21112 (Calculus 2) Lecture 14 - 2nd Order Partial Differentiation

Albert Cohen

March 2003

Just as we saw in the $1-d$ case, once we have differentiated a function, there is nothing stopping us from doing so again! But now, we have more than one direction that we can take a derivative in. So, in this lecture, we extend the concept of a second derivative to $2-d$ functions.

1 $f_{xx}, f_{xy}, f_{yx}, f_{yy}$

Since there are two directions, or variables, we can traverse, there should be two types of second order derivatives, ie the x derivative of the x derivative of $f(x, y)$, and the y derivative of the y derivative of $f(x, y)$. However,we can also *mix* these second order partial derivatives, and in fact it turns out the mixed partials are equal. Let's write this out:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \tag{1}$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \tag{2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \tag{3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{yx} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \tag{4}$$

It turns out , though, that for all functions you will encounter,

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \tag{5}$$

Let's try out our new knowledge:

2 Examples

Find all possible derivatives of the following:

1. $f(x, y) = x^2 + y^2$

2. $f(x, y) = xe^y + ye^x$

3. $f(x, y) = \ln(x^2 + y^2 + 1)$

4. $f(x, y) = \sin(x)\sin(y)$

What do these $2nd$ order partials mean, intuitively? In the $1 - d$ case, the sign of $f''(x_0)$ told us whether the function was convex or concave at the point $x = x_0$. The $2nd$ order partials tell us basically the same thing: the sign of $f_{xx}(x_0, y_0)$ tells us about the concavity in the x variable, and the sign of $f_{yy}(x_0, y_0)$ tells us about the concavity in the y variable. This will be very useful, but surprisingly incomplete, when we attempt to find minima/maxima of $2 - d$ functions later on.

3 More Examples

Going back to our new concepts here, let's try out some more $2nd$ order partials, and evaluate them at the point $(1,1)$:

1. $f(p,q) = p\ln(q) + q^2$
2. $f(x,y) = x^2 + 2xy + y^2$
3. $f(x,y) = \tan(xe^y)$
4. $f(x,y) = \frac{x+y}{x-y}$

4 Homework

p.403-4: $23, 24, 35$