21112 (Calculus 2) Lecture 8 - Applications of Improper Integrals

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In the last lecture, we defined the notion of an improper integral. In today's lecture, we use this new idea to define continuous probability densities and distributions.

A few diagrams in this lecture.

Probability Density/Distribution

In probability, we have events A which are elements of the set of all possible elements, and a function that assigns a number to that element A, $P(A)$, that tells how likely it is that A will actually occur. This set of all elements can take many forms: It can be finite (i.e. a switch is on (A) or not (A^c), a statement is TRUE (A) or FALSE (A^c), etc..), infinite, but countable (i.e. the set of all energy levels of an atom $(A = \{n\}, n \in 1, 2, ...)$) or infinite and uncountable (an electron shot out of a cathode ray tube will hit the T.V. screen a distance r from the centre, i.e. $A = \{dist. \leq r\}, 0 \leq r < \infty$.)

In this lecture, we will deal with the last case, where there are an uncountable number of possible events. Specifically, we will concern ourselves with the case of finding the result of some kind of experiment producing a number in the range $(a, b) \subset (-\infty, \infty)$.

So, for us, the event A will take the form 'Experiment A produces a number x such that $a \leq x \leq b$'. Associated with this set of possible of experiments is a function $f(x) : (-\infty, \infty) \rightarrow (0, \infty)$ which we use to calculate probabilities by integrating it out:

$$Prob(A : a \leq x \leq b) = \int_a^b f(x)dx \tag{1}$$

Now, since we now that a number between $-\infty$ and ∞ is produced with 100 percent certainty, we must have that

$$Prob(-\infty < x < \infty) = \int_{-\infty}^{\infty} f(x)dx = 1 \tag{2}$$

So, we can describe with some accuracy a process by it's probability density: we can at least predict it's probable outcome. Here are some interesting processes to look at:

1. The Normal or Gaussian process: $f(x) := \frac{1}{(2\pi)^{0.5}} e^{-0.5x^2}$
2. The $\lambda-$Exponenential process:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, x > 0, \\ 0, x < 0 \end{cases}$$

3. The Skewed Exponential process:

$$f(x) = \begin{cases} k^2 x e^{-kx}, x > 0, \\ 0, x < 0 \end{cases}$$

The last two examples show us that a process may have it's values with 100 percent certainty in an interval subset (A, B) of $(-\infty, infty)$. The only requirement then for $f(x)$ to be a probability density is that

$$\int_A^B f(x)dx = 1 \tag{3}$$

Example 1
Is

$$f(x) = \begin{cases} x^2, 0 < x < 1, \\ 0, \text{otherwise} \end{cases}$$

a probability density ?
Answer
Since

$$\int_0^1 x^2 dx = \frac{1}{3} \tag{4}$$

the answer is no. But

$$f(x) = \begin{cases} 3x^2, 0 < x < 1, \\ 0, \text{otherwise} \end{cases}$$

is.

Example 2
For what value(s) of k is the following a probability density ?

$$f(x) = \begin{cases} \frac{k}{x}, 1 < x < 4, \\ 0, \text{otherwise} \end{cases}$$

a probability density ?
Answer
For $f(x)$ to be a density, we need the following to hold:

$$1 = \int_1^4 \frac{k}{x} dx = k \int_1^4 \frac{1}{x} dx = k \ln(4) - k \ln(1) = k \ln(4) \tag{5}$$

and so we have a density when $k = \frac{1}{\ln(4)}$

Distributions

Moving on, we can define the probability for the set $\{A : -\infty < x < y\}$ as

$$F(y) := Prob(\{A : -\infty < x < y\}) = \int_{-\infty}^{y} f(x) dx \tag{6}$$

Example 3 Find the distribution function for the $\lambda-$Exponential process.

Answer

by definition,

$$F(y) = \int_{-\infty}^{0} 0 dx + \int_0^y \lambda e^{-\lambda x} dx = 1 - e^{-\lambda y} \tag{7}$$

Notice that $F(0) = 0$ and that $F(\infty) = 1$, as well as $F'(y) = \lambda e^{-\lambda y}$.

Going back a bit, let's calculate the probability that if A is an event following a $\lambda-$Exponential density/distribution, what is the probability that it produces an outcome x such that $1 < x < 2$? By definition, this is calculated by

$$Prob(A : 1 < x < 2) = \int_1^2 \lambda e^{-\lambda x} dx = e^{-\lambda} - e^{-2\lambda} \tag{8}$$

What if $\lambda = 1$? $\lambda = 4$? Could we have gotten this answer using the distribution $F(y)$?

Homework

pp.$383 - 4 : 1, 2, 10, 12$