21123 (Calculus of Approximation) Lecture 1 - Introduction to Sequences

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Most measurements we make in life are not continuous, but rather sampled discretely. In fact, we can count these measurements and label them the 1st, 2nd, 3rd, and so on. In doing so, we are really working in the space of sequences, or functions on the integers.

1 What is a sequence ?

Imagine we deposit an amount of money, and that we watch it grow as the intial amount is compounded yearly. At the end of the nth year, we will have

$$P_n = P_0\,(1+r)^n \tag{1}$$

dollars, where the interest paid is at a rate of $100r$ percent, n is the number of years since the initial investment, and P_0 is the initial amount invested. If we list the bank account balances in the order of months, then we have

$$\{P_0, P_1, P_2, P_3, ...\} = \{P_0, P_0(1+r), P_0(1+r)^2, P_0(1+r)^3, ...\} \tag{2}$$

Similarily, we could conduct an experiment where we take 10 measurements, which we **index** by the subscript n as

$$\{M_1, M_2, M_3, ..., M_n, ..., M_{10}\} = \{1, 4, 9, ..., n^2, ..., 100\} \tag{3}$$

The term indexed by n above gives us an idea what the pattern of the sequence is.

1.1 Practice

Can you guess the pattern for the following? What is M_n?

$$\begin{aligned} \{M_1, M_2, M_3, ...\} &= \{1, 8, 27, ...\} \\ \{M_1, M_2, M_3, ...\} &= \{1, 3, 5, ...\} \\ \{M_1, M_2, M_3, ...\} &= \{5, 10, 15, ...\} \\ \{M_1, M_2, M_3, ...\} &= \{e, e^2, e^3, ...\} \end{aligned} \tag{4}$$

2 Taking Limits

As we've seen above, a sequence can be defined simply by stating the form of the nth term. Just as we've done with functions $f(x)$ on the entire real line, we can now take "limits as $n \to \infty$". Formally, we **define a sequence** as a function M on the set of integers N, i.e.

$$M_n : n \in N \to \Re \tag{5}$$

just as

$$f(x) : x \in \Re \to \Re \tag{6}$$

takes real numbers and returns a real number value.

Let's draw, for example $f(x) = e^{-x}$:

We can make a similar drawing for $M_n = e^{-n}$:

Just as we see that $\lim_{x\to\infty} e^{-x} = 0$, it should be obvious that $\lim_{n\to\infty} e^{-n} = 0$. In fact, if we want to show that a sequence M_n converges, we can look at the *corresponding* function $f(x)$, and find that limit, which will coincide with the sequence limit.

As an example, consider

$$M_n = \frac{\ln(n)}{n} \tag{7}$$

It may not be clear how to find $\lim_{n\to\infty} M_n$, but we should recall that for

$$f(x) = \frac{\ln(x)}{x} \tag{8}$$

$$\lim_{x\to\infty} f(x) = 0 \tag{9}$$

How do we know this, you ask? Simple - L'Hopital's rule.

2.1 Practice

See if you can compute the limits as $n \to \infty$ for the following:

$$\begin{aligned} M_n &= \tfrac{n^2+2n+2}{n^2+n+2} \\ M_n &= \tfrac{n+2n+2}{n^2+n+2} \\ M_n &= e^{-n^2} \\ M_n &= \tfrac{\sin(n)}{n} \end{aligned} \tag{10}$$

3 Sequences of Functions

Sequences can have parameters inside them, like r in the example of compound interest above. Consequently, a sequence involving a parameter will have it's limit dependent on the parameter. As an example, consider

$$M_n(x) = x^n \tag{11}$$

The notation in this equation means that we have a sequence indexed by n, and that it has the free parameter x as well. Let's go through what the limit should be:

If $\mid x \mid < 1$, then $M_n(x) \to 0$. If you don't believe me, take out your calculator and start multiplying: $0.5 * 0.5 * 0.5 * 0.5 * ...$ or even $0.99 * 0.99 * 0.99 * 0.99 * ...$ What do you get?

If $x = 1$, then $1 * 1 * 1 * 1 * ... = 1$, and so our limit is simply 1. If $x = -1$, then our limit is undefined, as we oscillate between 1 and -1:

$$M_n(-1) = \{-1, 1, -1, 1...\} \tag{12}$$

and if $\mid x \mid > 1$, then $M_n(x) \to \infty$ as $n \to \infty$.

Collecting these, we have

$$\lim_{n\to\infty} M_n(x) := M(x) = \begin{cases} 0, \mid x \mid < 1, \\ \infty, \mid x \mid > 1, \\ 1, x = 1, \\ DNE, x = -1, \end{cases}$$

4 Practice, Practice, and more Practice!

Can you guess the pattern for the following? What is M_n ($M_n(x)$)?

$$\begin{aligned}
\{M_1, M_2, M_3, M_4...\} &= \{\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, ...\} \\
\{M_1, M_2, M_3, M_4...\} &= \{0, 1, 0, 1, ...\} \\
\{M_1, M_2, M_3, M_4...\} &= \{e^x, e^{2x}, e^{3x}, e^{4x}, ...\} \\
\{M_1, M_2, M_3, M_4...\} &= \{1, 2^x, 3^x, 4^x...\}
\end{aligned} \tag{13}$$

Find $\lim_{n\to\infty} M_n(x) := M(x)$ for the following:

$$\begin{aligned}
M_n(x) &= \tfrac{x^n}{2^n} \\
M_n(x) &= x^n 2^n \\
M_n(x) &= e^{-nx} \\
M_n(x) &= \sin(nx)
\end{aligned} \tag{14}$$