21112 (Calculus 2) Lecture 4 - Finding the area between curves

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Today, we apply the concept of linearity of definite integrals that we proved last lecture. Specifically, we use it to compute the integral of the difference of functions $f(x)$ and $g(x)$. Computing this integral allows us to find the area bounded between $f(x)$ and $g(x)$. In economics, we may know the marginal revenue $R'(x)$ and the marginal cost $C'(x)$, so we integrate the difference to find the total profit $P(b) - P(a) = \int_a^b (R'(x) - C'(x))\, dx$.

In this lecture, you will be filling in all the diagrams. This is instructive as I want you to get in the habit of drawing curves.

In all of our previous work, we have found the definite integral of $f(x)$, i.e. $\int_a^b f(x)dx$. This could be restated as

$$\int_a^b (f(x) - 0)\, dx \tag{1}$$

In other words, we are finding the area between the $y = f(x)$ curve and the $y = 0$ curve, assuming that $f(x) \geq 0$. However, this could be extended: What if $f(x)$ was bounded below by another function $g(x)$, and we wanted to find the area between them ? Then, all we need to do is to replace $y = 0$ in our first equation above with $y = g(x)$, *i.e.*

$$\int_a^b (f(x) - g(x))\, dx \tag{2}$$

Notice that is **not** necessary to have $g(x) \geq 0$, but rather only that $f(x) \geq g(x)$ in order to guarantee what we are integrating is indeed positive.

Example 1

Find the area in between $y = x$ and $y = x^2$.

ANSWER:

First off, we need to figure out which of the two functions is the top function, *i.e.* $f(x)$, and which is the bottom, *i.e.* $g(x)$.

To find out where they intersect, we set them equal

$$x = x^2 \Leftrightarrow x = 0, 1 \tag{3}$$

Now, we know that when $x \in [0, 1]$, squaring x makes it even smaller. So, it must be that $f(x) = x, g(x) = x^2$. The area in between the two is (Draw the graphs!)

$$\int_0^1 \left(x - x^2\right) dx = \frac{1}{2}(1)^2 - \frac{1}{3}(1)^3 = \frac{1}{6} \tag{4}$$

Now, what if we wanted to find the area between $y = x - \frac{1}{2}$ and $y = x^2 - \frac{1}{2}$? The answer is that the area is exactly the same, even though both graphs dip below the $x-$ axis for some x values. To see this, repeat the process above to see that the intersection points are the same: $x = 0, 1$ and that

$$\begin{aligned} Area &= \int_0^1 \left(\left(x - \frac{1}{2} \right) - \left(x^2 - \frac{1}{2} \right) \right) dx \\ &= \int_0^1 \left(x - x^2\right) dx \\ &= \frac{1}{2}(1)^2 - \frac{1}{3}(1)^3 = \frac{1}{6} \end{aligned}$$

Example 2

Find the area in between $y = x^3$ and $y = 2x^2 - x$.

ANSWER:

To find out the points of intersection, we set the two equal again!

$$x^3 = 2x^2 - x \Leftrightarrow x(x^2 - 2x + 1) = 0 \Leftrightarrow x = 0, 1 \tag{5}$$

To find out which is on top, take a test point. We will use $x = \frac{1}{2}$. Since $\frac{1}{2}^3 = \frac{1}{8} > 0 = 2(\frac{1}{2})^2 - \frac{1}{2}$, we obtain that $f(x) = x^3$ and $g(x) = 2x^2 - x$. So, the area in between is $Area = \int_0^1 (x^3 - 2x^2 + x)dx = \frac{1}{4}(1)^4 - \frac{2}{3}(1)^3 + \frac{1}{2}1^2 = \frac{1}{12}$.

Example 3

Two cars going in the same direction have a velocity functions $v_a(t)$ and $v_b(t)$ respectively. We have that

$$\begin{aligned} v_a(t) &= 10t^2 \\ v_b(t) &= e^{0.5t} \end{aligned}$$

How much farther ahead of car b is car a after 2 seconds ?

ANSWER

It is easy enough to see that car a always has a higher velocity than b when $t \in [0,2]$, and so it must go further. So, the total distance between them is simply $distance(a) - distance(b)$, which in integral terms is

$$\begin{aligned}
&\int_0^2 v_a(t)dt - \int_0^2 v_b(t)dt \\
&= \int_0^2 (v_a(t) - v_b(t))\, dt \\
&= \left(\frac{10}{3}(2)^3 + 2e^{0.5*2}\right) - \left(\frac{10}{3}(0)^3 + 2e^{0.5*0}\right) \\
&= \frac{80}{3} + 2e - 2
\end{aligned}$$

So, we can speak of the difference of the distance each traveled, or we can find the difference of velocities and integrate that.

Example 4

A company has marginal revenue of $R'(x) = \$10/$ unit and marginal costs of $C'(x) = e^{-x}$, also in \$/ unit. Find the increase in profits if production is increased from 1 unit to 10 units.

ANSWER:

Since $C' = e^{-x} \le 1 < 10 = R'$, we can see that we will have a positive difference to integrate. The profit increase is

$$\begin{aligned}
P(10) - P(0) &= \int_0^1 (R'(x) - C'(x))\, dx \\
&= \int_0^{10} \left(10 - e^{-x}\right) dx \\
&= \left(10(10) + e^{-10}\right) - \left(10(1) + e^{1}\right) \\
&= \$87.28
\end{aligned}$$

Note that there are marginal cost functions, mind you, that give a negative profit increase if we integrate to far. As an example, replace $C'(x)$ above with $C'(x) = 8x$ and then again find the profit change from 1 unit to 10 units.

Homework

pp.$351 - 53$ numbers $2, 7, 10, 17, 20, 27, 29$

REMEMBER: THERE IS A QUIZ THIS WEEK ON LAST WEEK'S MATERIAL!