21123 (Calculus of Approximation) Lecture 2 - More Computations concerning Sequences with Free Parameters

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Last lecture, we were introduced to sequences, which are simply functions whose domain is the positive integers. We ended off by looking at sequences that carried a free parameter in them, and saw that their limit as $n \to \infty$ also involved the free parameter. In this lecture, we continue along this line of inquiry and consider taking derivatives of such sequences (with respect to the free parameter x.)

1 Recall.....

Let us revisit the classic example of last lecture,

$$M_n(x) = x^n = \{x, x^2, x^3, x^4, ...\} \tag{1}$$

also known as the **Geometric Sequence** (and whose sum, the **Geometric Series**, we will discuss shortly.)

We can formally define the derivative of the Geometric Sequence as

$$\frac{d}{dx}M_n(x) = nx^{n-1} = \{1, 2x, 3x^2, 4x^3, ...\} \tag{2}$$

In fact, we can compute the derivative of any sequence $M_n(x)$ in a similar term by term fashion:

1.1 More Examples of Sequence Derivatives

Let's compute the derivatives of the following:

$$\begin{aligned} M_n(x) &= \tfrac{x^n}{2^n} \\ M_n(x) &= x^n 2^n \\ M_n(x) &= e^{-nx} \\ M_n(x) &= \sin(nx) \end{aligned} \tag{3}$$

How about computing the integrals of these sequences?

How about finding the limit of a derivative? Is it the derivative of the limit?

1.2 Interchanging Limit operations with Derivative operations

Take the sequence

$$M_n(x) = \frac{\sin(nx)}{n} = \{\sin(x), \frac{\sin(2x)}{2}, \frac{\sin(3x)}{3}, \frac{\sin(4x)}{4}, ...\} \tag{4}$$

Then

$$M(x) = \lim_{n\to\infty} M_n(x) = 0 \text{ for all } x \tag{5}$$

and so

$$\frac{d}{dx}M(x) = \frac{d}{dx}\lim_{n\to\infty} M_n(x) = 0 \text{ for all } x \tag{6}$$

Now,

$$\frac{d}{dx}M_n(x) = \frac{d}{dx}\frac{\sin(nx)}{x} = \cos(nx) \text{ for all } x \tag{7}$$

and so

$$\lim_{n\to\infty}\frac{d}{dx}M_n(x) = \begin{cases} 0, x = q\frac{\pi}{2}, q \text{ rational} \\ \infty, \text{ otherwise,} \end{cases}$$

What happened here? Our sequences seemed so smooth and nice? For the most part, we will be able to interchange limit and derivative operations - for sequnces that have rapidly increasing oscillations, or functions that bunch up at one point, such interchanges won't be possible. Let's consider our favourite 4:

$$
\begin{aligned}
M_n(x) &= \tfrac{x^n}{2^n} \\
M_n(x) &= x^n 2^n \\
M_n(x) &= e^{-nx} \\
M_n(x) &= \sin(nx)
\end{aligned} \tag{8}
$$

Obviously, we've discussed the last one. How about the first 3?

1.3 Shifting the parameter x

How about if we "shift" the x values? - What can we say about convergence of both $M_n(x)$ and $\frac{d}{dx}M_n(x)$?

$$
\begin{aligned}
M_n(x) &= \tfrac{(x-4)^n}{2^n} \\
M_n(x) &= (x-4)^n 2^n \\
M_n(x) &= e^{-n(x-4)} \\
M_n(x) &= \sin(n(x-4))
\end{aligned} \tag{9}
$$

2 Homework

Find the limits of the following sequences *and* their derivatives: **Note - a is a constant in the following questions**

$$
\begin{aligned}
M_n(x) &= \tfrac{(x-a)^n}{2^n} \\
M_n(x) &= \tfrac{x^2+nx}{e^{n^2 x}} \\
M_n(x) &= \arctan(nx) \\
M_n(x) &= 1 - e^{-nx} \\
M_n(x) &= \sin(\tfrac{e^{-nx}}{n}) \\
M_n(x) &= \ln(1 + \tfrac{e^{-nx}}{n})
\end{aligned}
\tag{10}
$$