21112 (Calculus 2) Lecture 15 - First order conditions and the $n-th$ dimension

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We are getting ever closer to being able to find maxima and minima of multivariable functions. Soon, we will be able to take experimental data and fit a best-fit line (linear regression) or another curve (non-linear regression). We will be able to take an economic model and determine the exact amount of labour and capital we will need to maximize productivity and profit of a corporation. We will be able to design physical objects (with constraints) that require the least amount of material and have the most volume. All of this is coming, but we need to understand the notion of a critical point in mulitvariable calculus (and be able to compute it) before we can classify it as a maximum or minimum point. Onwards!

1 What is a critical point in Multivariable Calculus?

In traditional $1-d$ calculus, we have that a maximum or minimum value of the function $f(x)$ occurs at the point $x=c$ that has $f'(c)=0$. In the last two classes, we have mentioned that there are more than one type of derivative to take in the multi-variable case. However, at the heart of the matter, we are still taking $1-d$ derivatives, in this case of slices of the function surface. So, it is natural (and correct) to believe that on *any* slice, we should have the slope $=0$. But what does this mean? Well, it turns out that if we are dealing with a $2-d$ function $f(x,y)$, then it is enough of a **necessary** condition to have that

$$f_x(x_0,y_0)=0$$
$$f_y(x_0,y_0)=0$$

If this is true, we have a start. However, we will still need a second derivative test, albeit a more involved one, to be able to determine if we have a max, min, or neither. Let's begin:

2 Examples

Compute the critical points of the following functions, and then let's graph them on Maple:

1. $f(x, y) = x^2 + y^2$
2. $f(x, y) = x^2 - y^2$
3. $f(x, y) = y \ln(x) - x$
4. $f(x, y) = e^{xy}$

Sometimes, we need some tricks for solving systems of equations, such as in example 4 above. Case in point:

Compute the critical points of the function $f(x, y) = x^2 - y^2 + 2xy$.

How did you do this ?

More tricks as we do more problems!

3 $3 - d$ and up

Now, let's add some more spice to the pot: What if we have three input variables? Can we still define a derivative? Of course. For example, what if $f = f(x, y, z)$? Then, we hold the other 2 variables constant as we take derivatives.

3.1 Examples

Find $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$, and $\frac{\partial f}{\partial z} = f_z$ for the following:

1. $f(x, y, z) = x^2 + y^2 + z^2$
2. $f(x, y) = x^2 y^2 z^2$
3. $f(x, y) = y \ln(x) - +x \ln(z)$
4. $f(x, y) = e^{xyz}$

Question: What do the "level curves" look like for 1.)? Does it give you any intuition for an object in four dimensions?

Keep playing, and study for your midterm!

4 Homework

Compute the critical points for the following:

1. $f(x, y) = xy + \ln(xy)$

2. $f(x, y) = x + y - \frac{x^3}{3} - \frac{y^3}{3}$
3. $f(x, y, z) = xyz$
4. $f(x, y, z) = ze^{xyz}$