f(x)dx \qquad (1)$$

and so we have given the idea for the following **Integral Test**, namely:

$\sum_{n=1}^{\infty} a_n$ **converges if and only if** $\int_1^{\infty} f(x) \quad dx$ **converges.**

The equation above that bounds the sum from $n+1$ to ∞ is in fact also very useful information, as it tells us how bad the error would be if we appproximated our infinite series by the same sum, only that we would stop at a *finite* n.

1.1 Practice

Let's apply the Integral Test to the following examples to determine convergence or divergence, and also the n^{th} remainder term:

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\ S &= \sum_{n=1}^{\infty} \frac{1}{n^{1.1}} \\ S &= \sum_{n=1}^{\infty} ne^{-n^2} \\ S &= \sum_{n=1}^{\infty} ne^{-n^2} \sin(e^{-n^2}) \end{aligned} \tag{2}$$

2 Homework

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