21112 (Calculus 2) Lecture 5 - Additional Applications of the Definite Integral

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Integration has many applications in Physics and Economics, as we have seen in the past few lectures. However, those examples were only the tip of the iceberg! In this lecture, we present methods to find the average height of an area bounded by the curve $f(x)$ (the $y-$value of the centre of mass for all you physicists!), the volume and surface area of the result of revolving $f(x)$ around the $x-$axis, and the consumer surplus for a commodity with price$-$demand curve $p = f(x)$ (where x is demand).

Once again, you will be filling in all the diagrams.

Average height of $f(x)$

Assuming that we have a function $f(x)$ that is defined on the interval $[0,1]$, we can find a grid of points $\{x_i\}_{i=1}^n$ where $x_1 = 0$, $x_n = 1$. Now, on this grid, if we take the corresponding $y-$values, $\{f(x_i)\}_{i=1}^n$, then we can find the average height of this *finite* set of $(x_i, f(x_i))$ points:

$$Average_n := \frac{f(x_1) + f(x_2) + ... + f(x_n)}{n} \tag{1}$$

However, this is the same as R_n, our n^{th} Riemann sum with $\Delta = \frac{1}{n}$. So, as $n \to \infty$, we have our definition of the average for the *continuous* curve $(x, f(x))$

$$\begin{aligned} Average_{f(x)}^{[0,1]} &:= \lim_{n\to\infty} Average_n \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta \\ &= \int_0^1 f(x)dx \end{aligned} \tag{2}$$

Now, what if we had the interval $[a,b]$ to deal with ? In this case, we have simply that

$$\begin{aligned} Average_n &:= \frac{f(x_1) + f(x_2) + ... + f(x_n)}{n} \\ &= \frac{1}{b-a}\left((f(x_1) + f(x_2) + ... + f(x_n))\,\frac{b-a}{n}\right) \\ &= \frac{1}{b-a} R_n \end{aligned} \tag{3}$$

where R_n is our n^{th} Riemann sum, except that $\Delta = \frac{b-a}{n}$, the more general case.

So, we have come to a general definition of the average of a function:

$$Average_{f(x)}^{[a,b]} = \frac{1}{b-a}\int_a^b f(x)dx \tag{4}$$

Example 1

Find the average of e^x over 1.) the interval $[0,1]$ and 2.) the interval $[-1,2]$.

ANSWER:

Over the interval $[0,1]$, we simply have

$$Average = \frac{1}{1-0}\int_0^1 e^x = \frac{e^1 - e^0}{1} = e - 1 \tag{5}$$

Over the interval $[-1,2]$,

$$Average = \frac{1}{2-(-1)}\int_{-1}^2 e^x = \frac{e^2 - e^{-1}}{2-(-1)} = \frac{e^2 - e^{-1}}{3} \tag{6}$$

Example 2

Find the average of x^2 over $[-1,1]$

ANSWER:

$$Average = \frac{1}{1-(-1)}\int_{-1}^1 x^2 dx = \frac{1}{2}\left(\frac{1}{3} - \frac{-1}{3}\right) = \frac{1}{3} \tag{7}$$

Solids of Revolution: Volume

In the first lecture, we began by defining the area under a curve as the limit of the sum of areas of approximating rectangles under the curve as the width of these rectangles gets smaller and smaller. Now, we can approximate the volume of a solid of revolution in the same way: Look at a right circular cone. It's cross-section is simply two straight lines starting at the origin, $y = ax$ and $y = -ax$ for some $a > 0$. In fact, if we just took this one curve from above, $y = ax$, then revolving it around the $x-$axis would give us our cone. Now, we can look at the cone as the limit of circular discs piled on each other. As the thickness of these discs approaches zero, we should get our volume. In symbols, choose a set $\{x_i\}_{i=1}^n$ as above, with $a = 0, b = h$ (h the height of the cone) and we have

$$Volume_{cone} := \lim_{n\to\infty} \sum_{i=0}^{n} \pi \left(ax_i\right)^2 \Delta = \int_0^h \pi \left(ax\right)^2 dx = \frac{1}{3}\pi a^3 h^3 \qquad (8)$$

but if the base of the cone is r, then $a = \frac{r}{h}$ and so the above formula reduces to the familiar $V = \frac{1}{3}\pi r^2 h$.

This idea can be generalized to the case of revolving the curve $y = f(x)$ around the $x-$axis:

$$Volume := \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(f(x_i)\right)^2 \Delta = \int_a^b \pi \left(f(x)\right)^2 dx \qquad (9)$$

Example 3: Find the volume of a paraboloid

If we revolve the parabola $y = x^2$ around the $x - axis$, then we get a *paraboloid*. The resulting volume from $x = a$ to $x = b$ is

$$Volume := \int_a^b \pi \left(x^2\right)^2 dx = \frac{\pi}{5}\left(b^5 - a^5\right) \qquad (10)$$

Consumer Surplus

Suppose we are in an open market where A (for Albert) units of books are demanded and a price B (for Big) is charged. Then, the total revenue is AB. But, suppose that there is an underlying price−demand relationship $p = f(x)$ where p is the price consumers will pay when they know x units are available. It is assumed then that $B = f(A)$. Since the price p goes up as supply x goes down, we expect that $f(x)$ is decreasing. So what if the publisher devised a scheme where he released a number of books x continuously, and kept decreasing the price ? In other words, first release $\frac{A}{n}$ number of books, charging $f(\frac{A}{n})$ dollars. Then, he would release another $\frac{A}{n}$ copies, charging this time $f(\frac{2A}{n})$ dollars. Let's say that he does this n times. So, there are n printings instead of just one, and everyone who waited for a later printing would get the book cheaper. However, let's pretend that the consumer doesn't know there is another printing planned, or rather she wants to get the book sooner and is willing to pay for this privilege. In this case, the revenue for the publisher from this plan is

$$\sum_{i=1}^{n} f\left(i\frac{A}{n}\right)\frac{A}{n} \tag{11}$$

dollars. If this scheme is carried out continuously, then the amount made is

$$\int_0^A f(x)dx \tag{12}$$

and so the amount the consumer market *saves* by having only one release is

$$\int_0^A f(x)dx - AB = \int_0^A (f(x) - B)\,dx \tag{13}$$

Example 4: Consumer Surplus

Suppose the demand (number of units x that market will purchase at price p) for Nintendo's Gamecube is $x(p) = 100,000\,(249.99 - p)$ and that presently, the price is \$149.99. Upon release, the Gamecube sold for \$199.99. How much more money could Nintendo have made if they had dropped the price *continuously* ?

ANSWER:

$$\begin{aligned} Difference &= -\int_{199.99}^{149.99} 100,000\,(249.99 - p - 149.99)\,dp \\ &= 100,000\int_{149.99}^{199.99} (249.99 - p - 149.99)\,dp \\ &= 100,000\left(100(199.99 - 149.99) - \frac{1}{2}\left((199.99)^2 - (149.99)^2\right)\right) \\ &= \$374,950,000 \end{aligned} \tag{14}$$

Homework

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