

21112 (Calculus 2) Lecture 16 - Finding Local Extrema of Multivariable Functions

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Picking up where we left off, a *necessary* condition for us to have the minimum or maximum of a multivariable function is that *all* of the first order partial derivatives must be zero. Upon computing these critical points, we can be sure that we must now prove that they indeed correspond to local maximum/minimum values. So, we will require a test to make sure that we have succeeded. For our purposes, we will only look at local extrema of functions of the form $f(x, y)$, although the following test can be extended to the general n -dimensional case.

1 What makes a Critical Point a Local Extrema ?

In 1 – d calculus, we make confirm that a critical point x_0 is in fact a local extrema when, if $f''(x_0)$ exists, and either

1. $f''(x_0) > 0$ (Local Minimum)
2. $f''(x_0) < 0$ (Local Maximum)
3. $f''(x_0) = 0$ (Saddle Point)

Now, we can use our intuition and extend this to the 2 variable case:

$$f_x(x_0, y_0) = 0 \tag{1}$$

$$f_y(x_0, y_0) = 0 \tag{2}$$

and

1. $f_{xx}(x_0, y_0) > 0, f_{yy}(x_0, y_0) > 0$ (Local Minimum)
2. $f_{xx}(x_0, y_0) < 0, f_{yy}(x_0, y_0) < 0$ (Local Maximum)
3. $f_{xx}(x_0, y_0) \geq 0, f_{yy}(x_0, y_0) \leq 0$ (Saddle Point)
4. $f_{xx}(x_0, y_0) \leq 0, f_{yy}(x_0, y_0) \geq 0$ (Saddle Point)

The last two cases certainly produce a saddle point. However, in the first two cases, it is not necessarily true that we indeed have a Minimum or Maximum. As an example, take

$$f(x, y) = x^2 + y^2 - 4xy \tag{3}$$

Then

$$f_x = 2x - 4y = 0 \tag{4}$$

$$f_y = 2y - 4x = 0 \tag{5}$$

implies that $(x_0, y_0) = (0, 0)$. It should be clear that

$$f_{xx}(0, 0) = 2 = f_{yy}(0, 0) \quad (6)$$

(In fact, $f_{xx}(x, y) = 2 = f_{yy}(x, y)$ for all (x, y) .)

But, if we graph this function, we see that we indeed have a saddle point at $(0, 0)$. So, how can we guarantee that we have a local minimum or maximum instead of a saddle point? The answer lies with the **Second Order Test for 2 – D functions**

2 Second Order Test for 2 – D functions

Theorem 1. *Assume that*

$$f_x(x_0, y_0) = 0 = f_y(x_0, y_0) \quad (7)$$

and $f_{xx}(x_0, y_0) > 0, f_{yy}(x_0, y_0) > 0$. If

$$f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0 \quad (8)$$

then we have a local Minimum at (x_0, y_0) . If

$$f_x(x_0, y_0) = 0 = f_y(x_0, y_0) \quad (9)$$

and $f_{xx}(x_0, y_0) < 0, f_{yy}(x_0, y_0) < 0$ and

$$f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0 \quad (10)$$

then we have a local Maximum at (x_0, y_0) .

If

$$f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 \leq 0 \quad (11)$$

then we have a Saddle Point at (x_0, y_0) .

3 Examples

3.1 $f(x, y) = x^2 + y^2 - 4xy$

Going back to the case of $f(x, y) = x^2 + y^2 - 4xy$, we see that, for all (x, y)

1. $f_{xx} = 2$
2. $f_{yy} = 2$
3. $f_{xy} = f_{yx} = -4$

so the 2nd Derivative Test shows us that we have a Saddle Point as $f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 2 * 2 - (-4)^2 = -12 < 0$.

3.2 A slightly strange example: $f(x, y) = xy - \ln(x) - \ln(y)$

In this case, we have that

$$f_x = y - \frac{1}{x} = 0 \tag{12}$$

$$f_y = x - \frac{1}{y} = 0 \tag{13}$$

so all we need for our critical point is that $(x_0, y_0) = (x_0, \frac{1}{x_0})$, $x_0 \neq 0$. So, for our Second Order test,

1. $f_{xx}(x_0, \frac{1}{x_0}) = \frac{1}{x_0^2}$
2. $f_{yy}(x_0, \frac{1}{x_0}) = \frac{1}{y_0^2} = \frac{1}{\frac{1}{x_0^2}} = x_0^2$
3. $f_{xy}(x_0, \frac{1}{x_0}) = f_{yx}(x_0, \frac{1}{x_0}) = 1$

$$\text{and so } f_{xx}(x_0, \frac{1}{x_0})f_{yy}(x_0, \frac{1}{x_0}) - f_{xy}(x_0, \frac{1}{x_0})^2 = \frac{1}{x_0^2}x_0^2 - (1)^2 = 0$$

So we in fact have a saddle point every where on this curve!

3.3 $f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - x - y$

In this example, we have that

$$f_x = x^2 - 1 = 0 \tag{14}$$

$$f_y = y^2 - 1 = 0 \tag{15}$$

so of course, we have 4 critical points to test out:

$$(x_0, y_0) = (1, 1), (1, -1), (-1, 1), (-1, -1) \quad (16)$$

Since

$$f_{xx} = 2x \quad (17)$$

$$f_{yy} = 2y \quad (18)$$

we see right away that $(1, -1)$ and $(-1, 1)$ are saddle points (as f_{xx}, f_{yy} differ in sign). Now, $(1, 1)$ is a candidate to be a local Minimum. For all (x, y) , we have that $f_{xy} = 0 = f_{yx}$, and so for $(1, 1)$,

$$f_{xx}(1, 1)f_{yy}(1, 1) - f_{xy}(1, 1)^2 = 2 * 2 - (0)^2 = 4 > 0 \quad (19)$$

and so we have a local Minimum. For $(-1, -1)$,

$$f_{xx}(-1, -1)f_{yy}(-1, -1) - f_{xy}(-1, -1)^2 = -2 * -2 - (0)^2 = 4 > 0 \quad (20)$$

and so we have a local Maximum.

4 Homework

Find and classify critical points (local Minimum, Local Maximum, or Saddle Point) for the following functions:

1. $f(x, y) = x^2y^2$
2. $f(x, y) = \sin(x) \sin(y)$
3. $f(x, y) = e^{x^2+y^2}$
4. $f(x, y) = (x - y)^2$