21112 (Calculus 2) Lecture 3 - Indefinite Integrals and their application to Definite Integrals: A Fundamental Theorem of Calculus

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Today, we combine the results of the first two lectures to find an easy way to compute definite integrals - after today, (except on your midterm and final :)), you will not need to use the limit defintion of the integral again. Instead, we will first compute the indefinite integral of $f(x)$, $\int f(x)dx$,and use that to compute the definite integral of $f(x)$, $\int_a^b f(x)dx$.

To begin, we notice that once again, integration, this time in the definite sense is linear:

Theorem 1. *Take c_1, c_2 real and $f(x), g(x)$ real valued functions. Then*

$$\int_a^b (c_1 f(x) + c_2 g(x))\, dx = c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx \tag{1}$$

Proof.

$$\begin{aligned}
\int_a^b (c_1 f(x) + c_2 g(x))\, dx &= \lim_{n\to\infty} \sum_1^n (c_1 f(x_i) + c_2 g(x_i))\, \Delta x \\
&= \lim_{n\to\infty} \left(\sum_1^n c_1 f(x_i)\Delta x + \sum_1^n c_2 g(x_i)\Delta x \right) \\
&= \lim_{n\to\infty} \left(\sum_1^n c_1 f(x_i)\Delta x \right) + \lim_{n\to\infty} \left(\sum_1^n c_2 g(x_i)\Delta x \right) \\
&= c_1 \lim_{n\to\infty} \left(\sum_1^n f(x_i)\Delta x \right) + c_2 \lim_{n\to\infty} \left(\sum_1^n g(x_i)\Delta x \right) \\
&= c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx
\end{aligned}$$

□

Up until now, we have assumed that $f(x) \geq 0$ in order to ascribe an intuitive meaning to the integral $\int_a^b f(x)dx$, that of area under the graph of $f(x)$. However, everything we have done holds true even if $f(x)$ dips below the $x-$axis. In fact, we can give another intuitive meaning: Say that $f(x) \geq 0$ for $x \in [a, c]$ and that $f(x) \leq 0$ for $x \in [c, b]$. Then

$$\int_a^b f(x)dx = Area(a, c) - Area(c, b) \tag{2}$$

where we define the $Area(a, b)$ to be the area under $f(x)$ and *above* the $x-$axis, and $Area(c, b)$ to be the area under the $x-$axis and above $f(x)$. So, for example,

$$\int_0^2 (1 - x)dx = 0 \tag{3}$$

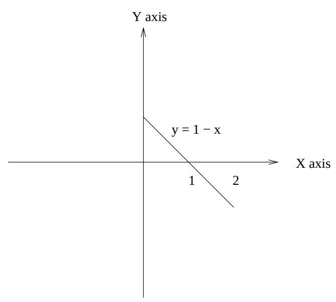

Figure 1: $f(x) = 1 - x$. Notice the two right triangles whose areas cancel each other out

We can see this as from 0 to 1, we have an area of $\frac{1}{2}$ *above* the $x-$axis, and from 1 to 2, we have an area of $\frac{1}{2}$ *below* the $x-$axis. (See Figure 1.) So, these cancel each other out to give an integral of value 0. Notice that this happens even though $f(x) \neq 0$ identically!

We are now ready for one of the **Fundamental Theorems of Calculus** :

Theorem 2. *If $f(x)$ is continuous on $[a, b]$ and $F(x) := \int f(x)dx$ is a definite integral of $f(x)$, the we have that*

$$\int_a^b f(x)dx = F(b) - F(a) \tag{4}$$

Proof. We can give an intuitive proof via the Mean value theorem:

By the definition of $F(x)$, we know that $F'(x) = f(x)$. So, by the Mean Value Theorem, if we let $b = a + \Delta x$, then there exists an $x_1 \in [a, a + \Delta x]$ such that

$$\frac{F(b) - F(a)}{b - a} = \frac{F(a + \Delta x) - F(a)}{\Delta x} = f(x_1) \tag{5}$$

A similar argument shows that

$$\frac{F(a + 2\Delta x) - F(a + \Delta x)}{\Delta x} = f(x_2) \tag{6}$$

and in fact

$$\frac{F(a+i\Delta x)-F(a+(i-1)\Delta x)}{\Delta x}=f(x_i) \tag{7}$$

So,

$$\begin{aligned}
&F(a_1)-F(a)=f(x_1)\Delta x\\
&F(a_2)-F(a_1)=f(x_2)\Delta x\\
&.\\
&.\\
&.\\
&F(a_i)-F(a_{i-1})=f(x_i)\Delta x\\
&.\\
&.\\
&.\\
&F(b)-F(a_{n-1})=f(x_n)\Delta x
\end{aligned}$$

Adding up the left and right sides individually, we get

$$F(b)-F(a)=f(x_1)\Delta x+f(x_2)\Delta x+....+f(x_n)\Delta x \tag{8}$$

If we take the limit as $n\to\infty$ on both sides of this previous equation, we get that

$$F(b)-F(a)=\lim_{n\to\infty}\sum_{1}^{n}f(x_i)\Delta x=\int_a^b f(x) \tag{9}$$

and we are done.

□

Let's do some examples!

Example 1

Evaluate $\int_{-2}^{1}(x^2-2e^{-x}+x^{-4})dx$

ANSWER:

Define

$$F(x) := \int (x^2 - 2e^{-x} + x^{-4})dx$$
$$= \frac{1}{3}x^3 + 2e^{-x} - \frac{1}{5}x^{-5} + C$$

So we have by our Fundamental Theorem that

$$\int_{-2}^{1} (x^2 - 2e^{-x} + x^{-4})dx = F(1) - F(-2)$$
$$= \left(\frac{1}{3}(1)^3 + 2e^{-1} - \frac{1}{5}(1)^{-5} + C\right) - \left(\frac{1}{3}(-2)^3 + 2e^{2} - \frac{1}{5}(-2)^{-5} + C\right)$$
$$= \left(\frac{1}{3}(1)^3 + 2e^{-1} - \frac{1}{5}(1)^{-5}\right) - \left(\frac{1}{3}(-2)^3 + 2e^{2} - \frac{1}{5}(-2)^{-5}\right)$$

It should be apparent that the Fundamental Theorem is the way to go when it comes to tackling problems like this. Also, note that the constants C drop out, so we need not worry about inserting the constant C when we will be using the indefinite integral to compute the definite integral. However, it must be included when computing *only* the indefinite integral.

Example 2

Denote the temperature of a star, in Kelvin, that is about to explode as $T(t)$, where t stands for time in hours. Furthermore, assume that the rate of temperature increase, $T'(t)$ is given by $T'(t) = 0.9 - e^{-t}$. Assume that the balloon will burst when the temperature *increase* hits 100 Kelvin. Remember, it's not the absolute temperature, but rather the temperature change that we are concerned about. So, will the star burst after 100 hours?

ANSWER:

By definition,

$$
\begin{aligned}
T(100) - T(0) &= \int_0^{100} T'(t)dt \\
&= \int_0^{100} \left(0.9 - e^{-t}\right) dt \\
&= \left(0.9(100) + e^{-100}\right) - \left(0.9(0) + e^{0}\right) \\
&= 899
\end{aligned}
$$

and so *no*, the star will not burst after 100 hours.

Homework

pp.$341 - 343$ number $7, 25, , 30, 38, 41, 43$