21112 (Calculus 2) Lecture 18 - More Applications of Optimization

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April 2003

Today, we will explore more examples as we solidify our understanding of Multivariable Calculus and Optimization.

1 The Closest Point to a Cone

Imagine that we have a cone centred at the origin $(0,0,0)$ described by the equation $z = f(x,y) = \sqrt{(}x^2 + y^2)$. Now, if you are standing at the point $(x_1, y_1, z_1) = (1,1,0)$, you would like to know where the closest point on the cone is towards you. This can be found by finding the point (x_0, y_0, z_0) that minimizes the formula:

$$d(x,y,z) = \sqrt{(}(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2) \tag{1}$$

but we also have the constraints $(x_1, y_1, z_1) = (1,1,0)$ and $z = \sqrt{(}x^2+y^2)$, so we rewrite d as

$$d(x,y) = \sqrt{(}(x-1)^2 + (y-1)^2 + (\sqrt{(}x^2+y^2))^2) \tag{2}$$

Now, we will actually minimize the square of the distance, not the actual distane. We do this simply because it is equivalent to minimizing the distance itself. So, we wish to minimize

$$D(x,y) = (x-1)^2 + (y-1)^2 + x^2 + y^2 \tag{3}$$

As usual, we will take the first order partials:

$$D_x(x,y) = 2(x-1) + 2x \tag{4}$$

$$D_y(x,y) = 2(y-1) + 2y \tag{5}$$

and so our critical point is $(x_0, y_0) = (\frac{1}{2}, \frac{1}{2})$. Now, the second order partials are

$$D_{xx}(x,y) = 4 \tag{6}$$

$$D_{yy}(x,y) = 4 \tag{7}$$

$$D_{xy}(x,y) = 0 \tag{8}$$

and so our test is $D_{xx}(\frac{1}{2}, \frac{1}{2}) D_{yy}(\frac{1}{2}, \frac{1}{2}) - D_{xy}(\frac{1}{2}, \frac{1}{2})^2 = 4 * 4 - 0^2 = 16 > 0$ which means we have a minimum at $x = y = \frac{1}{2}$,and so $z = \sqrt{(}\frac{1}{2}^2 + \frac{1}{2}^2) = \frac{1}{\sqrt{(}2)}$.

2 Optimal Utility based Consumption

In economics and Finance, **Utility functions** $U(x, y)$ are devised to predict the consumption (i.e. purchasing habits) of a consumer in the market. These models have built into them the points that

1. $U_x(0, 0) = \infty = U_y(0, 0)$, i.e. some consumption is "infinitely" better than none
2. $U_x(\infty, \infty) = 0 = U_y(\infty, \infty)$, i.e. after a while, more and more consumption becomes boring to the consumer.

It should be easy to see that $U(x, y) = \ln(x^2 + y^2)$ satisfies these conditions, and so we can of course extend this to the 3 variable case, ie $U(x, y, z) = \ln(x^2 + y^2 + z^2)$. Now, if we say that (x, y, z) are three separate objects that she can consume, then if we add the constraint that in total, she has a fixed amount of 10 units to consume, i.e.

$$x + y + z = 10 \tag{9}$$

then our Utility function that we want to maximize is the rewritten as

$$U(x, y) = \ln(x^2 + y^2 + (10 - x - y)^2) \tag{10}$$

So, to maximize this, we first take the first order partials:

$$U_x(x, y) = \frac{2x - 2(10 - x - y)}{x^2 + y^2 + (10 - x - y)^2} \tag{11}$$

$$U_y(x, y) = \frac{2y - 2(10 - x - y)}{x^2 + y^2 + (10 - x - y)^2} \tag{12}$$

and so we need only that

$$2x - 20 + 2x + 2y = 0 \tag{13}$$

$$2y - 20 + 2x + 2y = 0 \tag{14}$$

simplifying, we obtain:

$$4x + 2y = 20 \tag{15}$$

$$2x + 4y = 20 \tag{16}$$

Upon solving, we get $x = y = \frac{10}{3}$ (and so $z = \frac{10}{3}$).

Now, in the privacy of your own home (and this is FOR HOMEWORK!), find the second partial derivatives and show that we indeed have a maximum.

3 Homework

1. Finish the proof of the second example
2. p.412 - 7
3. p.412 - 15
4. p.412 - 27