21123 (Calculus of Approximation) Lecture 4 - Rigourous Ideas and Definitions in Sequences and Series

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As we have for the last 3 classes, today is a day for getting our hands dirty with sequences and series. However, today, we put on more solid ground the concepts we've learned via definitions and theorems that will enable us to see even further. Onward!

1 More on Infinite Series

From last class, we saw that an infinite series is what we end up with when we allow the upper limit of a finite sum to approach infinity. Strictly speaking, we define

$$\sum_{k=0}^{\infty} c_k := \lim_{n\to\infty} \sum_{k=0}^{n} c_k = \lim_{n\to\infty} S_n \tag{1}$$

where S_n is our sequence of partial sums. We can now state and prove a tiny but extremely useful lemma:

Lemma 1. *If our series is to converge, i.e. if our sequence of partial sums S_n is to converge, then the terms being summed, c_n, must converge to zero, that is $c_n \to 0$*

Proof. If $S_n \to 0$, then $S_{n+1} \to 0$ and so $S_{n+1} - S_n \to 0$. But $S_{n+1} - S_n = c_{n+1}$. □

This is a necessary *but not sufficient* condition for our partial sums to converge; if c_n **does not** converge to 0, then S_n **does not** converge. However, we will see that the converse is not true; we can have series where $c_n \to 0$, but **still** S_n **does not** converge. More soon. Let's test out this lemma as it allows us to tell right away if a series won't converge:

1.1 Examples

Let's talk about the convergence properties of the following:

$$\begin{aligned} S_\infty &= \textstyle\sum_{k=0}^{\infty} \arctan(k) \\ S_\infty &= \textstyle\sum_{k=1}^{\infty} \ln(k) \\ S_\infty &= \textstyle\sum_{k=0}^{\infty} \frac{k^2}{1+k^2} \\ S_\infty &= \textstyle\sum_{k=0}^{\infty} x^k \end{aligned} \tag{2}$$

2 What about some Lemmas on Sequences?

Since the convergence of an infinite series is defined as the convergence of the sequence of it's partial sums, it would be nice to have some more ammo in the form of theorems regarding convergence of sequences.

Theorem 1. *Any sequence of real numbers that is always increasing (decreasing) and is bounded above (below) converges to to it's least upper (greatest lower) bound.*

Proof. (Idea) Assume without loss of generality that our sequence X_n is always increasing and has it's least upper bound as M. Imagine we don't have convergence - then X_n is always some fixed distance ϵ away from M. But since X_n is always increasing, this means that M is not the least upper bound, and so we have a contradiction. □

Once again, let's test this theorem out!

2.1 Examples

Let's talk about the convergence properties of the following:

$$\begin{aligned} S_\infty &= \textstyle\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1} \\ S_\infty &= \textstyle\sum_{k=2}^{\infty} \frac{e^{-k}}{k} - \frac{e^{-(k-1)}}{k-1} \\ S_\infty &= \textstyle\sum_{k=1}^{\infty} \arctan(k) - \arctan(k-1) \\ S_\infty &= \textstyle\sum_{k=1}^{\infty} x^k - x^{k-1} \end{aligned} \tag{3}$$

3 Properties of Convergent Series

If we have a couple of series that we know are convergent, then we have added on some more nice properties, albeit ones that we would expect:

Theorem 2. *Assume that $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are convergent series, and that c is a constant. Then the following properties hold (and are proven via the corresponding properties of limits of sequences) :*

1. $\sum_{k=0}^{\infty} ca_k = c\sum_{k=0}^{\infty} a_k$
2. $\sum_{k=0}^{\infty} (a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$
3. $\sum_{k=0}^{\infty} (a_k - b_k) = \sum_{k=0}^{\infty} a_k - \sum_{k=0}^{\infty} b_k$

3.1 Examples

Find the following sums:

$$\begin{aligned} S_\infty &= \sum_{k=1}^{\infty} \tfrac{1}{k} - \tfrac{1}{k+1} + 0.5^k \\ S_\infty &= \sum_{k=2}^{\infty} \tfrac{e^{-k}}{k} - \tfrac{e^{-(k-1)}}{k-1} + 0.25^k \\ S_\infty &= \sum_{k=1}^{\infty} \arctan(k) - \arctan(k-1) - 0.4^k \\ S_\infty &= \sum_{k=1}^{\infty} x^k - x^{k-1} - e^{-kx} \end{aligned} \tag{4}$$

4 Homework

Section 11.2 - pp 720-721, Ex. 21,23,27,28,30,35,50,53