21112 (Calculus 2) Lecture 17 - Applications of Optimization

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Today, we will put into use our new skills in finding maxiumum/minimum values for functions. The real benefit of optimization is that once we have a real-life model, we can move on and find the optimal strategy to maximize profit or minimize energy expended. What we will do today is, in essence, the beginning of what you are all here to learn. Everything else in this section is just a work-up to today.

1 What is the Minimum Amount of Material needed to build a Box?

Imagine we need to ship 1000 cm^3 of material overseas, and we want to build the box to ship it in. As much as possible, we would like to minimize the amount of material needed to construct the box. (The objects we are sending can be easily re-packed given any dimensions of the box.) Upon drawing the box with dimensions x, y and z, we see that the function we wish to minimize is

$$S(x, y, z) = 2xy + 2xz + 2yz \tag{1}$$

However, this is a 3 variable problem, and we only know how to find Maximums and Minimums of 2 variable functions. All is not lost, as we see that we have a constraint to consider:

$$V = xyz = 1000 \tag{2}$$

or, in other words:

$$z = \frac{1000}{xy} \tag{3}$$

Substiting this last equation into expression for S, we obtain:

$$S(x, y) = 2xy + \frac{2000}{x} + \frac{2000}{y} \tag{4}$$

To begin, we find the first order partials to be:

$$S_x(x, y) = 2y - \frac{2000}{x^2} \tag{5}$$

$$S_y(x, y) = 2x - \frac{2000}{y^2} \tag{6}$$

Setting these above two expressions to 0, we obtain that

$$1000 = x^2y \tag{7}$$

$$1000 = xy^2 \tag{8}$$

So, dividing one equation by the other, we obtain that $x = y$ and then

$$1000 = x^3 = y^3 \tag{9}$$

which of course means that

$$1000^{\frac{1}{3}} = x = y \tag{10}$$

Now, the second order partials are such that

$$S_{xx}(x, y) = 2\frac{2000}{x^3} \tag{11}$$

$$S_{yy}(x, y) = 2\frac{2000}{y^3} \tag{12}$$

$$S_{xy}(x, y) = 2 \tag{13}$$

And so for any (x, y), our Second Order test value is

$$T(x, y) = 4\frac{2000^2}{x^3y^3} - 4 \tag{14}$$

so, to test our critical point, we have:

$$T(1000^{\frac{1}{3}}, 1000^{\frac{1}{3}}) = 4\frac{2000^2}{1000^2} - 4 = 4(4) - 4 = 12 > 0 \tag{15}$$

which means that our critical point corresponds to a minimum value of S as

$$S_{xx}(1000^{\frac{1}{3}}, 1000^{\frac{1}{3}}) = 2\frac{2000}{1000} = 4 > 0 \tag{16}$$

$$S_{yy}(1000^{\frac{1}{3}}, 1000^{\frac{1}{3}}) = 2\frac{2000}{1000} = 4 > 0 \tag{17}$$

2 Maximum Volume allowed by U.S. Postal Regulations

According to U.S. Postal regulations, the maximum girth allowed for a rectangular box to be shipped is 84, i.e.

$$84 = z + 2x + 2y \tag{18}$$

and we wish to maximize the volume:

$$V(x, y, z) = xyz \tag{19}$$

After substituting in the girth constraint, we get that we wish to maximize

$$V(x, y) = xy(84 - 2x - 2y) = 84xy - 2x^2y - 2xy^2 \tag{20}$$

To begin, the first order partials are

$$V_x(x, y) = 84y - 4xy - 2y^2 = y(84 - 4x - 2y) \tag{21}$$

$$V_y(x, y) = 84x - 2x^2 - 4xy = x(84 - 2x - 4y) \tag{22}$$

Now, it seems that there should be 4 possible critical points. However, if *either* $x = 0$ or $y = 0$, then we don't have a physicaly reasonable answer for a box (i.e. a box of a side length zero ?) So, we really have only one candidate, and that comes from solving the following:

$$2x + 4y = 84 = 4x + 2y \tag{23}$$

Once again, $y = x$ and so we get that

$$x = y = \frac{84}{6} \tag{24}$$

Now, the second partials are such that

$$V_{xx}(, y) = -4y \tag{25}$$

$$V_{yy}(x, y) = -4x \tag{26}$$

$$V_{xy}(x, y) = 84 - 4x - 4y \tag{27}$$

For our critical point,

$$V_{xx}(\frac{84}{6}, \frac{84}{6}) = -4\left(\frac{84}{6}\right) = -48 < 0 \tag{28}$$

$$V_{yy}(\frac{84}{6},\frac{84}{6}) = -4\left(\frac{84}{6}\right) = -48 < 0 \tag{29}$$

$$V_{xy}(\frac{84}{6},\frac{84}{6}) = 84 - 4x - 4y = 84 - 48 - 48 = -12 \tag{30}$$

and so the test reduces to

$$V_{xx}(\frac{84}{6},\frac{84}{6})V_{yy}(\frac{84}{6},\frac{84}{6}) - \left(V_{xy}(\frac{84}{6},\frac{84}{6})\right)^2 = (-48)(-48) - (-12)^2 > 0 \tag{31}$$

Therefore, our dimensions $x = \frac{84}{6}, y = \frac{84}{6}, z = \frac{84}{3}$ correspond to the maximum volume possible according to U.S. Postal regulations.