21112 (Calculus 2) Lecture 19 - Constrained Optimization

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April 14 2003

In life, we make do with what we have and try to make the best of it. In essence, we would like to optimize many things, but we have constraints: we'd like to make the most amount of money, but we only have 24 hours a day. We'd like to make the biggest box possible, but we only have so much cardboard availible. All of this can be done with the concept of constrained optimization, where we introduce a new function to be optimized, and a new variable to optimize over.

1 Lagrange Multipliers

Imagine that we would like to minimize the square of the distance from the origin $(0,0,0)$ to some surface $z = f(x,y)$:

$$\begin{aligned} &\min\left(x^2+y^2+z^2\right) \\ &\text{given } z = f(x,y) \end{aligned} \tag{1}$$

In this case, we may substitute in the function for z:

$$\min\left(x^2+y^2+f^2(x,y)\right) \tag{2}$$

and go about finding the minimum. However, another way that is more general and allows us to compute the solution for a wide range of problems with constraints is as follows: Define $L(x,y,z,\lambda)$

$$L(x,y,z,\lambda) := x^2+y^2+z^2-\lambda(z-f(x,y)) \tag{3}$$

Then we have that our partials require that

$$\begin{aligned} 0 &= L_x = 2x+\lambda f_x \\ 0 &= L_y = 2y+\lambda f_y \\ 0 &= L_z = 2z+\lambda f_z \\ 0 &= L_\lambda = -(z-f(x,y)) \end{aligned} \tag{4}$$

and we would then go on to solve for these equations. The natural question now is "why does this work?"

2 Explanation of Lagrangian

The Lagrangian method works for any $n-$variable problem, so let's test it out on the $2-d$ case.

Imagine we wish to minimize the distance from $(0,0)$ to the line $y = 1-x$. Let's draw this constraint (that we *must* be on the line $y = 1-x$) and "grow" the level sets of the function we wish to minimize, $x^2 + y^2$

So it seems that our minimum happens when our level curves just touch the constraint. When this happens, we have a tangency condition between our function we wish to minimize and the constraint function. In general terms, if we want

$$\begin{aligned} &\min f(x,y,z) \\ &\text{given } g(x,y,z) = 0 \end{aligned} \tag{5}$$

we define the **Lagrangian** as $L(x,y,z,\lambda) := f(x,y,z) - \lambda g(x,y,z)$ and solving the constrained problem corresponds to solving for the following system of equations:

$$\begin{aligned} 0 &= L_x = f_x - \lambda g_x \\ 0 &= L_y = f_y - \lambda g_y \\ 0 &= L_z = f_z - \lambda g_z \\ 0 &= L_\lambda = g(x,y,z) \end{aligned} \tag{6}$$

3 Examples

Solve the following problems:

$$\begin{aligned} &\min\,(2xy + 2yz + 2xz) \\ &\text{given } xyz - 1 = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} &\min\,(x + y - z)^2 \\ &\text{given } z - \ln(x) - 2\ln(y) = 0 \end{aligned} \tag{8}$$

$$\begin{aligned} &\min\left(x^2 + y^2 + z^2\right) \\ &\text{given } \frac{z}{(1 + x^2 + y^2)^{\frac{1}{2}}} - 1 = 0 \end{aligned} \tag{9}$$