On graph irregularity strength

Alan Frieze*, Ronald J. Gould†, Michał Karoński†,‡  
and Florian Pfender†

Abstract

An assignment of positive integer weights to the edges of a simple graph $G$ is called irregular if the weighted degrees of the vertices are all different. The irregularity strength, $s(G)$, is the maximal weight, minimized over all irregular assignments. In this paper we show, that $s(G) \leq c_1 n/\delta$, for graphs with maximum degree $\Delta \leq n^{1/2}$ and minimum degree $\delta$, and $s(G) \leq c_2 (\log n)n/\delta$, for graphs with $\Delta > n^{1/2}$, where $c_1$ and $c_2$ are explicit constants. To prove the result, we are using a combination of deterministic and probabilistic techniques.

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*Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA, Research supported in part by NSF grant CCR-9818411, alan@random.math.cmu.edu
†Department of Mathematics and Computer Science, Emory University, Atlanta GA 30322, USA, rg@mathcs.emory.edu, miche@mathcs.emory.edu, fpfend@mathcs.emory.edu
‡Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, Poland, karonski@amu.edu.pl
1 Introduction:

Perhaps the second oldest "fact" in graph theory is that in a simple graph, two vertices must have the same degree. This fact no longer holds for multi-
graphs. By an irregular multigraph we mean one in which each vertex has
different degree. Hence, a natural question would be: What is the least
number of edges we would need to add to a graph in order to convert a simple
graph into an irregular multigraph?

Another way to view this question is via an assignment of integer weights
to the edges of the graph. Given a simple graph $G$ of order $n$, an assignment
$f : E(G) \to \{1, \ldots, w\} = [w]$ of positive integers weights to the edges of $G$ is
called irregular if the weighted degrees, $f(v) = \sum_{u \in N(v)} f(uv)$ of the vertices
are all different. The irregularity strength, $s(G)$, is the maximal weight $w$,
minimized over all irregular weight assignments, and is set to $\infty$ if no such
assignment is possible. Clearly, $s(G) < \infty$ if and only if $G$ contains no
isolated edges and at most one isolated vertex.

The irregularity strength was introduced in [3] by Chartrand et al. . The
irregularity strength of regular graphs was considered by Faudree and Lehel
in [4]. They showed that if $G$ is a $d$-regular graph of order $n$, $d \geq 2$, then
$s(G) \leq [n/2] + 9$, and they conjectured that $s(G) = \left[ \frac{n+d-1}{d} \right] + c$ for some
constant $c$. This conjecture comes from the lower bound $s(G) \geq \left[ \frac{n+d-1}{d} \right]$. For
general graphs with finite irregularity strength, Aigner and Trisch [1] showed
that $s(G) \leq n - 1$ if $G$ is connected and $s(G) \leq n + 1$ otherwise. Nierhoff
[8] refined their method to show $s(G) \leq n - 1$ holds for all graphs with
finite irregularity strength, except for $K_3$. We will provide an improvement
of both the Faudree-Lehel bound and the Aigner-Trisch-Nierhoff bound in
this paper.

For a review of other results and open problems in this area we refer the
reader to a survey paper by Lehel [7].

In this paper all graphs are simple of order $n$. The degree of a vertex $v$
is denoted by $d_v$ or $\text{deg}(v)$, we shall denote the minimum degree of $G$ by $\delta$
and the maximum degree by $\Delta$. For terms not found here see [2] or [6]. Our
upper bounds on $s(G)$ involve a function of $n$ and $\delta$ or both $\delta$ and $\Delta$, and
are stated in the next Theorem.
Theorem 1 Let $G$ be a graph with no isolated vertices or edges.

(a) If $\Delta \leq \lfloor (n/\ln n)^{1/4} \rfloor$, then $s(G) \leq 7n \left(\frac{1}{\delta} + \frac{1}{\Delta} \right)$.

(b) If $\lfloor (n/\ln n)^{1/4} \rfloor + 1 \leq \Delta \leq \lfloor n^{1/2} \rfloor$, then $s(G) \leq 60n/\delta$.

(c) If $\Delta \geq \lfloor n^{1/2} \rfloor + 1$, $\delta \geq \lceil 6 \log n \rceil$ then $s(G) \leq 336(\log n)n/\delta$.

For regular graphs, we get the following Theorem with improved constants.

Theorem 2 Let $G$ be a $d$-regular graph with no isolated vertices or edges.

(a) If $d \leq \lfloor (n/\ln n)^{1/4} \rfloor$, then $s(G) \leq 10n/d + 1$.

(b) If $\lfloor (n/\ln n)^{1/4} \rfloor + 1 \leq d \leq \lfloor n^{1/2} \rfloor$, then $s(G) \leq 48n/d + 1$.

(c) If $d \geq \lfloor n^{1/2} \rfloor + 1$, then $s(G) \leq 240(\log n)n/d + 1$.

Observe that both (a) and (b) give bounds of the correct order of magnitude. If $\Delta \geq \lfloor n^{1/2} \rfloor + 1$ and $\delta < \lfloor 6 \ln n \rfloor$, Theorem 1 does not apply, but we can still make the following statement:

Theorem 3 Let $G$ be a graph with no isolated vertices or edges. If $n$ is sufficiently large, then $s(G) \leq 14n/\delta^{1/2}$.

To explain the main technique used to prove all results let us define

$$m_g = \max_{X \subseteq V(G)} \{|X| : g(v) = g(u) \text{ for all } v, u \in X\},$$

where $g$ is defined as a weight assignment, i.e., $g : E(G) \to \{1, 2, \ldots, w\} = [w]$, for some integer $w$. In the deterministic part of our proof (see Lemma 4) we show that $s(G) \leq 3(w+1)m_g$. Next, we use probabilistic tools to establish bounds on $m_g$. Here the idea is to assign weights to edges from the set $\{1, 2\}$ or $\{1, 2, 3\}$, and show that for such weightings, there exist assignments with $m_g$ of the order $n/\delta$ or $n \log n/\delta$ (see Lemmas 7, 8 and 9).
2 Deterministic Lemmas

The next two Lemmas will be fundamental to our results. Their proofs follow below.

**Lemma 4** Let $G$ be a graph without isolated vertices or isolated edges. Let $g : E(G) \to [w]$ be a weight assignment. Then there exists an irregular assignment $f : E(G) \to \{2m_g, \ldots, (3w + 1)m_g\}$.

**Lemma 5** Let $G$ be a $d$-regular graph without isolated vertices or isolated edges. Let $g : E(G) \to [w]$ be a weight assignment. Then there exists an irregular assignment $f : E(G) \to [(3w - 1)m_g + 1]$.

We begin with a lemma needed to prove Lemma 4. We will call a tree with at most one vertex of degree greater than two, and $k$ vertices of degree one, a generalized $k$-star.

**Lemma 6** Let $G$ be a graph without isolated vertices or isolated edges. Then $G$ has a factor consisting of generalized stars of order at least three.

**Proof:** Let $T$ be a spanning tree of a component of $G$. Note that $|V(T)| \geq 3$ by our hypothesis. We show that $T$ can be broken into disjoint generalized stars that together span $V(T)$. Then repeating this argument on each component produces the result.

To do this we induct on $|U|$, where $U = \{u \in V(T) | \text{deg}_T(u) \geq 3\}$. If $|U| \leq 1$ we are done, as $T$ is itself a generalized star. Now assume the result holds on any tree $T$ with $|U| = l \geq 1$ and suppose $T$ is a tree with $|U| = l + 1$. Now root $T$ at $u \in U$ and select any vertex $v \in U$, $v \neq u$, such that the distance in $T$ between $u$ and $v$ is maximum over all vertices of $U$. Let $T_v$ be the subtree of $T$ rooted at $v$ and consider $T' = T \setminus T_v$. This tree has $|U| = l$ and by the induction hypothesis, we can find generalized stars in $T'$ that span $V(T')$. Further, the tree $T_v$ is, by our choice of $v$, a generalized star of order at least three. This star, together with the collection of stars that spans $T'$, spans $T$, completing the proof.

**Proof of Lemma 4.** Denote the weight class of a vertex $v \in V(G)$ as

$$C_v = \{u \in V(G) : g(u) = g(v)\}.$$
Define a new weight function \( \hat{f} : E \to [3m_g w] \) by \( \hat{f}(e) = 3m_g g(e) \). Note that the weight classes are unchanged under this function. Let \( S \) be a generalized star factor of \( G \), guaranteed by Lemma 6. We select one generalized star \( S \) from \( S \). Let \( u \) be a vertex of maximum degree in \( S \) and suppose that \( S \) consists of \( t \) paths rooted at \( u \). Let \( u_1, u_2, \ldots, u_t \) be the neighbors of \( u \) in \( S \). Consider the first branch (path) of \( S \), say \( v_1, v_2, \ldots, v_r \), where \( v_1 = u_1 \) and \( r \geq 2 \) (if such a branch of \( S \) exists). Now begin with the last edge \( v_r v_{r-1} \). We change the weight of this edge as follows. Put \( f(v_r v_{r-1}) = f(v_r v_{r-1}) + x \), where \( x \) is selected from the set \( L = \{0, -1, \ldots, -(m_g - 1)\} \) in such a way that \( f(v_r) \), its new weighted degree, is different from the current weighted degrees of any vertex from \( C_{v_r} \setminus \{v_r\} \). Since \( |C_{v_r}| \leq m_g \), it is always possible to select an appropriate \( x \). We now repeat this process to the edges \( v_{r-1} v_{r-2}, v_{r-2} v_{r-3}, \ldots, v_1 \), thus making \( f(v_{r-1}), f(v_{r-2}), \ldots, f(v_2) \) unique also. To complete the first phase, repeat the procedure on the paths emanating from \( u_2, u_3, \ldots, u_t \), in this order.

It remains to adjust the weights of the star centered at \( u \). So, we change the weights of the edges \( uu_1, uu_2, \ldots, uu_{t-1} \), one by one, starting at \( uu_1 \). Let \( f(uu_i) = \hat{f}(uu_i) + y_i \), where \( y_i \) is chosen from the set \( L' = \{-m_g, -(m_g - 1), \ldots, m_g - 1, m_g\} \), in such a way that \( f(u_i), i = 1, 2, \ldots, t-1 \), the new weighted degree of \( u_i \), is different from the current weighted degrees of any vertex from \( C_{u_i} \setminus \{u_i\} \) and, additionally, such that \( \sum_{k=1}^{t} y_i \) belongs to the set \( (L \cup \{-m_g\}) \setminus \{f(u_i v) - \hat{f}(uu_i)\} \), where \( v \) is the second vertex of the path starting in \( u_i \) (if no such vertex \( v \) exists, use instead \( (L \cup \{-m_g\}) \setminus \{0\} \)). Now we are left with \( uu_t \). Observe that \( u \) and \( u_t \) have different weighted degrees at this time. Now let \( f(uu_t) = \hat{f}(uu_t) + x \), where \( x \in L' \setminus \{-m_g\} \), such that both \( f(u) \) and \( f(u_t) \) are unique in their respective classes. This is possible, since there are \( 2m_g \) options, and \( C_u \) and \( C_{u_t} \) can only block \( 2(m_g - 1) \) of these. Finally, repeat the process for all remaining stars \( S \in S \).

Now for every weight class \( C_u \) all vertices have different weighted degrees under \( f \). The weighted degrees were altered from \( \hat{f} \) by total values from the range \( \{-2m_g + 1, \ldots, m_g\} \), the different classes were at least \( 3m_g \) apart from each other under \( \hat{f} \), so \( f \) is an irregular assignment to the set \( \{2m_g, 2m_g + 1, \ldots, 3m_g w + m_g\} \).

**Proof of Lemma 5.** Use Lemma 4 to get an irregular weight assignment \( f' : E(G) \to \{2m_g, 2m_g + 1, \ldots, 3m_g w + m_g\} \). Now define \( f : E(G) \to [(3w - 1)m_g + 1] \) by \( f(e) = f'(e) - 2m_g + 1 \). This assignment is irregular,
since the weighted degree of every vertex is reduced by $d(2m_g - 1)$.

3 Probabilistic Lemmas

The following two lemmas will be used to get bounds on the irregularity strength of graphs with maximal degree $\Delta \leq n^{1/2}$. Again, the proofs follow below.

**Lemma 7** Let $G$ be a graph. If $\Delta \leq (n/ \ln n)^{1/4}$, then $\exists g : E(G) \rightarrow \{1, 2\}$ such that $m_g \leq \frac{n}{3} + \frac{n}{\Delta}$.

**Lemma 8** Let $G$ be a graph. If $\Delta \leq n^{1/2}$, then $\exists g : E(G) \rightarrow \{1, 2, 3\}$ such that $m_g \leq 6n/\delta$.

The next lemma is used for graphs with $\Delta > n^{1/2}$.

**Lemma 9** Let $G$ be a graph. If $n \geq 10$ and $\delta \geq 10 \log n$, then $\exists g : E(G) \rightarrow \{1, 2\}$ such that $m_g \leq 48(\log n)n/\delta$.

Finally, we state the lemma which provides bounds on $m_g$, without any restrictions on vertex degrees of a graph $G$, but for sufficiently large $n$ only.

**Lemma 10** Let $G$ be a graph. If $n$ is sufficiently large, then $\exists g : E(G) \rightarrow \{1, 2\}$ such that $m_g \leq 2n/\delta^{1/2}$.

Since the proofs of both Lemma 7 and Lemma 9 use the same model of assigning weights to the edges, at random, we will present their proof together.

**Proof of Lemmas 7 and 9.**

Let $X_v, v \in V$ be independent random variables with uniform distribution over the interval $[0, 1]$, and then for $e = uv \in E$, let

$$g(e) = \begin{cases} 2 & \text{if } X_u + X_v \geq 1 \\ 1 & \text{if } X_u + X_v < 1 \end{cases}.$$

For the non-negative integer $y \in \{0, 1, \ldots, d_v\}$,

$$\Pr(g(v) = d_v + y) = \int_{x=0}^{1} \left( \frac{d_v}{y} \right) x^y (1 - x)^{d_v-y} dx = \frac{1}{d_v + 1} \leq \frac{1}{\delta + 1}. \quad (1)$$
It follows for every $y$ with $\delta \leq y \leq 2\Delta$ and $Z_y = |\{v \in V : g(v) = y\}|$ that
\[
E(Z_y) \leq \frac{n}{\delta + 1}.
\] (2)

To prove Lemma 7, we assume that $G$ is a graph with maximum degree $\Delta \leq (n/\log n)^{1/4}$.

We apply the Hoeffding-Azuma inequality, see e.g. Janson, Łuczak and Ruciński [6]. Changing the value of an $X_v$ can only change the value of $Z_y$ by at most $\Delta + 1$. It follows that for $t > 0$,
\[
Pr(Z_y \geq E(Z_y) + t) \leq \exp \left\{ -\frac{t^2}{2n(\Delta + 1)^2} \right\}.
\] (3)

Putting $t = \frac{n}{\Delta + 1}$ and using (2) we see that
\[
Pr(Z_y \geq E(Z_y) + t) < \frac{1}{2\Delta},
\]
and thus
\[
Pr(\exists y : Z_y \geq \frac{n}{\delta} + \frac{n}{\Delta}) < 1,
\]
and Lemma 7 follows.

We now prove Lemma 9. We use the Markov inequality for $t, k > 0$ and any event $\mathcal{E}$, to obtain
\[
Pr(Z_y > t \mid \mathcal{E}) \leq \frac{E \left( \left[ \frac{Z_y}{k} \right] \mid \mathcal{E} \right)}{t}.
\] (4)

But
\[
E \left( \left[ \frac{Z_y}{k} \right] \mid \mathcal{E} \right) = \sum_{|S|=k} Pr(g(v) = y, v \in S \mid \mathcal{E}).
\] (5)

Now fix $S = \{v_1, v_2, \ldots, v_k\}$ in (5). For $v \in S$ let $N_S(v) = N(v) \setminus S$, and let $\mu(v) = |N_S(v)|$. Note that $d_v - \mu(v) \leq k - 1$. For $v \in S$ let $\xi_1 < \xi_2 < \cdots < \xi_d_v$ be the values of $X_u, u \in N(v)$, sorted in increasing order and let $\eta_1 < \eta_2 < \cdots < \eta_{\mu(v)}$ be the values of $X_{u}, u \in N_S(v)$, also sorted in increasing order.
Note that, in general, if \( \xi_1 < \xi_2 < \cdots < \xi_s \) is the sequence of order statistic from the uniform distribution over \([0, 1]\), then \( \xi_i \) has the same distribution as \( (Y_1 + Y_2 + \cdots + Y_i)/(Y_1 + Y_2 + \cdots + Y_{s+1}) \) where \( Y_1, Y_2, \ldots, Y_{s+1} \) is a sequence of independent random variables, each having exponential distribution with mean one, see for example Ross, Theorem 2.3.1 [9].

To prove the lemma we need to show the following general statement.

**Lemma 11** Let \( Y_1, Y_2, \ldots, Y_s \) be a sequence of independent random variables, each having exponential distribution with mean one. Then for any real \( a > 0, \ 0 < b < 1 \) we have

\[
\Pr(Y_1 + \ldots + Y_s \geq (1 + a)s) \leq ((1 + a)e^{-a})^s
\]

\[
\Pr(Y_1 + \ldots + Y_s \leq (1 - b)s) \leq ((1 - b)e^b)^s.
\]

**Proof:**

\[
\Pr(Y_1 + \ldots + Y_s \geq t) \leq \Pr(e^{-\lambda(Y_1 + \ldots + Y_s - t)} \geq 1)
\]

\[
\leq e^{-\lambda t} \mathbb{E}(e^{\lambda(Y_1 + \ldots + Y_s)})
\]

\[
= \frac{e^{-\lambda t}}{(1 - \lambda)^s},
\]

provided \( \lambda \in (0, 1) \).

So putting \( t = (1 + a)s \), we see that

\[
\Pr(Y_1 + \ldots + Y_s \geq (1 + a)s) \leq \left( \frac{e^{-\lambda(1+a)} - \lambda}{1 - \lambda} \right)^s = ((1 + a)e^{-a})^s
\]

on putting \( \lambda = a/(1 + a) \).

A similar argument shows that

\[
\Pr(Y_1 + \ldots + Y_s \leq (1 - b)s) \leq ((1 - b)e^b)^s,
\]

completing the proof of Lemma 11.

Let \( k = \lfloor \log n \rfloor \) and

\[
\mathcal{E} = (\Theta < (16 \log n)/\delta),
\]

where

\[
\Theta = \max_{v \in V} \Theta_v, \quad \text{and} \quad \Theta_v = \max_{0 \leq t \leq d_k - 2k + 1} \xi_{i+2k} - \xi_i.
\]

Here, by default, we take \( \xi_0 = 0 \) and \( \xi_{d_k+1} = 1 \).
Now, observe that \( g(v) = y \) implies

\[
1 - X_v \in [\xi_{2d_n-y}, \xi_{2d_n-y+1}] \subset [\eta_{2d_n-y-k+1}, \eta_{2d_n-y+1}] \subseteq [\xi_{2d_n-y-k+1}, \xi_{2d_n-y+k}].
\]

In the above formula, we take \( \xi_j = \eta_j = 0 \) for \( j \leq 0 \), and \( \xi_{d_n+j} = \eta_{\nu(v)+j} = 1 \) for \( j \geq 1 \).

Applying Lemma 11 to the order statistics defining \( \Theta \), we see that

\[
\Pr(-\mathcal{E}) = \Pr \left( \exists v \in V : \Theta_v \geq \frac{16 \log n}{\delta} \right)
\leq n \Pr \left( \exists 0 \leq i \leq \Delta - 2k + 1 : \frac{Y_i + \cdots + Y_{i+2k-1}}{Y_1 + \cdots + Y_{\delta+1}} \geq \frac{16 \log n}{\delta} \right)
\leq n \Pr(Y_1 + \cdots + Y_{\delta+1} \leq \delta/2) + n^2 \Pr(Y_1 + \cdots + Y_{2k} \geq 8k)
\leq n(e^{1/2}/2)^{\delta+1} + n^2 (4e^{-3})^{2k}
\leq 1/10.
\]

(6)

Further,

\[
\Pr(g(v) = y, v \in S \mid \mathcal{E}) \leq \Pr(1 - X_{v_i} \in [\eta_{2d_n-y-k+1}, \eta_{2d_n-y+1}], i = 1, 2, \ldots, k \mid \mathcal{E})
\leq 2 \Pr(1 - X_{v_i} \in [\eta_{2d_n-y-k+1}, \eta_{2d_n-y-k+1} + \frac{16 \log n}{\delta}], i = 1, 2, \ldots, k)
\leq 2 \left( \frac{16 \log n}{\delta} \right)^k.
\]

From (4) and (5) we obtain

\[
\Pr(\exists y : Z_y > t \mid \mathcal{E}) \leq 2n \left( \frac{t}{k} \right)^{-1} \left( \frac{n}{k} \right) \left( \frac{16 \log n}{\delta} \right)^k.
\]

Putting \( t = 48(\log n)n\delta^{-1} \) together with (6) establishes

\[
\Pr(\exists y : Z_y > t) \leq \Pr(\exists y : Z_y > t \mid \mathcal{E}) + \Pr(-\mathcal{E}) < 1,
\]

proving Lemma 9.
**Proof of Lemma 8.** For every vertex \(v\) independently assign a number \(W_v\) from \(\{0, \ldots, d_v\}\) uniformly at random. Now pick a random subset \(N \subseteq N(v)\) of size \(W_v\), and for every \(u \in N\), set \(v_u = 1\), and for every \(u \in N(v) \setminus N\), set \(v_u = 0\).

Let \(g : E \rightarrow [3]\) as follows: For \(uv \in E\), let \(g(uv) = 1 + v_u + u_v\). For a vertex \(v\), let \(g(v) = \sum_{u \in N(v)} g(uv)\). For some integer \(\delta \leq y \leq 3\Delta\), let \(Z_y = |\{v \in V : g(v) = y\}|\). Then

\[
E(Z_y) \leq \frac{n}{\delta},
\]

since

\[
\Pr(g(v) = y) = \Pr(W_v = y - d - \sum_{u \in N(v)} u_v) \leq \frac{1}{d_v + 1}.
\]

By the symmetry of the construction we know that \(\forall x \in V, v, u \in N(x)\):

\[
\begin{align*}
\Pr(x_v = 1) &= 1/2, \\
\Pr(x_v = x_u = 1) &= \Pr(x_v = x_u = 0) = 1/3, \\
\Pr(x_v = 1, x_u = 0) &= \Pr(x_v = 0, x_u = 1) = 1/6.
\end{align*}
\]

To use Chebyshev's inequality, we have to bound the variance of \(Z_y\):

\[
\text{Var}(Z_y) = \sum_{v \in V} \sum_{u \in V} (\Pr(g(v) = g(u) = y) - \Pr(g(v) = y)\Pr(g(u) = y)).
\]

Fix a \(v \in V\), and consider

\[
S_v = \sum_{u \in V} (\Pr(g(v) = g(u) = y) - \Pr(g(v) = y)\Pr(g(u) = y)).
\]

Divide \(V\) into three classes \(V_1, V_2, V_3\), and consider the partial sums

\[
S_i = \sum_{u \in V_i} (\Pr(g(v) = g(u) = y) - \Pr(g(v) = y)\Pr(g(u) = y)).
\]

**Class 1:** \(V_1 = \{v\}\).

\[
S_1 \leq \Pr(g(v) = y) \leq \frac{1}{d_v} \leq \frac{\Delta}{\delta^2}.
\]
Class 2: $V_2 = N(v)$.

$$S_2 \leq d_v \Pr(g(v) = g(u) = y)$$

$$\leq d_v \Pr \left( W_v = y - d_v - \sum_{x \in N(v)} x_v \mid g(u) = y \right) \Pr \left( W_u = y - d_u - \sum_{x \in N(u)} x_u \right)$$

$$\leq \frac{d_v}{(d_v + 1)(d_u + 1)} < \frac{2 \Delta}{\delta^2}.$$  \hspace{1cm} (10)

Class 3: $V_3 = V \setminus \{v\} \cup N(v)$.

Let $u \in V_3$, and let $c = |N(v) \cap N(u)|$. For the sake of the analysis, pick a random subset $A$ from $\{x \in N(u) \cap N(v) : x_u = x_v\}$, by choosing each vertex with probability $1/2$. So, using (8), for every vertex $x \in N(u) \cap N(v)$,

$$\Pr(x_u = x_v = 1 \wedge x \in A) = \Pr(x_u = x_v = 1 \wedge x \notin A) =$$

$$\Pr(x_u = x_v = 0 \wedge x \in A) = \Pr(x_u = x_v = 0 \wedge x \notin A) =$$

$$\Pr(x_u = 0 \wedge x_v = 1) = \Pr(x_u = 1 \wedge x_v = 0) = 1/6,$$

and

$$\Pr(x \in A) = 1/3.$$  

Let $A \subseteq N(u) \cap N(v)$, and let $a = |A|$.

Then, for every vertex $x \in N(u) \cap N(v)$,

$$\Pr(x_u = x_v = 1 \mid A = A \wedge x \notin A) = \frac{\Pr(x_u = x_v = 1 \wedge A = A \mid x \notin A)}{\Pr(A = A \mid x \notin A)} =$$

$$= \frac{(1/6)(1/3)^a(2/3)^{c-a-1}}{(1/3)^a(2/3)^{c-a}} = \frac{1}{4}.$$  

By symmetry, we get

$$\Pr(x_u = x_v = 0 \mid A = A) = \Pr(x_u = 0, x_v = 1 \mid A = A) =$$

$$\Pr(x_u = 1, x_v = 0 \mid A = A) = 1/4.$$  

Thus, given $x \notin A$ and $A = A$, the events $(x_u = 1)$ and $(x_v = 1)$ are independent. For $x \in A$, we get

$$\Pr(x_u = x_v = 1 \mid A = A \land x \in A) = \Pr(x_u = x_v = 0 \mid A = A \land x \in A) = 1/2.$$
We introduce the following notation:

\[
P_A = \Pr(g(v) = g(w) = y \mid \mathcal{A} = A) - \Pr(g(v) = y \mid \mathcal{A} = A) \Pr(g(w) = y \mid \mathcal{A} = A)
\]

since \( \Pr(g(v) = y) \) is independent from the choice of \( \mathcal{A} \). In particular,

\[
P_\emptyset = \Pr(g(v) = g(w) = y \mid \mathcal{A} = \emptyset) - \Pr(g(v) = y) \Pr(g(w) = y) = 0.
\]

(11)

For \( A \neq \emptyset \), pick any \( x \in A \). We want to bound the difference \( P_A - P_{A \setminus x} \). Let

\[
b_v = d_v + \sum_{z \in N(v) \setminus x} z_v, b_u = d_u + \sum_{z \in N(u) \setminus x} z_u.
\]

Now consider the difference between \( P_A \) and \( P_{A \setminus x} \), given that \( b_v = l \) and \( b_u = r \), and denote it by

\[
P^l_A \setminus x - P^l_{A \setminus x} = \]

\[
= \Pr(g(v) = g(w) = y \mid \mathcal{A} = A \wedge b_v = l \wedge b_u = r)
\]

\[
- \Pr(g(v) = g(w) = y \mid \mathcal{A} = A \setminus x \wedge b_v = l \wedge b_u = r)
\]

\[
= \left[ \Pr(x_u = x_v = 1 \mid \mathcal{A} = A) - \Pr(x_u = x_v = 1 \mid \mathcal{A} = A \setminus x) \right]
\]

\[
\times \Pr(W_v = y - l - 1) \Pr(W_u = y - r - 1)
\]

\[
+ \left[ \Pr(x_u = x_v = 0 \mid \mathcal{A} = A) - \Pr(x_u = x_v = 0 \mid \mathcal{A} = A \setminus x) \right]
\]

\[
\times \Pr(W_v = y - l) \Pr(W_u = y - r)
\]

\[
+ \left[ \Pr(x_u = 1 \wedge x_v = 0 \mid \mathcal{A} = A) - \Pr(x_u = 1 \wedge x_v = 0 \mid \mathcal{A} = A \setminus x) \right]
\]

\[
\times \Pr(W_v = y - l) \Pr(W_u = y - r - 1)
\]

\[
+ \left[ \Pr(x_u = 0 \wedge x_v = 1 \mid \mathcal{A} = A) - \Pr(x_u = 0 \wedge x_v = 1 \mid \mathcal{A} = A \setminus x) \right]
\]

\[
\times \Pr(W_v = y - l - 1) \Pr(W_u = y - r)
\]

\[
= \frac{1}{4} \left[ \Pr(W_v = y - l - 1) \Pr(W_u = y - r - 1) + \Pr(W_v = y - l) \Pr(W_u = y - r) \right.
\]

\[
- \Pr(W_v = y - l) \Pr(W_u = y - r - 1) - \Pr(W_v = y - l - 1) \Pr(W_u = y - r)
\].

Therefore,

\[
P^l_A \setminus x - P^l_{A \setminus x} = \begin{cases} 
1/[4(d_v + 1)(d_u + 1)] & \text{if } (r = y - d_u - 1 \land l = y - d_v - 1) \\
& \text{or } (r = y \land l = y), \\
-1/[4(d_v + 1)(d_u + 1)] & \text{if } (r = y - d_u - 1 \land l = y) \\
& \text{or } (r = y \land l = y - d_v - 1), \\
0 & \text{otherwise.}
\end{cases}
\]
Thus, summing over all possible values of $l$, $r$ and $t = \{z \in A \mid x : z_u = z_v = 1\}$,

$$P_A = P_{A \setminus x} \leq \frac{1}{4(d_v + 1)(d_u + 1)} \times \left[ \Pr(b_u = y - d_u - 1 \land b_v = y - d_v - 1) + \Pr(b_u = y \land b_v = y) \right]$$

$$\leq \frac{1}{4(d_v + 1)(d_u + 1)} \times \left[ \sum_{t=0}^{a-1} \binom{a-1}{t} 2^{-a+1} \left( \frac{d_u - a}{y - 2d_u - 1 - t} \right) \left( \frac{d_v - a}{y - 2d_v - 1 - t} \right)^{2-d_u-d_v+2a} + \sum_{t=0}^{a-1} \binom{a-1}{t} 2^{-a+1} \left( \frac{d_u - a}{y - d_u - t} \right) \left( \frac{d_v - a}{y - d_v - t} \right)^{2-d_u-d_v+2a} \right]$$

$$\leq \frac{1}{(d_v + 1)(d_u + 1)} \left( \frac{d_u - a}{(d_u - a)/2} \right) \left( \frac{d_v - a}{(d_v - a)/2} \right)^{2-d_u-d_v+a} \sum_{t=0}^{a-1} \binom{a-1}{t} \left( \frac{a-1}{t} \right).$$

Suppose first that $1 \leq a \leq \delta/3$. Then,

$$P_A - P_{A \setminus x} \leq \frac{2^{2-a} - d_v - d_u + 2a - 1}{(d_v + 1)(d_u + 1)} \left( \frac{2d_u - a + 1}{(d_v - a)^{1/2}} \right)^{2} \left( \frac{2d_u - a + 1}{(d_u - a)^{1/2}} \right)^{2} \leq \frac{3}{d_v \delta^2}.$$

Hence,

$$P_A = \frac{3a}{d_v \delta^2} \leq \frac{3c}{d_v \delta^2}. \quad (12)$$

Note that for all $A$,

$$\Pr(g(v) = g(u) = y \mid A = A) \leq \frac{1}{(d_v + 1)(d_u + 1)},$$

hence, for $a > \delta/3$,

$$P_A \leq \Pr(g(v) = g(u) = y \mid A = A) \leq \frac{3a}{d_v \delta^2} \leq \frac{3c}{d_v \delta^2}. \quad (13)$$

Therefore, combining (11), (12) and (13),

$$\Pr(g(v) = g(u) = y) - \Pr(g(v) = y)\Pr(g(u) = y) \leq \sum_{A \subseteq N(v) \cap N(u)} (3c/d_v \delta^2) \Pr(A = A) = \frac{3|N(v) \cap N(u)|}{d_v \delta^2}.$$
Now notice that \( \sum_{u \in V} |N(v) \cap N(u)| \) counts the number of walks of length two starting in \( v \), thus \( \sum_{u \in V} |N(v) \cap N(u)| \leq d_v \Delta \), and therefore

\[
S_3 \leq \sum_{u \in V_3} \frac{3|N(v) \cap N(u)|}{d_v \delta^2} \leq \frac{3 \Delta}{\delta^2}. \tag{14}
\]

Altogether, we get from (9), (10) and (14),

\[
S_v = S_1 + S_2 + S_3 \leq \frac{6 \Delta}{\delta^2},
\]

and thus,

\[
\text{Var}(Z_v) = \sum_{v \in V} S_v \leq \frac{6n \Delta}{\delta^2}.
\]

By Chebyshev’s inequality and (7) we get

\[
\Pr(Z_v > 6n/\delta) \leq \frac{\text{Var}(Z_v)}{(5n/\delta)^2} < \frac{1}{3\Delta},
\]

and thus,

\[
\Pr(\exists y : Z_y > 6n/\delta) < 1,
\]

finishing the proof.

\[\phantom{1}\]

**Proof of Lemma 10.**

Choose \( g \) randomly from \( \{1, 2\}^E \). Observe that \( g(v) - d_v \) has the binomial distribution \( Bi(d_v, 1/2) \). For a non-negative integer \( y \) let

\[
V_y = \{v : |y - \frac{3}{2}d_v| \leq (2d_v \log n)^{1/2}\}.
\]

The Chernoff bounds for the tails of the binomial (see for example [6]) imply that for any \( t > 0 \),

\[
\Pr(|g(v) - \frac{3}{2}d_v| \geq t) \leq e^{-2t^2/d_v}.
\]

Hence,

\[
\Pr(g(v) = y) \leq \frac{1}{n^4} \quad \text{if } v \notin V_y. \tag{15}
\]
Now consider $v \in V_y$. Clearly,
\[
Pr(g(v) = y) = 0 \quad \text{if } d_v < y/2. \tag{16}
\]

**Case 1:** $y \geq n^{1/4}$

If $d_v \geq y/2 \geq n^{1/4}/2$ then we can use Stirling’s inequality or apply Feller [5], Chapter VII (2.7) to get
\[
Pr(g(v) = y) = \frac{1}{2^{d_v}} \left( \frac{d_v}{y - d_v} \right) \approx \sqrt{\frac{2}{\pi d_v}} e^{-z^2/2}, \tag{17}
\]
where $z = 2(y - \frac{3}{2}d_v)/d_v^{1/2}$.

Let $Z_y = |\{v : g(v) = y\}|$. It follows from (15), (16) and (17) that
\[
E(Z_y) \leq \frac{|V_y|}{\delta^{1/2}}. \tag{18}
\]

Let
\[Z_y^1 = |\{v \in V_y : g(v) = y\}| \text{ and } Z_y^2 = |\{v \notin V_y : g(v) = y\}|.
\]

It follows from (15) that
\[
Pr(Z_y^2 \neq 0) \leq \frac{1}{n^3}. \tag{19}
\]

Note also that $v \in V_y$ implies that
\[
y = \frac{3}{2}d_v + O \left( (d_v \log n)^{1/2} \right). \tag{20}
\]

Now for $t > 0$ and $k = (\log n)^2$ we use the Markov inequality to obtain
\[
Pr(Z_y^1 > t) \leq \frac{E \left( \binom{Z_y^1}{k} \right)}{\binom{t}{k}}. \tag{21}
\]

But
\[
E \left( \binom{Z_y^1}{k} \right) = \sum_{S \subseteq V_y, |S| = k} Pr(g(v) = y, v \in S)
\]
\[
= \sum_{S \subseteq V_y, |S| = k} \sum_{\xi \in \{1, 2\}} \Pr(g(v) = y, v \in S \mid g(S) = \xi) \Pr(g(E_S) = \xi) \tag{22}
\]

15
where $E_S = \{ e \in E : e \subseteq S \}$.

Now fix $S$ in (22). For $v \in S$ let

$$A_v = \{ e = uv \in E : u \notin S \} \text{ and } B_v = \{ e = uv \in E : u \in S \}.$$ 

Then, if $|g(B_v)|$ denotes $\sum_{u \in B_v} g(u)$,

$$\Pr( g(v) = y \mid g(E_S) = \xi) = \Pr( |g(A_v)| = y - |g(B_v)| )$$

$$= 2^{-|A_v|} \left( y - |g(B_v)| - |A_v| \right).$$

Therefore,

$$\frac{\Pr( |g(A_v)| = y - |g(B_v)| )}{\Pr( g(v) = y )} = 2^{|A_v|} \left( \frac{|A_v|}{y - |g(B_v)| - |A_v|} \right)$$

$$= 2^{|B_v|} |A_v| (|A_v| - 1) \cdots (2|A_v| + |g(B_v)| - y + 1)$$

$$\frac{1 \times 2 \times \cdots \times (y - |g(B_v)| - |A_v|)}{d_v (2d_v - y + 1) \cdots (|A_v| + 1)}$$

$$\frac{(y - d_v)(y - d_v - 1) \cdots (y - |g(B_v)| - |A_v| + 1)}{d_v (2d_v - y + 1) \cdots (|A_v| + 1)}$$

$$= \log^{|B_v|} \left( 1 + O \left( k \left( \frac{\log n}{d_v} \right)^{1/2} \right) \right)^{1/2}$$

$$= \left( \frac{1}{2} \right)^{|g(B_v)| - |B_v|} \cdot \left( \frac{1}{2} \right)^{|g(B_v)| - |A_v|}$$

$$\left( 1 + O \left( k \left( \frac{\log n}{d_v} \right)^{1/2} \right) \right)^{1/2}$$

$$= \left( \frac{1}{2} \right)^{|g(B_v)| - |B_v|} \cdot \left( \frac{1}{2} \right)^{|g(B_v)| - |A_v|}$$

$$\left( 1 + O \left( k \left( \frac{\log n}{d_v} \right)^{1/2} \right) \right)^{1/2}$$

Now we use

$$|A_v| + |B_v| = d_v \text{ and } |B_v| \leq |g(B_v)| \leq 2|B_v| \leq 2k$$

and (20) to verify that

$$\frac{1 \times 2 \times \cdots \times (y - d_v)}{1 \times 2 \times \cdots \times (y - |g(B_v)| - |A_v|)}$$

$$= \left( \frac{1}{2} \right)^{|g(B_v)| - |B_v|} \cdot \left( \frac{1}{2} \right)^{|g(B_v)| - |A_v|}$$

$$\left( 1 + O \left( k \left( \frac{\log n}{d_v} \right)^{1/2} \right) \right)^{1/2}$$

and

$$\frac{|A_v| (|A_v| - 1) \cdots (2|A_v| + |g(B_v)| - y + 1)}{d_v (2d_v - y + 1) \cdots (|A_v| + 1)}$$

$$= \left( \frac{1}{2} \right)^{|g(B_v)| - |B_v|} \cdot \left( \frac{1}{2} \right)^{|g(B_v)| - |A_v|}$$

$$\left( 1 + O \left( k \left( \frac{\log n}{d_v} \right)^{1/2} \right) \right)^{1/2}.$$
Plugging (25) and (26) into (24) we see that

\[
\frac{\Pr(|g(A_u)| = y - |g(B_v)|)}{\Pr(g(v) = y)} = 1 + O \left( k \left( \frac{\log n}{d_v} \right)^{1/2} \right).
\]

So from (22) and (23) we see that

\[
\mathbb{E} \left( \left( \frac{Z_y^1}{k} \right)^k \right) \leq \sum_{S \subseteq V_y, |S| = k} \sum_{\xi \in \{1, \ldots, k\}} \prod_{v \in S} \left( 1 + O \left( k \left( \frac{\log n}{d_v} \right)^{1/2} \right) \right) \Pr(g(v) = y) \Pr(g(E_S) = \xi)
\]
\[
\leq \left( 1 + O \left( k^2 \frac{\log n}{n^{1/8}} \right) \right) \sum_{S \subseteq V_y, |S| = k} \prod_{v \in S} \Pr(g(v) = y)
\]
\[
\leq (1 + o(1)) \frac{1}{k!} \left( \sum_{S \subseteq V_y, |S| = k} \Pr(g(v) = y) \right)^k
\]
\[
= (1 + o(1)) \frac{\mathbb{E}(Z_y^1)^k}{k!}.
\]

So (18), (21) imply

\[
\Pr \left( Z_y^1 > 2 \frac{n}{\delta^{1/2}} \right) \leq (1 + o(1)) \frac{\mathbb{E}(Z_y^1)^k}{(2n/\delta^{1/2})^k} \leq (1 + o(1))2^{-k}
\]

and then together with (19) we get

\[
\Pr \left( \exists y : Z_y > 2 \frac{n}{\delta^{1/2}} \right) \leq 2n((1 + o(1))2^{-k} + n^{-3}) = o(1). \tag{27}
\]

**Case 2:** \( y \leq n^{1/4} \).

Assume that \( V_y \neq \emptyset \). We apply the Hoeffding-Azuma inequality. Changing the value of \( g \) on a single edge can only change the value of \( Z_y^1 \) by at most 2. Also, \( Z_y^1 \) is determined by the outcome of at most

\[
\sum_{v \in V_y} d_v \leq |V_y|(y + (\log n)^2)
\]
random choices. It follows that for $t > 0$,

$$
\Pr(Z_y^1 \geq \mathbb{E}(Z_y^1) + t) \leq \exp \left\{ - \frac{t^2}{2|V_y| (y + (\log n)^2)} \right\}. \tag{28}
$$

Putting $t = n/\delta^{1/2}$ and observing that $V_y \neq \emptyset$ implies $\delta \leq n^{1/4}$ and $y\delta \leq n^{1/2}$, and applying (18), (19), (28), we see that

$$
\Pr \left( Z_y^1 > 2 \frac{n}{\delta^{1/2}} \right) \leq e^{-n^{1/2}/3}. \tag{29}
$$

The lemma follows from (19), (27) and (29).

\section{Proofs of Theorems}

We are now able to prove the Theorems.

\textbf{Proof of Theorem 1.} Let $\Delta \leq n^{1/2}$. By Lemma 8, there exists a weight assignment $g : E \rightarrow [w]$ with $m_g \leq 6n/\delta$ and $w = 3$. Now by Lemma 4, $s(G) \leq 3m_gw + m_g \leq 60n/\delta$, proving (b). Similar arguments, using Lemma 7 and Lemma 9 in place of Lemma 8, provide part (a) and (c).

\textbf{Proof of Theorem 2.} The proof is similar to the proof of Theorem 1, just use Lemma 5 in place of Lemma 4.

\textbf{Proof of Theorem 3.} The proof is similar to the proof of Theorem 1, just use Lemma 4 and Lemma 10.

\section*{References}


