21112 (Calculus 2) Lecture 7 - Improper Integrals: What happens when we go too far?

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For us, an integral can have two forms: definite or indefinite. In the definite case, we are computing the integral from $x = a$ to $x = b$. A question arises regarding the infinite case; what happens if $a = -\infty$ and/or $b = \infty$? In this lecture, we shall define the *improper* integral for this case using the concept of limits at infinity.

A few diagrams in this lecture.

Integral of $\int_1^b \frac{1}{x^n}dx$ for $n > 1$ and $n = 1$

Recall that our power rule says that $\int x^n dx = \frac{x^{n+1}}{n+1}$. So, we can compute the above as

1. $n > 1$: $\int_1^b \frac{1}{x^n}dx = \frac{b^{-n+1}-1}{-n+1}$
2. $n = 1$: $\int_1^b \frac{1}{x^n}dx = \int_1^b \frac{1}{x^1}dx = \ln(b) - \ln(1) = \ln(b)$

Draw in the curves here. As you can see, they look qualitatively the same. Now, what happens as $b \to \infty$? In the specific cases,

1. $n > 1$: $\lim_{b\to\infty} \int_1^b \frac{1}{x^n}dx = \lim_{b\to\infty} \left(\frac{b^{-n+1}-1}{-n+1}\right) = \frac{1}{n-1}$
2. $n = 1$: $\lim_{b\to\infty} \int_1^b \frac{1}{x^n}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^1}dx = \lim_{b\to\infty} \ln(b) = \infty$

There is a name for the above: the **Improper Integral from 1 to ∞**, or in symbol notation, we define

$$\int_1^\infty x^{-n}dx = \lim_{b\to\infty} \int_1^b x^{-n}dx \tag{1}$$

We can do this for any *general* function $f(x)$ and lower limit a:

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx \tag{2}$$

and similarily,

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx \tag{3}$$

In the above example of $\int_1^\infty x^{-n}dx$, we can see that $\frac{1}{x}$ decays *too slowly* for it's improper integral to exist, but this is not a problem when $n > 1$.

Let's try some other examples now:

Find, if they exist, the following improper integrals

1. $\int_0^\infty e^{-kx}dx$

2. $\int_0^\infty xe^{-x}dx$

3. $\int_0^\infty xe^{-x^2}dx$

4. $\int_{-\infty}^\infty xe^{-x^2}dx$

How about some harder ones ? :)

1. $\int_2^\infty \frac{1}{x\ln(x)}dx$

2. $\int_0^\infty \frac{x}{(x^2+1)}dx$

3. $\int_0^\infty \frac{2x}{(x^2+1)^2}$

4. $\int_{-\infty}^0 x^2 e^{x^3}dx$

Homework

pp.$377-8: 22, 37, 43, 44, 48, 49$