21112 (Calculus 2) Lecture 6 - Fun with Integrals: Stupid Integration Tricks

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Having spent the past few lectures on applications of integration, we return to the nuts and bolts of the matter. In this lecture, we learn how to tackle more complicated integrals using the techniques of integration by parts and by substitiution.

No diagrams in this lecture!

Integration by Substitution

When we have a composition of functions, ie $f(g(x))$, the derivative is taken in layers, in the sense that $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. So, we can work this backwards in the case of integration, i.e.

$$\int f'(g(x))g'(x)dx = f(g(x)) \tag{1}$$

As a specific example, notice that

$$\frac{d}{dx}e^{x^2} = 2xe^{x^2} \tag{2}$$
$$\frac{d}{dx}\ln\left(x^2+1\right) = \frac{2x}{x^2+1} \tag{3}$$
$$\frac{d}{dx}\left(x+x^2\right)^2 = 2\left(x+x^2\right)(1+2x) \tag{4}$$

so we can say that

$$e^{x^2} + C = \int 2xe^{x^2}dx \tag{5}$$
$$\ln\left(x^2+1\right) + C = \int \frac{2x}{x^2+1}dx \tag{6}$$
$$\left(x+x^2\right)^2 + C = \int 2\left(x+x^2\right)(1+2x)\,dx \tag{7}$$
$$\tag{8}$$

The trick is to find the $g(x)$, or the $''$ inner $''$ part and it's derivative $g'(x)$. This takes practice. Go ahead and see if you can spot the following $g(x)$ and $g'(x)$:

$$\int 4x^3 e^{x^4} dx \tag{9}$$

$$\int (\ln(x))^2 \frac{1}{x} dx \tag{10}$$

$$\int 2\left(1+e^x\right) e^x dx \tag{11}$$

Some books will use the notation $\int f(u)du$, which translates for us as $u = g(x)$ and $du = g'(x)dx$. Do some more practice questions at home!

Integration by Parts

Just like integration by Substitution, integration by Parts is derived from a derivative rule, namely the Product Rule:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x) \tag{12}$$

So, as we did in deriving the method of Substitution, we reverse this derivativerule to obtain

$$f(x)g(x) = \int f'(x)g(x) + f(x)g'(x)dx \tag{13}$$

or, in it's complete form,

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx \tag{14}$$

For Parts, we must choose wisely which fucntion we choose to label $f'(x)$ and which one we choose to label $g(x)$. An example is in order:

Example 1

Compute $\int x\ln(x)dx$

ANSWER:

As a first attempt, label $f'(x) = \ln(x)$ and $g(x) = x$. Then we have $g'(x) = 1$, but $f(x) = \int \ln(x)dx$ is not easily computable. (In fact, this integral itself can only be done by parts. More on this later.) For our second attempt, we label $f'(x) = x$ and $g(x) = \ln(x)$. So, we have that $f(x) = \frac{x^2}{2}$ and $g'(x) = \frac{1}{x}$. Using the formula above, we have that

$$\int x \ln(x) dx = \frac{x^2}{2} \ln(x) - \int \frac{x^2}{2} \frac{1}{x} dx$$
$$= \frac{x^2}{2} \ln(x) - \int \frac{x}{2} \frac{1}{x} dx$$
$$= \frac{x^2}{2} \ln(x) - \frac{x^2}{4} + C$$

Example 2

Compute $\int xe^x dx$

ANSWER:

Try $f'(x) = x$ and $g(x) = e^x$. Then, by the Parts formula we have that

$$\int xe^x dx = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x dx \tag{15}$$

which is of no use to us. So, we try $f'(x) = e^x$ and $g(x) = x$. Then, we obtain that

$$\int xe^x dx = xe^x - \int e^x dx$$
$$= xe^x - e^x + C$$

Example 3

A company sells it's product for $p = 1$, and it notices that the demand $q(x)$ when x units are made is $q(x) = \ln(x)$. What is the total revenue made when the production is increased from 1 unit to 10 units ?

ANSWER: Revenue $= pq(x) = 1 * \ln(x)$. So, we have that

$$Revenue = \int_1^{10} 1 * \ln(x) dx \tag{15}$$

This is obtained by integration by Parts: Let $f'(x) = 1$, and $g(x) = \ln(x)$. Then we have that

$$\int \ln(x)dx = x\ln(x) - \int x\frac{1}{x}dx = x\ln(x) - \int 1dx = x\ln(x) - x + C \quad (15)$$

So, our revenue increase (in dollars) is simply

$$Revenue = \int_1^{10} \ln(x)dx = (10\ln(10) - 10) - (1\ln(1) - 1) = 10\ln(1) - 9 \quad (15)$$

Homework
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