21112 (Calculus 2) Lecture 1 - Introduction to Integration

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January 13 2003

Today, you begin the second half of your single-variable calculus education. In the first half, you learned how to take a function and find it's derivative. Now, we will show you how to reverse this - namely, if given a function $f(x)$ can we find *a* function $g(x)$ such that $g'(x) = f(x)$? We will spend today's lecture, however, by looking at a specific definition of the integral. Just as we gave a formal definition of the derivative at a specific point $x = a$, *i.e.* $f'(a)$, we give the formal definition of the definite integral $\int_a^b f(x)dx$. So, sit straight and hold on - this will be a fun ride!

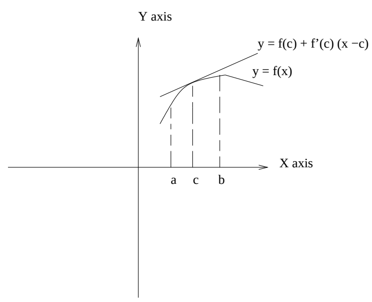

Figure 1: $f(x)$ and it's tangent line

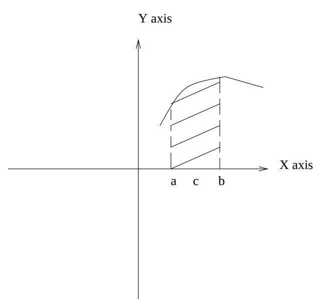

Figure 2: The area underneath $f(x)$ from $x = a$ to $x = b$.

Introduction

Visually, the derivative $f'(a)$ is the slope of the tangent line to the curve $f(x)$ at $x = a$. (See Figure 1.)

Now, another (related) geometric question that we could ask is what is the area underneath the curve from $x = a$ to $x = b$? (See Figure 2.)

If we are to interpret the y axis as velocity, and the x axis as time, then the physical interpretation is that $f'(c)$ is the instantaneous acceleration at time $= c$, whereas the area under the curve from time $= a$ to time $= b$ is the *total distance* traveled from time a to b. There are two questions that should pop into our minds:

1. When can we define such an area ? For example, should $f(x)$ be con-

tinuous, or even differentiable ? Should $f(x)$ be defined at every single point, or could we skip a (countable) few ? As a physical interpretation, if we knew the velocity vs. time curve $f(x)$ at every point x for our racecar except at a specific time $x = t_1$, could we still reconstruct the total distance traveled in the interval $x = a < t_1 < b$?

2. How do we calculate this area for a specific function $f(x)$?

It turns out that none of the conditions in 1.) are needed. Finding the area, or in mathematical terms ***integrating*** the function $f(x)$ from a to b is a very nice operation. In fact, it smooths out any discontinuity or kink, and in some sense fills in missing values. The way this area is calculated is fundamentally through a limit process. As a technical point which will be revealed later, we assume that $f(x) \geq 0$. Now, the area is calculated as follows: This is known as the **Riemann Sum definition of an Integral**

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_1^n f(x_i)\Delta x \tag{1}$$

with

$$\Delta x := \frac{b-a}{n} \tag{2}$$

$$a_i := a + i * \Delta x \tag{3}$$

$$x_i \in [a_{i-1}, a_i] \tag{4}$$

Picture-wise, this means that we approximate the area under the curve by the area of rectangles slightly above or below the curve. As the width of these rectangles gets smaller, the difference between this approximate area and the real area also gets smaller. Hence, if we take the limit of this process as the width $\Delta x \to 0$, we should have the real area. Notice that we only require $x_i \in [a_{i-1}, a_i]$. For most applications, especially software written for industry, the choices are usually $x_i = a_{i-1}, \frac{a_{i-1}+a_i}{2}$, or a_i.

Example 1

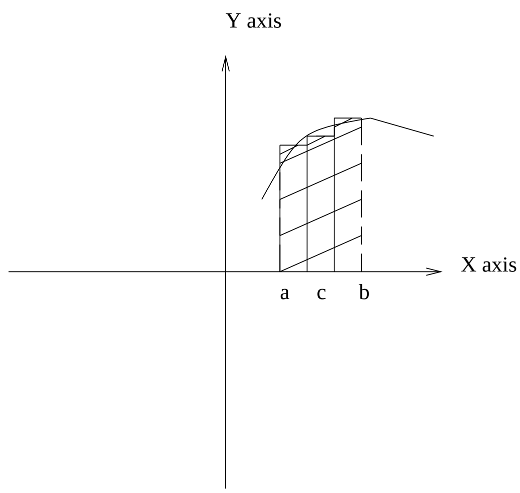

Figure 3: A Riemann Sum with $n = 3$. In your mind's eye, continue the subdivision and convince yourself that the area of the rectangles indeed does indeed approach that of the area underneath the curve.

A Riemann Sum approximation R_n is a sum of the form $R_n := \sum_1^n f(x_i)\Delta x$.

If we are to take a right angle triangle of side-length 1, then the area of such a triangle is $\frac{1}{2}$. So, correspondingly,

$$\int_0^1 x dx = \frac{1}{2} \tag{5}$$

Now, if we are to take a Riemann Sum approximation with

$$n = 4 \tag{6}$$
$$x_i = a_i \tag{7}$$

then we have that

$$\int_0^1 x dx \sim 0 * \frac{1}{4} + \frac{1}{4} * \frac{1}{4} + \frac{2}{4} * \frac{1}{4} + \frac{3}{4} * \frac{1}{4} = \frac{6}{16} = 0.375 \tag{8}$$

If we take $n = 8$, then we have

$$\int_0^1 xdx \sim 0*\frac{1}{8}+\frac{1}{8}*\frac{1}{8}+\frac{2}{8}*\frac{1}{8}+\frac{3}{8}*\frac{1}{8}+\frac{4}{8}*\frac{1}{8}+\frac{5}{8}*\frac{1}{8}+\frac{6}{8}*\frac{1}{8}+\frac{7}{8}*\frac{1}{8} = \frac{28}{64} = 0.4375 \tag{9}$$

in fact, the n^{th} Riemann Sum R_n is

$$R_n := \frac{\sum_1^{n-1} i}{n^2} = \frac{1}{2} - \frac{1}{2n} \tag{10}$$

Of course, as $n \to \infty$, we have that $\frac{1}{2} - \frac{1}{2n} \to \frac{1}{2}$ and so our Riemann Sum definition holds.

Example 2

This time, let's try

$$f(x) = x^2 \tag{11}$$

$$a = 0 \tag{12}$$

$$b = 1 \tag{13}$$

$$x_i = a_i = \frac{1 * i}{n} \tag{14}$$

and we shall now try to find the general Riemann Sum R_n.

$$\begin{aligned} R_n &:= \sum_1^n a_i^2 * \Delta x = \sum_1^n \left(\frac{i}{n}\right)^2 * \frac{1}{n} = \frac{1}{n^3}\sum_1^n i^2 = \frac{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}}{n^3} \\ &= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \to \frac{1}{3} \end{aligned} \tag{15}$$

Hence, we have shown that $\int_0^1 x^2 dx = \frac{1}{3}$.

Homework

Read Ch. 6.2
Work on questions $11, 12, 17, 21$ on $pp.330 - 331$