A note on random minimum length spanning trees

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Abstract
Consider a connected r-regular n-vertex graph G with random independent edge lengths, each uniformly distributed on [0, 1]. Let mst(G) be the expected length of a minimum spanning tree. We show in this paper that if G is sufficiently highly edge connected then the expected length of a minimum spanning tree is ∼ \frac{n}{r} \zeta(3). If we omit the edge connectivity condition, then it is at most ∼ \frac{n}{r} (\zeta(3) + 1).

1 Introduction

Given a connected simple graph G = (V, E) with edge lengths x = (x_e : e ∈ E), let mst(G, x) denote the minimum length of a spanning tree. When X = (X_e : e ∈ E) is a family of independent random variables, each uniformly distributed on the interval [0, 1], denote the expected value E(mst(G, X)) by mst(G). Consider the complete graph Kn. It is known (see [2]) that, as n → ∞, mst(Kn) → ζ(3). Here ζ(3) = \sum_{j=1}^{∞} j^{-3} ∼ 1.202. Beveridge, Frieze and McDiarmid [1] proved two theorems that together generalise the previous results of [2], [3], [5].

Theorem 1 For any n-vertex connected graph G,

\[ mst(G) \geq \frac{n}{\Delta} (\zeta(3) - \epsilon_1) \]

where Δ = Δ(G) denotes the maximum degree in G and ε1 = ε1(Δ) → 0 as Δ → ∞.

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For an upper bound we need expansion properties of $G$.

**Theorem 2** Let $\alpha = \alpha(r) = O(r^{-1/3})$ and let $\rho = \rho(r)$ and $\omega = \omega(r)$ tend to infinity with $r$. Suppose that the graph $G = (V,E)$ is connected and satisfies

$$r \leq \delta \leq \Delta \leq (1 + \alpha)r,$$

where $\delta = \delta(G)$ denotes the minimum degree in $G$. Suppose also that

$$|(S : \bar{S})|/|S| \geq \omega r^{2/3} \log r$$

for all $S \subseteq V$ with $r/2 < |S| \leq \min\{\rho r, |V|/2\}$,

(2)

where $(S : \bar{S}) = \{(x, y) \in E : x \in S, y \in \bar{S} = E \setminus S\}$. Then

$$|\text{mst}(G) - \frac{n}{r} \zeta(3)| \leq \epsilon_2 \frac{n}{r}$$

where the $\epsilon_2 = \epsilon_2(r) \to 0$ as $r \to \infty$.

For regular graphs we of course take $\alpha = 0$.

The expansion condition in the above theorem is probably not the “right one” for obtaining $\text{mst}(G) \sim \frac{n}{r} \zeta(3)$. We conjecture that high edge connectivity is sufficient: Let $\lambda = \lambda(G)$ denote the edge connectivity of $G$.

**Conjecture 1**

Suppose that (1) holds. Then,

$$|\text{mst}(G) - \frac{n}{r} \zeta(3)| \leq \epsilon_3 \frac{n}{r}$$

where $\epsilon_3 = \epsilon_3(\lambda) \to 0$ as $\lambda \to \infty$.

Note that $\lambda \to \infty$ implies $r \to \infty$.

Along these lines, we prove the following theorem.

**Theorem 3** Assume $\alpha = \alpha(r) = O(r^{-1/3})$ and (1) is satisfied. Suppose that $r \geq \lambda(G) \geq \omega r^{2/3} \log n$ where $\omega = \omega(r)$ tends to infinity with $r$. Then

$$|\text{mst}(G) - \frac{n}{r} \zeta(3)| \leq \epsilon_4 \frac{n}{r}$$

where the $\epsilon_4 = \epsilon_4(r) \to 0$ as $r \to \infty$.

**Remark:** It is worth pointing out that it is not enough to have $r \to \infty$ in order to have the result of Theorem 2, that is, we need some extra condition such as high edge connectivity. For consider the graph $\Gamma(n, r)$ obtained from $n/r$ $r$-cliques $C_1, C_2, \ldots, C_{n/r}$ by deleting an edge $(x_i, y_i)$ from $C_i$, $1 \leq i \leq n/r$ then joining the cliques into a cycle of cliques by adding edges $(y_i, x_{i+1})$ for $1 \leq i \leq n/r$. It is not hard to see that

$$\text{mst}(\Gamma(n, r)) \sim \frac{n}{r} \left(\zeta(3) + \frac{1}{2}\right)$$

if $r \to \infty$ with $r = o(n)$. We repeat the conjecture from [1] that this is the worst-case, i.e.
Conjecture 2 Assuming only the conditions of Theorem 1,
\[ mst(G) \leq \frac{n}{\delta} \left( \zeta(3) + \frac{1}{2} + \epsilon_5 \right) \]
where \( \epsilon_5 = \epsilon_5(\delta) \to 0 \) as \( \delta \to \infty \).

We prove instead

**Theorem 4** If \( G \) is a connected graph then
\[ mst(G) \leq \frac{n}{\delta} (\zeta(3) + 1 + \epsilon_6) \]
where the \( \epsilon_6 = \epsilon_6(\delta) \to 0 \) as \( \delta \to \infty \).

We finally note that high connectivity is not necessary to obtain the result of Theorem 2. Since if \( r = o(n) \) then one can tolerate a few small cuts. For example, let \( G \) be a graph which satisfies the conditions of Theorem 2 and suppose \( r = o(n) \). Then taking 2 disjoint copies of \( G \) and adding a single edge joining them we obtain a graph \( G' \) for which \( mst(G') \sim \frac{1}{2} + \frac{n'}{r} \zeta(3) \sim \frac{n'}{r} \zeta(3) \) where \( n' = 2n \) is the number of vertices of \( G' \).

## 2 Proof of Theorem 3

Given a connected graph \( G = (V, E) \) with \( |V| = n \) and \( 0 \leq p \leq 1 \), let \( G_p \) be the random subgraph of \( G \) with the same vertex set which contains those edges \( e \) with \( X_e \leq p \). Let \( \kappa(G) \) denote the number of components of \( G \). We shall first give a rather precise description of \( mst(G) \).

**Lemma 1** [1]
For any connected graph \( G \),
\[ mst(G) = \int_{p=0}^{1} E(\kappa(G_p)) dp - 1. \]  \hspace{1cm} (3)

\[ \square \]

We substitute \( p = x/r \) in (3) to obtain
\[ mst(G) = \frac{1}{r} \int_{x=0}^{r} E(\kappa(G_{x/r})) dx - 1. \]

Now let \( C_{k,x} \) denote the total number of components in \( G_{x/r} \) with \( k \) vertices. Thus
\[ mst(G) = \frac{1}{r} \int_{x=0}^{r} \sum_{k=1}^{n} E(C_{k,x}) dx - 1. \]  \hspace{1cm} (4)
Proof of Theorem 3

In order to use (4) we need to consider three separate ranges for \( x \) and \( k \), two of which are satisfactorily dealt with in [1]. Let \( A = (r/\omega)^{1/3} \), \( B = [(Ar)^{1/4}] \) so that each of \( B\alpha \), \( AB^2/r \) and \( A/B \to 0 \) as \( r \to \infty \). These latter conditions are needed for the analysis of the first two ranges.

**Range 1:** \( 0 \leq x \leq A \) and \( 1 \leq k \leq B \) - see [1].

\[
\frac{1}{r} \int_{x=0}^{A} \sum_{k=1}^{B} E(C_{k,x})dx \leq (1 + o(1)) \frac{n}{r} \zeta(3).
\]

**Range 2:** \( 0 \leq x \leq A \) and \( k > B \) - see [1].

\[
\frac{1}{r} \int_{x=0}^{A} \sum_{k=B}^{n} E(C_{k,x})dx = o(n/r).
\]

**Range 3:** \( x \geq A \).

We use a result of Karger [4]. A cut \( (S : \overline{S}) = \{(u, v) \in E : u \in S, v \notin S\} \) of \( G \) is \( \gamma \)-minimal if \( |(S : \overline{S})| \leq \gamma \lambda \). Karger proved that the number of \( \gamma \)-minimal cuts is \( O(n^{2\gamma}) \). We can associate each component of \( G_p \) with a cut of \( G \). Thus

\[
\sum_{k=1}^{n} E(C_{k,x}) \leq O \left( \sum_{s=\lambda}^{\infty} n^{2s/\lambda} \left( 1 - \frac{x}{r} \right)^s \right) = O \left( \sum_{s=\lambda}^{\infty} (n^{2r/\lambda}e^{-x})^{s/r} \right)
= O \left( \int_{s=\lambda}^{\infty} (n^{2r/\lambda}e^{-x})^{s/r} ds \right) = O \left( \frac{r n^{2} e^{-x} \lambda/r}{x - \frac{2r}{\lambda} \log n} \right),
\]

and

\[
\frac{1}{r} \int_{x=A}^{r} \sum_{k=1}^{n} E(C_{k,x})dx = O \left( \int_{x=A}^{r} \frac{n^{2} e^{-x} \lambda/r}{x - \frac{2r}{\lambda} \log n} dx \right) = o \left( \frac{n^{2} e^{-A} \lambda/r}{r} \right) = o(n/r).
\]

We complete the proof by applying Lemma 1. \( \square \)

3 Proof of Theorem 4

We keep the definitions of \( A, B \) and Ranges 1,2, but we split Range 3 and let \( \delta = r \).

**Range 3a:** \( x \geq A \) and \( k \leq (1 - \epsilon)r \), \( 0 < \epsilon < 1 \), arbitrary - see [1] (here \( \epsilon = 1/2 \) but the argument works for arbitrary \( \epsilon \)).

\[
\frac{1}{r} \int_{x=A}^{r} \sum_{k=1}^{(1-\epsilon)r} E(C_{k,x})dx = o(n/r).
\]

**Range 3b:** \( x \geq A \) and \( k > (1 - \epsilon)r \).
Clearly
\[ \sum_{k=(1-\epsilon)r}^{n} C_{k,x} \leq \frac{n}{(1-\epsilon)r} \]
and hence
\[ \frac{1}{r} \int_{x=A}^{r} \sum_{k=(1-\epsilon)r}^{n} \mathbb{E}(C_{k,x}) \, dx \leq \frac{n}{(1-\epsilon)r}. \]

We again complete the proof by applying Lemma 1. \qed

References


