21123 (Calculus of Approximation) Lecture 3 - Introduction to Series

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May 19 2004

Now that we've seen sequences with free parameters, i.e. sequences of functions, the next step is to compute series involving such free parameters. In this lecture, we reaquaint ourselves with series via partial sums, and move forward to compute the sum of a Geometric Series. This will provide the foundation of our theoretical work later on, including ideas like the Ratio Test and Root Test we will see shortly.

1 Gauss's Idea...

Imagine we wanted to sum up the first n integers:

$$S_n := 1 + 2 + 3 + ... + n \tag{1}$$

As a young boy, Gauss was given this problem along with his schoolmates as a form of busywork. Unlike his schoolmates, however, Gauss found a neat argument for computing this sum rather quickly:

$$\begin{aligned} S_n &= 1 + 2 + 3 + ... + n \\ S_n &= n + (n-1) + (n-2) + ...1 \end{aligned} \tag{2}$$

Adding these two lines together gives us twice the sum as

$$2S_n = (n+1) + (n+1) + (n+1) + ... + (n+1) = n\,(n+1) \tag{3}$$

and so $S_n = \frac{n(n+1)}{2}$.

2 Geometric Series

Now that we've computed our first sum, let's look at what will be the most important example of a series....

$$S_n(z) := 1 + z + z^2 + ... + z^n \tag{4}$$

What does this sum come out to be? Well, let's take z and multiply it by $S_n(z)$:

$$zS_n(z) := z + z^2 + ... + z^n + z^{n+1} \tag{5}$$

and now let's subtract:

$$S_n(z) - zS_n(z) = 1 - z^{n+1} \tag{6}$$

and so we now have a *sequence* of **partial sums** $S_n(z)$:

$$S_n(z) = \frac{1 - z^{n+1}}{1 - z} \tag{7}$$

Since we have a sequence of functions of z, we can now answer the following questions using our skills from the last two classes: What happens when $n \to \infty$?

3 Series as Limit of Partial Sums

We can use the notation

$$\sum_{k=0}^{n} a_n := a_0 + a_1 + ... + a_n \tag{8}$$

and so

$$S_n(z) := \sum_{k=0}^{n} z^n \tag{9}$$

One the one hand, we have that our $S_n(z)$ is a sequence of functions, on the other, it is the sum of a *finite* number of elements. What happens as $n \to \infty$ is that we move towards an *infinite* sum, i.e.

$$\lim_{n\to\infty} S_n = S_\infty = \sum_{k=0}^{\infty} a_n \tag{10}$$

and for our specific example,

$$\lim_{n\to\infty} S_n(z) = \sum_{k=0}^{\infty} z^n \tag{11}$$

From our bag of tricks, we now know that

$$\lim_{n\to\infty} S_n(z) = \sum_{k=0}^{\infty} z^n = \lim_{n\to\infty} \frac{1 - z^{n+1}}{1 - z} \tag{12}$$

and so we see the connection between sequences of function, their limit as $n \to \infty$, and infinite series.

3.1 Calculations involving the Geometric Series

So, what does happen as $n \to \infty$?

We see that we are left with a piecewise function,

$$\lim_{n\to\infty} S_n(z) = \begin{cases} \frac{1}{1-z}, \mid z \mid< 1 \\ \infty, \text{ otherwise,} \end{cases}$$

3.2 What about other functions?

How about the sum of

$$S_n(z) = \sum_{n=0}^{\infty} z^{2n} \tag{13}$$

What can we do here?

4 Homework

Find the limits of the following sequences:

$$\begin{aligned} S_n(x) &= \textstyle\sum_{n=0}^{\infty} e^{-nx} \\ S_n(x) &= \textstyle\sum_{n=0}^{\infty} 2^{-n} \\ S_n(x) &= \textstyle\sum_{n=0}^{\infty} x^{2n} \\ S_n(x) &= \textstyle\sum_{n=0}^{\infty} (1-x)^n \end{aligned} \tag{14}$$