6

ON SMALL SUBGRAPHS OF RANDOM GRAPHS*

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6.1. INTRODUCTION

Let H be some fixed graph with $r \geq 3$ vertices and s edges. H is assumed to be *strictly balanced*, that is,

$$\frac{s}{r} > \frac{\mu(H')}{\nu(H')}$$

for all nontrivial subgraphs H' of H, $H' \neq H$, where $\nu(H')$ and $\mu(H')$ are the numbers of vertices and edges in H', respectively. (From now on $H' \subset H$ will always mean such subgraphs.) Note that this implies H is connected.

Consider now the random graph $G_{n,m}$ chosen uniformly from $\mathscr{G}_{n,m} = \{$graphs with vertex set $[n] = \{1, 2, \ldots, n\}$ and m edges$\}$ and let X_H denote the number of copies of H in $G_{n,m}$. Suppose now $m = \frac{1}{2}\omega n^{2-r/s}$, where $\omega = \omega(n)$. Erdős and Rényi (1960) showed that

$$\Pr(X_H = 0) = 1 - o(1), \quad \text{if } \omega \to 0,$$
$$\Pr(X_H \neq 0) = 1 - o(1), \quad \text{if } \omega \to \infty.$$

*Research supported in part by NSF grant CCR-89-00112.

Random Graphs, Volume 2. Edited by Alan Frieze and Tomasz Łuczak.
 ISBN: 0471-57292-6.

Here, as usual, we consider limits and so forth as $n \to \infty$. Using $a(n) \sim b(n)$ to stand for $a(n) = (1 - o(1))b(n)$, we remark that

$$E(X_H) \sim \frac{\omega^s}{\alpha} = \lambda, \quad \text{say,}$$

where α denotes the number of automorphisms of H.

Erdős and Rényi's result has been refined in many ways. In particular, Bollobás (1981) and Karoński and Ruciński (1983) independently showed that if ω tends to a constant and k is a fixed nonnegative integer, then

$$\Pr(X_H = k) \sim e^{-\lambda}\frac{\lambda^k}{k!}. \tag{6.1}$$

The aim of this paper is to show that the Poisson expression (6.1) is good for $\omega \to \infty$ sufficiently slowly. In particular we prove the following theorem.

Theorem 6.1. Let H be strictly balanced and λ be as previously defined. Then there exists a positive real constant $\theta = \theta(H)$ such that if $\omega \to \infty$ and $\omega = o(n^\theta)$, then

$$\Pr(X_H = k) \sim e^{-\lambda}\frac{\lambda^k}{k!} \quad \text{for all } 0 \le k \le (1 + \epsilon_1)\lambda, \tag{6.2}$$

where

$$\epsilon_1 = \frac{A_1(\log n)^{r/(2r-1)}}{\lambda^{(r-1)/(2r-1)}} \quad \text{for some constant } A_1 > 0.$$

$$\Pr(X = k) \gg e^{-\lambda}\frac{\lambda^k}{k!} \quad \text{for all } (1 + \epsilon_2)\lambda \le k \le \lambda \log n \tag{6.3}$$

where $\epsilon_2 = A_2(\log n/\lambda^{1-2/r})^{r/2(r-1)}$ for some constant $A_2 > 0$, provided $\epsilon_2 \le 1$. [The notation $a(n) \gg b(n)$ is used for $a(n)/b(n) \to \infty$.]

Remarks

1. We could also allow ω tends to a constant, but this is the well known case we are extending.
2. We are not able to obtain the largest possible values for $\theta(H)$ although we hope to refine our analysis for particular graphs, for

example, triangles. It is possible that the largest value coincides with that for Poisson convergence; see Ruciński (1988).

3. Observe that $\epsilon_1 \lambda \gg \lambda^{1/2}$ and so (6.2) is valid into the tails of the Poisson distribution.
4. A somewhat stronger result for $k = 0$ and $G_{n,p}$ has been proved independently by Boppana and Spencer (1989) and Janson, Łuczak, and Ruciński (1990). Janson (1991) has extended this result to estimate $\Pr(X_H \le k)$ for $k \le E(X_H)$. See also Suen (1991).
5. See Ruciński (1991) for a recent survey on the distribution of the number of copies of small subgraphs of random graphs.

6.2. PROOF OF THEOREM 6.1

We will not specify $\theta(H)$ immediately, but upper bounds for it will be derived along with the proof. We will use $A, A_1, A_2, \ldots$ to denote absolute constants whose values may or may not be explicitly stated. Throughout the paper, stated inequalities are only claimed to hold for n sufficiently large.

We distinguish between *isolated* copies of H and *nonisolated* copies. Here a copy of H in $G_{n,m}$ is isolated if it shares no edge with any other copy of H.

Now let

$$\pi_{k,l} = \Pr(G_{n,m} \text{ contains exactly } k \text{ isolated and } l \text{ nonisolated copies of } H)$$

and

$$q_l = \sum_{k=0}^{\infty} \pi_{k,l} = \Pr(G_{n,m} \text{ contains exactly } l \text{ nonisolated copies of } H),$$

$$p_k = \sum_{l=0}^{k} \pi_{k-l,l} = \Pr(G_{n,m} \text{ contains exactly } k \text{ copies of } H).$$

The main work involved in the proof is to justify the following inequalities:

$$n^{-A_3 l^{2/r}} \le q_l \le n^{-A_4 l^{1/r}}, \qquad 2 \le l \le \lambda_0 = \lfloor \lambda (\log n)^4 \rfloor, \quad (6.4)$$

$$\Pr(G_{n,m} \text{ contains at least } \lambda_0 \text{ isolated copies of } H) = o(e^{-\lambda_0}) \quad (6.5)$$

and more importantly

$$\frac{\pi_{k,l}}{\pi_{k-1,l}} = (1+\epsilon_{k,l})\frac{\lambda}{k}, \qquad 0 \le k-1, l \le \lambda_0, \tag{6.6}$$

where $|\epsilon_{k,l}| = o(\lambda_0^{-1})$.

We devote the remainder of this section to showing how our theorem follows from (6.4)–(6.6), and we prove these inequalities later on.

Suppose now that $0 \le l \le \lambda_0$. It follows from (6.6) that

$$\pi_{i,l} = (1+o(1))\pi_{0,l}\frac{\lambda^i}{i!}, \qquad 0 \le i \le \lambda_0, \tag{6.7}$$

and so

$$\begin{aligned} q_l &= (1+o(1))\pi_{0,l}\sum_{i=0}^{\lambda_0}\frac{\lambda^i}{i!} + \sum_{i>\lambda_0}\pi_{i,l} \\ &= (1+o(1))\pi_{0,l}(e^{\lambda} - o(e^{-\lambda_0})) + o(e^{-\lambda_0}) \end{aligned}$$

on using (6.5). Hence

$$\pi_{0,l} = (1+o(1))(q_l - o(e^{-\lambda_0}))e^{-\lambda}$$

and, by (6.7),

$$\pi_{i,l} = (1+o(1))q_l e^{-\lambda}\frac{\lambda^i}{i!} + o\left(\frac{\lambda^i}{i!}e^{-\lambda-\lambda_0}\right), \qquad 0 \le i \le \lambda_0.$$

Thus

$$p_k = (1+o(1))\sum_{l=0}^{k} q_l e^{-\lambda}\frac{\lambda^{k-l}}{(k-l)!} + o(e^{-\lambda_0}), \qquad 0 \le k \le \lambda_0.$$

Now

$$\begin{aligned} p_k &\ge (1+o(1))q_k e^{-\lambda} + o(e^{-\lambda_0}) \\ &\ge n^{-A_3(\lambda_0)^{2/r}}e^{-\lambda} + o(e^{-\lambda_0}) \gg e^{-\lambda_0}, \quad \text{since } r \ge 3, \end{aligned}$$

and so

$$p_k \sim \sum_{l=0}^{k} q_l e^{-\lambda} \frac{\lambda^{k-l}}{(k-l)!} \qquad (0 \le k \le \lambda_0)$$

$$= e^{-\lambda} \frac{\lambda^k}{k!}\left(q_0 + \sum_{l=2}^{k} \frac{(k)_l}{\lambda^l} q_l\right), \tag{6.8}$$

where $(k)_l = k(k-1)\cdots(k-l+1)$.

To proceed from here we need to show $q_0 = 1 - o(1)$. Assume this for the moment so that we can verify Theorem 6.1. With this done we will prove $q_0 = 1 - o(1)$.

Suppose first that $0 \le k \le \lambda$. Then for θ sufficiently small,

$$1 - o(1) \le q_0 + \sum_{l=2}^{k} \frac{(k)_l}{\lambda^l} q_l \le q_0 + \sum_{l=2}^{k} q_l \le 1. \tag{6.9}$$

Now let $k = (1+\epsilon)\lambda$ where $0 \le \epsilon \le \epsilon_1 = A_1(\log n)^{r/(2r-1)}/\lambda^{(r-1)/(2r-1)}$. Then, using (6.4),

$$u_l = \frac{(k)_l}{\lambda^l} q_l \le 2\left(\frac{k}{\lambda}\right)^l e^{-l^2/2k} n^{-A_4 l^{1/r}}$$

$$\le 2\exp\left\{\epsilon l - \frac{l^2}{2k} - A_4 l^{1/r}\log n\right\}.$$

Case 1

$l \ge 3\epsilon\lambda$ (and hence $\epsilon \le \frac{1}{2}$).

$$u_l \le 2n^{-A_4 l^{1/r}}$$

Case 2

$l < 3\epsilon\lambda$.

$$u_l \le 2\exp\{l^{1/r}(\epsilon l^{1-1/r} - A_4\log n)\}$$

$$\le 2\exp\{l^{1/r}(3^{1-1/r}\epsilon^{2-1/r}\lambda^{1-1/r} - A_4\log n)\}$$

$$\le 2\exp\{l^{1/r}\log n(3^{1-1/r}A_1^{2-1/r} - A_4)\}.$$

So if we make A_1 small enough so that $A_4 \geq 4A_1^2$, then we have

$$u_l \leq 2n^{-A_1^2 l^{1/r}},$$

which is also valid for Case 1.

Hence if $\lambda \leq k \leq (1+\epsilon_1)\lambda$ and θ is sufficiently small

$$\begin{aligned} 1 - o(1) \leq q_0 + \sum_{l=2}^{k} \frac{(k)_l}{\lambda^l} q_l &\leq 1 + 2\sum_{l=2}^{\infty} n^{-A_1^2 l^{1/r}} \\ &= 1 + o(1). \end{aligned}$$

This together with (6.9) proves the first part of the theorem.

Suppose now that $k = (1+\epsilon)\lambda$ where

$$1 \geq \epsilon \geq \epsilon_2 = A_2\left(\log n/\lambda^{1-2/r}\right)^{r/2(r-1)}.$$

Then by (6.8),

$$\begin{aligned} p_k \Big/ \left(\frac{e^{-\lambda}\lambda^k}{k!}\right) &\geq (1-o(1))\frac{k!}{\lambda^{k-\lfloor\lambda\rfloor}\lfloor\lambda\rfloor!} q_{k-\lfloor\lambda\rfloor} \\ &\geq A\left(\frac{k}{e\lambda}\right)^k e^{\lambda} n^{-A_3(\epsilon\lambda+1)^{2/r}} \\ &\geq A e^{\epsilon^2\lambda/3} n^{-2A_3(\epsilon\lambda)^{2/r}} \\ &= A\exp\left\{\frac{\epsilon^2\lambda}{3}\left(1 - 6A_3\epsilon^{2/r-2}\lambda^{2/r-1}\log n\right)\right\} \\ &\geq A\exp\left\{\frac{\epsilon^2\lambda}{3}\left(1 - 6A_3A_2^{2/r-2}\right)\right\}. \end{aligned}$$

Now $\epsilon^2\lambda \to \infty$ and we are free to choose A_2 so that

$$1 - 6A_3A_2^{2/r-2} = \tfrac{1}{2}$$

and the result is proved for this case.

When $k \geq 2\lambda$ we use

$$\frac{(k+1)!}{\lambda^s(k+1-s)!} q_s \geq \frac{k!}{\lambda^s(k-s)!} q_s$$

to reduce to the previous case.

We of course have to prove that $q_0 = 1 - o(1)$. To prove this we need a lemma on the edge density of intersecting copies of H. We need a general version of this to prove (6.4) and we prove this here. Let

$$\theta_1 = \min_{H' \subset H} \left(\frac{2s - \mu(H')}{2r - \nu(H')} \right) - \frac{s}{r} > 0.$$

Note that $\theta_1 > 0$ follows from the fact that H is strictly balanced. A collection $H_1, H_2, \dots, H_k$ of copies of H in $G_{n,m}$ is said to be *linked* if for each i there is $j \neq i$ such that H_i, H_j share an edge.

Lemma 6.1. Let $H_1, H_2, \dots, H_k$, $k \geq 2$, be a linked collection of copies of H. Let $K = \bigcup_{i=1}^{k} H_i$. Then

$$\mu(K) \geq \left(\theta_1 + \frac{s}{r} \right) \nu(K).$$

Proof. Assume w.l.o.g. that $H_i \not\subseteq \bigcup_{j \neq i} H_j$ for $i = 1, 2, \dots, k$. We prove the result by induction on k. We discuss the base case and the inductive step in tandem. Let $K' = \bigcup_{i=1}^{k-1} H_i$. Then

$$\frac{\mu(K)}{\nu(K)} = \frac{\mu(H_k) + \mu(K') - |E(H_k) \cap E(K')|}{\nu(H_k) + \nu(K') - |V(H_k) \cap V(K')|}. \tag{6.10}$$

Furthermore,

$$uv \in E(H_k) \cap E(K') \to u, v \in V(H_k) \cap V(K')$$

and so if $H' = (V(H_k) \cap V(K'),\ E(H_k) \cap E(K'))$, then H' is a nontrivial proper subgraph of H and, by (6.10),

$$\frac{\mu(K)}{\nu(K)} = \frac{s + \mu(K') - \mu(H')}{r + \nu(K') - \nu(H')}.$$

Base case: $k = 2$

Here $K' = H_2$ and $\mu(K)/\nu(K) \geq \theta_1 + s/r$ follows from the definition of θ_1.

Inductive step

Write

$$\frac{\mu(K)}{\nu(K)} = \frac{2s - \mu(H') + (\mu(K') - s)}{2r - \nu(H') + (\nu(K') - r)}$$

and observe that

$$(\mu(K') - s) - \left(\theta_1 + \frac{s}{r}\right)(\nu(K') - r)$$
$$= \left(\mu(K') - \left(\theta_1 + \frac{s}{r}\right)\nu(K')\right) + r\theta_1 > 0$$

by induction. □

It is always more pleasant to do computation in the independent model $G_{n,p}$, $p = m/N$, $N = \binom{n}{2}$. We quote the following simple results [see Bollobás (1981), Section 1.1]. Let $\mathscr{A}$ be any property of graphs. Then

$$\Pr(G_{n,m} \in \mathscr{A}) \leq 3m^{1/2} \Pr(G_{n,p} \in \mathscr{A}) \tag{6.11}$$

and if $\mathscr{A}$ is monotone, then

$$\text{a.e. } G_{n,p} \in \mathscr{A} \rightarrow \text{a.e. } G_{n,m} \in \mathscr{A}. \tag{6.12}$$

Lemma 6.2. If

$$\theta < \theta_1 r^2/(s^2 + \theta_1 rs), \tag{6.13}$$

then $q_0 = 1 - o(1)$.

Proof. If $G_{n,m}$ has a pair of edge intersecting copies of H, then it contains a set of $r \leq k \leq 2r - 1$ vertices which span at least $\lceil k(s/r + \theta_1)\rceil$ edges. Now this property is monotone and

$$\Pr(G_{n,p} \text{ contains a pair of edge intersecting copies of } H)$$
$$\leq \sum_{k=r}^{2r-1} \binom{n}{k} 2^{\binom{k}{2}} p^{k(s/r+\theta_1)}$$
$$\leq \sum_{k=r}^{2r-1} 2^{\binom{k}{2}} \omega^{k(s/r+\theta_1)} n^{-k\theta_1 r/s}$$
$$= o(1).$$

Now use (6.12). □

6.3. PROOF OF (6.4) AND (6.5)

The upper bound in (6.4) follows fairly easily from Lemma 6.1. Indeed suppose $G_{n,m}$ contains exactly l nonisolated copies of H. Let K denote the graph induced by the union of these copies. If K has ρ vertices then, by Lemma 6.1, it has at least $\tau\rho$ edges where $\tau = \theta_1 + s/r$. Note that

$$l^{1/r} \le \rho \le rl \le r\lambda_0,$$

where the lower bound on ρ is from $(\rho)_r \ge l$. Hence, on using (6.11),

$$\begin{aligned} q_l &\le 3m^{1/2} \sum_{\rho = l^{1/r}}^{rl} \binom{n}{\rho} \binom{\binom{\rho}{2}}{\tau\rho} p^{\tau\rho} \\ &\le 3m^{1/2} \sum_{\rho = l^{1/r}}^{rl} \left(\frac{ne}{\rho}\right)^{\rho} \left(\frac{\rho^2 ep}{2\tau\rho}\right)^{\tau\rho} \\ &\le 3m^{1/2} \sum_{\rho = l^{1/r}}^{rl} \left(\frac{A\rho^{(\tau-1)^+}\omega^{\tau}}{n^{\tau r/s - 1}}\right)^{\rho} \quad \left[(\tau-1)^+ = \max\{0, \tau-1\}\right] \\ &\le 3m^{1/2} \sum_{\rho = l^{1/r}}^{rl} \left(\frac{A'\omega^{s(\tau-1)^+ + \tau}(\log n)^{4(\tau-1)^+}}{n^{r\theta_1/s}}\right)^{\rho} \end{aligned} \tag{6.14}$$

and the upper bound in (6.4) follows provided l is sufficiently large and

$$\theta\left(s(\tau-1)^+ + \tau\right) < r\theta_1/s.$$

For small l one can use the proof of Lemma 6.2.

It is convenient to stop and prove a similar inequality which is needed later. Let $\lambda_1 = \lfloor \omega^{rs}(\log n)^{4r+1} \rfloor$. It follows from (6.14) that provided

$$\theta\left(rs(\tau-1)^+ + \tau\right) < r\theta_1/s, \tag{6.15}$$

$$\sum_{l=\lambda_1}^{2\lambda_1} q_l' = o(e^{-2\lambda_0}), \tag{6.16}$$

where q_l' is the probability that $G_{n,2m}$ contains precisely l nonisolated copies. Furthermore, if $G_{n,2m}$ contains more than $2\lambda_1$ nonisolated copies of H, then we can choose λ_1 of them. For each chosen copy of H that does not share an edge with another chosen copy we choose a further copy that does share an edge. In this way we build a linked collection of between λ_1 and $2\lambda_1$ copies. It then follows by (6.16) that

$$\sum_{l=2\lambda_1+1}^{\infty} q_l' = o(e^{-2\lambda_0}), \quad \text{also.} \tag{6.17}$$

To prove the lower bound of (6.4) we consider the probability of the existence of a collection of disjoint complete subgraphs of specific sizes. Thus let $\sigma_t = \binom{t}{r} r!/\alpha$ for $t \ge r$ and observe that K_t contains σ_t distinct copies of H. For a given a define $\tau = \tau(a)$ by $\sigma_{\tau+1} > a \ge \sigma_\tau$. Next let $l_1 = l$ and $l_{i+1} = l_i - \sigma_{\tau(l_i)}$ and $T_i = \sum_{j=1}^{i} \tau(l_j)$ for $i = 1, 2, \dots, k$, where $l_k \ge (r+1)!/\alpha > l_{k+1}$.

Now let $\mathscr{E}$ denote the event that

$G_{n,m}$ contains complete subgraphs with vertex set $[T_1], [T_2] \setminus [T_1], \dots, [T_k] \setminus [T_{k-1}]$ (6.18a)

and

l_{k+1} copies of H containing the edge $\{1, 2\}$ but otherwise disjoint from all other copies. We assume some single choice among the many possibilities for our choice of l_{k+1} possibilities. Let their vertices belong to $[T] \setminus [T_k]$ where $T - T_k = (r-2)l_{k+1}$ (6.18b)

and

there are no other edges in $[T]$ (this assumption simplifies the calculations but may be a bit drastic!) (6.18c)

and

there are no other nonisolated copies of H is $G_{n,m}$. (6.19)

Thus if $\mathscr{E}$ occurs, then $G_{n,m}$ contains exactly l nonisolated copies of H. We can write

$$\Pr(\mathscr{E}) = \pi_1 \pi_2,$$

where

$$\pi_1 = \Pr((6.18)) \quad \text{and} \quad \pi_2 = \Pr((6.19)|(6.18)).$$

But

$$\pi_1 = \binom{N - \binom{T}{2}}{m-u} \Big/ \binom{N}{m} = \left(\frac{m}{N}\right)^u \left(1 - O\left(\frac{mT^2}{N} + \frac{u^2}{m}\right)\right),$$

where

$$u = \sum_{i=1}^{k} \binom{\tau(l_i)}{2} + (s-1)l_{k+1}.$$

So

$$\pi_1 = \left(\frac{\omega}{n^{r/s}}\right)^u \left(1 - O\left(\frac{mT^2}{N} + \frac{u^2}{m} + \frac{u}{n}\right)\right)$$

$$= \left(\frac{\omega}{n^{r/s}}\right)^u (1 - o(1)), \tag{6.20}$$

since we show later that

$$\sum_{i=1}^{k} \tau(l_i)^x = O(l^{x/r}) \quad \text{for any fixed positive integer } x, \tag{6.21}$$

and we assume

$$\theta < \min\left\{\frac{r(2s-r)}{4s^2}, \frac{r^2}{2s^2}\right\}. \tag{6.22}$$

We show next that $\pi_2 = 1 - o(1)$. Note that (6.19) given (6.18) is monotone and so we can use the $G_{n,p}$ model to estimate π_2. Now by the FKG inequality

$$\pi_2 \geq \pi_2' \pi_2'',$$

where

$\pi_2' = \Pr($there are no nonisolated copies of H which have no edge in $[T])$

and

$$\pi_2'' = \Pr(\text{there are no extra copies of } H \text{ which share an edge with those defined in (6.18)}).$$

Now $\pi_2' = 1 - o(1)$ if (6.13) holds and

$$\begin{aligned}\pi_2'' &\geq 1 - E\,(\text{number of such copies of } H)\\ &\geq 1 - \sum_{H' \subset H} (n)_{r-\nu(H')} \binom{\binom{r}{2}}{s-\mu(H')} p^{s-\mu(H')} \left(\sum_{i=1}^{k} (\tau(l_i))_{\nu(H')} + O(1) \right)\\ &= 1 - O\left(\sum_{H' \subset H} n^{r-\nu(H')} \frac{\omega^{s-\mu(H')}}{n^{r-(r/s)\mu(H')}} l^{\nu(H')/r} \right)\end{aligned}$$

on using (6.21) to simplify the second summation

$$= 1 - o(1)$$

provided

$$\theta < \min_{H' \subset H} \frac{\nu(H') - (r/s)\mu(H')}{s - \mu(H') + \nu(H')(s/r)}. \tag{6.23}$$

The proof of (6.4) is completed once we have proved (6.21). For then (6.20) implies

$$\pi_1 \geq \left(\frac{\omega}{n^{r/s}} \right)^{O(l^{2/r})} (1 - o(1)).$$

Proof of (6.21). When $a \geq \sigma_r = r!/\alpha$ is large we have, where $\tau = \tau(a)$,

$$\begin{aligned} a - \sigma_\tau &\leq \sigma_{\tau+1} - \sigma_\tau\\ &= r(\tau)_{r-1}\alpha^{-1}\\ &< r\tau^{r-1}.\end{aligned}$$

But

$$\begin{aligned} a \geq \sigma_\tau &\rightarrow \binom{\tau}{r} \leq a\\ &\rightarrow \left(\frac{\tau}{r}\right)^r \leq a\\ &\rightarrow \tau \leq ra^{1/r} \end{aligned} \tag{6.24}$$

and so

$$a - \sigma_\tau \le r^r a^{1-1/r}.$$

Recalling that $l_1 = l$ and $l_{i+1} = l_i - \sigma_{\tau(l_i)}$ we see that

$$l_{i+1} \le r^{ir} l^{(1-1/r)^i}, \qquad 1 \le i \le k. \tag{6.25}$$

Now let $i_0 = \lceil r \log r \rceil$ and assume first that $i_0 \le k$ (6.21). Then (6.25) implies

$$l_{i_0} \le A l^{1/r}, \tag{6.26}$$

where $A = r^{i_0 r}$.

Now $\tau(l_1) \le r l^{1/r}$ and τ is monotone increasing and so

$$\sum_{i=1}^{i_0} \tau(l_i)^x \le i_0 r^x l^{x/r}. \tag{6.27}$$

When $i_0 > k$ we may replace i_0 by k in (6.27) to obtain (6.21). We may thus assume $i_0 \le k$ for the remainder of the proof of (6.21). On the other hand, it is easy to see that

$$\sigma_\tau \ge \tau \quad \text{for } \tau \ge r+1$$

and thus

$$\begin{aligned} l &= (l_1 - l_2) + (l_2 - l_3) + \cdots + (l_k - l_{k+1}) + l_{k+1} \\ &= \sigma_{\tau(l_1)} + \sigma_{\tau(l_2)} + \cdots + \sigma_{\tau(l_k)} + l_{k+1} \\ &\ge \tau(l_1) + \tau(l_2) + \cdots + \tau(l_k) \end{aligned}$$

and so replacing l by l_{i_0} above

$$\tau(l_{i_0+1}) + \cdots + \tau(l_k) \le l_{i_0+1}.$$

Hence

$$\begin{aligned} \sum_{i=i_0+1}^{k} \tau(l_i)^x &\le \left(\sum_{i=i_0+1}^{k} \tau(l_i) \right)^x \\ &\le l_{i_0+1}^x \\ &= O(l^{x/r}) \quad \text{by (6.26)}. \end{aligned} \tag{6.28}$$

(6.21) follows from (6.27) and (6.28) and this completes the proof of (6.4). □

We now turn to the proof of (6.5). For positive integer t,

$$\Pr(\exists\ t \text{ isolated copies of } H \text{ in } G_{n,p}) \le \frac{1}{t!}\binom{n}{r}^t\left(\frac{r!}{\alpha}\right)^t p^{ts}$$
$$\le \left(\frac{e}{t}\cdot\frac{n^r}{r!}\cdot\frac{r!}{\alpha}\cdot p^s\right)^t$$
$$\le \left(\frac{3\omega^s}{t\alpha}\right)^t.$$

Now put $t = \lambda_0$ and apply (6.11).

The same argument gives

$$\Pr(G_{n,2m} \text{ contains at least } \lambda_1 \text{ isolated copies}) = o(e^{-2\lambda_0}) \quad (6.29)$$

and so, using (6.16) and (6.17), we find

$$\Pr(G_{n,2m} \text{ contains } 2\lambda_1 \text{ or more copies of } H) = o(e^{-2\lambda_0}). \quad (6.30)$$

6.4. PROOF OF (6.6)

This section contains the main ideas of the proof of Theorem 6.1.

Let $\mathscr{A}_{k,l} = \{G \in \mathscr{G}_{n,m}\colon G$ has k isolated copies and l nonisolated copies of $H\}$. Let $a_{kl} = |\mathscr{A}_{k,l}|$ so that (6.6) is actually concerned with the ratio $a_{k,l}/a_{k-1,l}$.

Now for $k > 0$, $l \ge 0$, let $BP_{k,l}$ denote the bipartite graph with vertex partition $\mathscr{A}_{k,l}$, $\mathscr{A}_{k-1,l}$ and edge set $\mathscr{E}_{k,l}$ where $G_1G_2 \in \mathscr{E}_{k,l}$, $G_1 \in \mathscr{A}_{k,l}$, $G_2 \in \mathscr{A}_{k-1,l}$ if the edge sets of G_1, G_2 are related by

$$E(G_2) = (E(G_1)\setminus\{e\}) \cup \{f\},$$

where e is an edge of some isolated copy of H in G_1 and f is some edge which does not create a new copy of H when added to G_1/e.

If $G \in \mathscr{A}_{k,l} \cup \mathscr{A}_{k-1,l}$, let $d(G)$ denote its degree in $BP_{k,l}$. Then

$$G \in \mathscr{A}_{k,l} \text{ implies } ks(N - m - \xi(G)) \leq d(G) \leq ks(N - m),$$
$$\xi(G) = \sum_{H'} a_\xi(H'). \tag{6.31}$$

The sum here is over all copies in G of graphs H' of the form $H - x$, $x \in E(H)$. $a_\xi(H')$ is the number of different ways of adding an edge to H' (without adding vertices) to create a copy of H. [For example, if $H = K_4 \setminus e$ and $H' = C_4$, then $a_\xi(H') = 2$.]

This is because we have ks choices for edge e in an isolated copy of H. Then of the $N - m$ possible edge replacements f there are at most $\xi(G - e) - 1$ choices which create a new H when added. Finally observe that $\xi(G - e) - 1 \leq \xi(G)$.

Also

$$G \in \mathscr{A}_{k-1,l} \text{ implies}$$
$$(m - s(k + l) - \zeta'(G))(\xi(G) - s(k + l) - 2\zeta(G) - \zeta''(G))$$
$$\leq d(G) \leq m\xi(G), \tag{6.32}$$
$$\zeta(G) = \sum_{H'} a_\zeta(G).$$

Here we sum over all copies in G of graphs H' of the form $(H_1 \cup H_2) - x$ where H_1, H_2 are distinct copies of H which have at least one edge in common and $x \in E(H_1) \cup E(H_2)$, and a_ζ is defined analogously to a_ξ. $\zeta'(G) = s\xi(G)$ if $s > r$ and 0 otherwise and $\zeta''(G) = 0$ if $s > r$ and $sr!\Delta(G)^{r-1}$ otherwise.

To see this we overestimate the number of choices of f by m and the number of choices of e by $\xi(G)$. To underestimate $d(G)$ we underestimate the number of choices of f by $m - s(k + l) - \zeta'(G)$ since we do not wish to touch a copy of H and for a further reason to be explained. The number of choices δ_f for e, given f is

$$|\{e \notin E(G): h_e(G - f) \geq 1 \text{ and adding } e \text{ creates no new intersecting pair } H_1, H_2\}|,$$

where $h_e(G - f) =$ the number of copies of H created when adding e to $G - f$. Now $h_e(G) \geq 1$ implies $h_e(G - f) \geq 1$ unless f belongs to some copy of $H - x$ in G and $x = e$. When $s > r$ we eliminate such f by subtracting $\xi(G)$. When $s \leq r$ we underestimate δ_f by subtracting an upper bound $(sr!\Delta(G)^{r-1})$ on the number of possible

xs in copies of $H - x$ that contain f. So

$$\begin{aligned}\delta_f &\geq \left|\{e \notin E(G): h_e(G) \geq 1\}\right| - \zeta(G) - \zeta''(G)\\ &\geq \xi(G) - s(k+l) - \sum_{e \notin E(G)} \max\{0, h_e(G) - 1\} - \zeta(G) - \zeta''(G)\\ &\qquad \left[\text{since } \xi(G) \leq \sum_{e \notin E(G)} h_e(G) + s(k+l)\right]\\ &\geq \xi(G) - s(k+l) - \sum_{e \notin E(G)} \binom{h(G)}{2} - \zeta(G) - \zeta''(G)\\ &\geq \xi(G) - s(k+l) - 2\zeta(G) - \zeta''(G)\end{aligned}$$

and the lower bound follows.

The equation

$$\sum_{G \in \mathscr{A}_{k,l}} d(G) = \sum_{G \in \mathscr{A}_{k-1,l}} d(G),$$

(6.31) and (6.32) lead to

$$\begin{aligned}&\frac{\left(m - s(k+l) - \bar{\zeta}'_{k-1,l}\right)\left(\xi_{k-1,l} - s(k+l) - 2\bar{\zeta}_{k-1,l} - \bar{\zeta}''_{k-1,l}\right)}{ks(N-m)}\\ &\qquad \leq \frac{a_{k,l}}{a_{k-1,l}} \leq \frac{m\bar{\xi}_{k-1,l}}{ks\left(N - m - \bar{\xi}_k\right)},\end{aligned} \tag{6.33}$$

where $\bar{\xi}_{k,l}$, $\bar{\zeta}_{k,l}$, $\bar{\zeta}'_{k,l}$ and $\bar{\zeta}''_{k,l}$ denote the expectations of $\xi(G)$, $\zeta(G)$, $\zeta'(G)$ and $\zeta''(G)$ over $\mathscr{A}_{k,l}$. It only remains now to estimate these quantities. Let $\mathscr{N}(G) = \{e \in E(\overline{G}): h_e > 0\}$ and $\eta(G) = |\mathscr{N}(G)|$ $[h_e = h_e(G)]$. Let λ_1 be as prior to (6.15).

Lemma 6.3. Let $G = G_{n,m}$.

(a) $\Pr(\exists\ e \in E(\overline{G}): h_e \geq 2\lambda_1) = o(n^2 e^{-2\lambda_0})$.

(b) $\Pr(\eta(G) \geq n^{r/s}\lambda_1 \log n) = o(e^{-2\lambda_0})$.

(c) $\Pr(\Delta(G) \geq \lambda_0) = o(e^{-2\lambda_0})$ for $s \leq r$.

Proof. Let $\mathscr{E}$ denote the event $\{G_{n,2m}$ has at least $2\lambda_1$ copies of $H\}$. Think of $G_{n,2m}$ as $G_{n,m}$ plus m random edges.

(a) Let $\mathscr{E}_a = \{\exists\ e \in E(\overline{G}) \text{ s.t. } h_e \geq 2\lambda_1\}$. Then

$$\begin{aligned}\Pr(\mathscr{E}) &\geq \Pr(\mathscr{E}|\mathscr{E}_a)\Pr(\mathscr{E}_a)\\ &\geq \frac{m}{N}\Pr(\mathscr{E}_a).\end{aligned}$$

Part (a) now follows from (6.30).

(b) Let $\lambda_2 = n^{r/s}\lambda_1 \log n$ and $\mathscr{E}_b = \{\eta(G) \geq \lambda_2\}$. Then

$$\Pr(\mathscr{E}) \geq \Pr(\mathscr{E}|\mathscr{E}_b)\Pr(\mathscr{E}_b)$$

and (b) follows if we show that $\Pr(\mathscr{E}|\mathscr{E}_b) \geq \frac{1}{2}$. But to see this observe that the expected number of copies of H created by adding the second m edges is at least $(m/N)\eta(G_{n,m})$ and

$$\begin{aligned}\frac{m}{N}\lambda_2 &\approx \omega\lambda_1 \log n\\ &\gg \lambda_1.\end{aligned}$$

Note that we see now that the actual number added, given $\mathscr{E}_b$, majorizes a hypergeometrically distributed random variable with mean $\gg \lambda_1$.

(c) In $G_{n,p}$,

$$\begin{aligned}\Pr(\Delta \geq \lambda_0) &\leq n\binom{n}{\lambda_0}\left(\frac{\omega}{n}\right)^{\lambda_0}\\ &\leq n\left(\frac{e\omega}{\lambda_0}\right)^{\lambda_0}.\end{aligned}$$

Now apply (6.11). □

Let us now return to the consideration of (6.33). Suppose $l \leq \lambda_0$. It follows from (6.4) and (6.5) that there exists $k_0 \leq \lambda_0$ such that

$$\pi_{k_0,l} \geq n^{-A_3 l^{2/r}}(2\lambda_0)^{-1}.$$

We prove that

$$\begin{aligned}\pi_{k,l} &\geq \left(1 - \frac{1}{\lambda_0}\right)^{|k-k_0|} n^{-A_3 l^{2/r}}(2\lambda_0)^{-1} \qquad (0 \leq k \leq \lambda_0)\\ &\geq e^{-\lambda_0/10}. \qquad (6.34)\end{aligned}$$

This is true for $k = k_0$ and assume inductively that it is true for some $0 < k \leq k_0$. $k > k_0$ will be dealt with subsequently and this is why we are assuming that $k_0 > 0$. We will be able to verify (6.6) as we proceed with the induction.

We will estimate $\bar{\xi}_{k,l}, \bar{\zeta}_{k,l}$ by similar methods, and to do this we let Γ denote a generic graph of the form $H - x$ or $H_1 \cup H_2 - x$. For $G \in \mathscr{A}_{k,l}$ we let $EH(G)$ denote the edges of G which lie in some copy of H contained in G. Let Γ_0 denote some fixed copy of Γ in K_n. For $X \subseteq Y \subseteq E_0 = E(\Gamma_0) = \{e_1, e_2, \ldots, e_u\}$ we let

$$\mathscr{A}_{k,l,X} = \{G \in \mathscr{A}_{k,l}\colon EH(G) \cap E_0 = X \text{ and } (E_0 - E(G)) \cap \mathscr{N}(G) = \varnothing\},$$
$$\mathscr{A}_{k,l,X,Y} = \{G \in \mathscr{A}_{k,l,X}\colon Y \subseteq E(G)\}.$$

Observe now that if Z_Γ denotes the number of copies of Γ in G chosen randomly from $\mathscr{A}_{k,l}$, then

$$E_{k,l}(Z_\Gamma) = \binom{n}{\nu(\Gamma)} \frac{r!}{\alpha_\Gamma} \sum_{X \subseteq E_0} \frac{|\mathscr{A}_{k,l,X,E_0}|}{|\mathscr{A}_{k,l}|}, \tag{6.35}$$

where $E_{k,l}$ denotes expectation over $\mathscr{A}_{k,l}$ and α_Γ is the number of automorphisms of Γ.

For if $\Gamma_0 \subseteq E(G)$, then $G \in \mathscr{A}_{k,l,X,E_0}$ where $X = E_0 \cap EH(G)$.

The following lemma deals with the relative sizes of these sets. Let

$$\theta_2 = \min_{H' \subset H} \left\{\nu(H') - \frac{r}{s}\mu(H')\right\} > 0.$$

Lemma 6.4.

(a) $$1 - \frac{2s\lambda_2}{N} \leq \frac{|\mathscr{A}_{k,l,\varnothing,\varnothing}|}{|\mathscr{A}_{k,l}|} \leq 1 - \frac{sk}{N}.$$

(b) $$\frac{|\mathscr{A}_{k,l,X,Y}|}{|\mathscr{A}_{k,l,X,Y'}|} = \frac{m}{N}\left(1 + O\left(\frac{m + \lambda_2}{N}\right)\right),$$

if $Y \supseteq Y'$ and $|Y - Y'| = 1$ and $|\bigcup_Y \mathscr{A}_{k,l,X,Y}| \geq e^{-\lambda_0}|\mathscr{A}_{k,l}|$.

(c) $$n^{\nu(\Gamma)} \frac{|\mathscr{A}_{k,l,X,E_0}|}{|\mathscr{A}_{k,l}|} \leq A\lambda_0^{2r}\omega^{2s-1}n^{r/s - \theta_2}$$

unless $X = \varnothing$ and Γ is of the form $H - x$.

Proof. (a) Note first that if $G \in \mathscr{A}_{k,l}$ and $\phi(G)$ is an isomorphic image of G (i.e., obtained by relabelling vertices), then $\phi(G) \in \mathscr{A}_{k,l}$ too. So if $\mathscr{A}'_{k,l} = \mathscr{A}_{k,l} - \mathscr{A}_{k,l,\phi,\phi}$, then

$$\Pr_\phi(\mathscr{E}_1) \le \frac{|\mathscr{A}'_{k,l}|}{|\mathscr{A}_{k,l}|} \le \Pr_\phi\left(\bigcup_{i=1}^{u} \mathscr{E}_i \cup \bigcup_{i=1}^{u} \mathscr{E}'_i\right),$$

where (i) $\Pr_\phi$ refers to randomly choosing a member G of $\mathscr{A}_{k,l}$ and then choosing a random permutation of the vertices (this does choose a random member of $\mathscr{A}_{k,l}$); (ii) $\mathscr{E}_i = \{e_i \in EH(\phi(G))\}$; (iii) $\mathscr{E}'_i = \{e_i \in \mathscr{N}(\phi(G))\}$. But since each edge of G is mapped, by ϕ, to a randomly chosen edge of K_n,

$$\frac{sk}{N} \le \Pr_\phi(\mathscr{E}_i) \le \frac{s(k+l)}{N},$$

$$\Pr_\phi(\mathscr{E}'_i) \le E_{k,l}\left(\frac{\eta(G)}{N}\right).$$

Thus

$$\frac{sk}{N} \le \frac{|\mathscr{A}'_{k,l}|}{|\mathscr{A}_{k,l}|} \le E_{k,l}\left(\frac{(2s-1)(\eta(G) + s(k+l))}{N}\right).$$

Now (6.34) and Lemma 6.4(b) imply $E_{k,l}(\eta(G)) \le \lambda_2 + o(Ne^{-\lambda_0})$ and (a) follows on tidying up.

(b) Consider the bipartite graph $BP = BP_{k,l,X,Y,Y'}$ with bipartition $\mathscr{A}_{k,l,X,Y}, \mathscr{A}_{k,l,X,Y'}$ and an edge G_1G_2 for $G_1 \in \mathscr{A}_{k,l,X,Y}, G_2 \in \mathscr{A}_{k,l,X,Y'}$ if G_2 can be obtained from G_1 by deleting the unique $e \in Y - Y'$ and adding a new edge f. Using d to denote degree in BP we have

$$G \in \mathscr{A}_{k,l,X,Y} \text{ implies } N - m - \eta(G) \le d(G) \le N - m. \quad (6.36)$$

There are at most $N - m$ choices for f which gives the upper bound. On the other hand, if $f \notin E(G) \cup \mathscr{N}(G)$, then $G - e + f \in \mathscr{A}_{k,l,X,Y'}$. To see this we first note that $G + f$ has the same $k + l$ copies of H as G. But this implies $e \notin \mathscr{N}(G - e + f)$ and then if $e' \in \mathscr{N}(G - e + f)$ for some $e' \in E_0 - Y$, we find that e' belongs to a copy of H in $G + f$ and hence in G, which is disbarred by $G \in \mathscr{A}_{k,l,X}$.

$$G \in \mathscr{A}_{k,l,X,Y'} \text{ implies } m - s(k+l) \le d(G) \le m. \quad (6.37)$$

There are at most m choices for f and if we choose to delete an f which is not in any copy of H, then $G + e - f$ is in $\mathscr{A}_{k,l,X,Y}$. The latter fact following from $e \notin \mathscr{N}(G)$.

Hence we have, analogously to (6.33),

$$\frac{m - s(k+l)}{N} \le \frac{|\mathscr{A}_{k,l,X,Y}|}{|\mathscr{A}_{k,l,X,Y'}|} \le \frac{m}{N - m - \overline{\eta}_{k,l,X,Y}}, \quad |Y - Y'| = 1. \tag{6.38}$$

Our assumption on the size of $\bigcup_{X \subseteq Y \subseteq E_0} \mathscr{A}_{k,l,X,Y}$ implies the existence of $Y_0 \supseteq X$ such that

$$|\mathscr{A}_{k,l,X,Y_0}| \ge \left(\tfrac{1}{2}\right)^{2s} e^{-\lambda_0} |\mathscr{A}_{k,l}|.$$

Now (6.38) implies that $|\mathscr{A}_{k,l,X,Y}| / |\mathscr{A}_{k,l,X,Y'}| \ge m/2N$ and so if $Y \supseteq Y_0$,

$$|\mathscr{A}_{k,l,X,Y}| \ge \left(\frac{1}{2}\right)^{2s} \left(\frac{m}{2N}\right)^{|Y - Y_0|} e^{-\lambda_0} |\mathscr{A}_{k,l}| \ge e^{-3\lambda_0/2} \binom{N}{m},$$

and hence we see from this and Lemma 6.3(b) that $\overline{\eta}_{k,l,X,Y} \le 2\lambda_2$ for $Y \supseteq Y_0$. But this then implies that for $Y \supseteq Y_0$, $|Y - Y'| = 1$,

$$\left(1 - \frac{2s\lambda_0}{m}\right) \frac{m}{N} \le \frac{|\mathscr{A}_{k,l,X,Y}|}{|\mathscr{A}_{k,l,X,Y'}|} \le \left(1 + \frac{3(m + \lambda_2)}{N}\right) \frac{m}{N}. \tag{6.39}$$

But if $Y_0 \ne X$ and $|Y - \tilde{Y}| = 1$, we see from (6.38) that $|\mathscr{A}_{k,l,X\tilde{Y}}| \ge (N/2m) |\mathscr{A}_{k,l,X,Y}|$. This and Lemma 6.3(b) implies an upper bound of $2\lambda_2$ on $\overline{\eta}_{k,l,X,\tilde{Y}}$ and then substitution in (6.38) yields (6.39) for $Y = \tilde{Y}$. Clearly we can repeat this argument to show that (6.39) holds for $Y \supsetneqq X$, which completes the proof of (b).

(c) We use the extra randomization ϕ as in part (a). Let H' denote the subgraph of Γ_0 induced by X. If $X \subseteq EH(G)$, then ϕ must map some $\nu(H')$ vertices of the $k + l$ copies of H onto the vertices of H'. The probability of this happening is at most

$$\frac{(s(k+l))_{\nu(H')}}{(n)_{\nu(H')}} \le A \left(\frac{\lambda_0}{n}\right)^{\nu(H')}.$$

Since $\mu(H') = |X|$ we can apply part (b) to conclude that

$$\frac{|\mathscr{A}_{k,l,X,E_0}|}{|\mathscr{A}_{k,l}|} \le A\lambda_0^{2r}\omega^{2s-1}n^{-((r/s)(\mu(\Gamma)-\mu(H'))+\nu(H'))}. \tag{6.40}$$

To prove the inequality in the statement of (c) we must, by (6.40) find a bound for

$$\Delta = \nu(\Gamma) - \frac{r}{s}(\mu(\Gamma) - \mu(H')) - \nu(H').$$

We first consider the case where Γ is of the form $H - x$. Then $\nu(\Gamma) = r$, $\mu(\Gamma) = s - 1$, and H' is a proper subgraph of H. Hence, in this case

$$\begin{aligned}\Delta &= \frac{r}{s} - \left(\nu(H') - \frac{r}{s}\mu(H')\right)\\ &\le \frac{r}{s} - \theta_2.\end{aligned} \tag{6.41}$$

The following proof of the last part of this lemma is due to Andrzej Ruciński.

For a graph G let $f(G) = \nu(G) - (r/s)\mu(G)$. Suppose now that $\Gamma = H_1 \cup H_2 - x$ and $K = H_1 \cup H_2$, so that

$$\Delta - \frac{r}{s} = f(K) - f(H'). \tag{6.42}$$

Let $H_0 = H_1 \cap H_2$, $H_i' = H' \cap H_i$, $i = 1,2$, and $H'' = H_1' \cap H_2' = H_i' \cap H_0$, $i = 1,2$. Then (i) $f(H_i') \ge 0$, $i = 1,2$; (ii) $f(H_i' \cup H_0) = f(H_i') + f(H_0) - f(H'') \ge 0$; (iii) at least two of the above four inequalities are strict, by an amount θ_2. Hence, adding these inequalities,

$$2(f(H_1') + f(H_2') + f(H_0) - f(H'')) \ge 2\theta_1. \tag{6.43}$$

But

$$f(H') = f(H_1') + f(H_2') - f(H'') \tag{6.44}$$

and

$$f(K) = f(H_1) + f(H_2) - f(H_0) = -f(H_0). \tag{6.45}$$

(6.42)–(6.45) imply $\Delta - r/s \le -\theta_2$, as was to be shown. □

Let us now consider $\xi(G)$. Fix Γ of the form $H - x$. Let $\Omega = \{X: |\bigcup_Y \mathscr{A}_{k,l,X,Y}| \geq e^{-\lambda_0}|\mathscr{A}_{k,l}|\}$. Then $\varnothing \in \Omega$ and (6.35) and Lemma 6.4(a), (b) give

$$\begin{aligned} E_{k,l}(Z_\Gamma) &\geq \binom{n}{r}\frac{r!}{\alpha_\Gamma}\frac{|\mathscr{A}_{k,l,\varnothing,E_0}|}{|\mathscr{A}_{k,l}|} \\ &\geq \binom{n}{r}\frac{r!}{\alpha_\Gamma}\left(\frac{m}{N}\right)^{s-1}\left(1 - O\left(\frac{m+\lambda_2}{N}\right)\right) \\ &= \frac{\omega^{s-1}n^{r/s}}{\alpha_\Gamma}\left(1 - O\left(\frac{m+\lambda_2}{N}\right)\right). \end{aligned}$$

Conversely,

$$\begin{aligned} &E_{k,l}(Z_\Gamma) \\ &\quad= \binom{n}{r}\frac{r!}{\alpha_\Gamma}\left(\frac{|\mathscr{A}^*_{k,l,\varnothing,E_0}|}{|\mathscr{A}_{k,l}|} + \sum_{\substack{X\in\Omega \\ X\neq\varnothing}}\frac{|\mathscr{A}^*_{k,l,X,E_0}|}{|\mathscr{A}_{k,l}|} + \sum_{X\notin\Omega}\frac{|\mathscr{A}^*_{k,l,X,E_0}|}{|\mathscr{A}_{k,l}|}\right) \\ &\quad\leq \binom{n}{r}\frac{r!}{\alpha_\Gamma}\left(\frac{m}{N}\right)^{s-1}\left(1 + O\left(\frac{m+\lambda_2}{N}\right)\right) \\ &\qquad + O\left(\lambda_1^{2r}n^{r/s-\theta_2}\right) + O(n^r e^{-\lambda_0}) \\ &\quad= \frac{\omega^{s-1}n^{r/s}}{\alpha_\Gamma}\left(1 + O\left(\frac{\lambda_1^{2r}}{n^{\theta_2}}\right)\right). \end{aligned}$$

Observe that if Λ_ξ denotes the set of possible Γ,

$$\sum_{\Gamma\in\Lambda_\xi}\frac{r!a_\xi(\Gamma)}{\alpha_\Gamma} = \frac{sr!}{\alpha},$$

since we obtain all copies of graphs of the form $H - x$ in K_r by taking all copies of H and deleting an edge. Thus we can write

$$\bar{\xi}_{k,l} = \frac{s\omega^{s-1}}{\alpha}n^{r/s}\left(1 + O\left(\frac{\lambda_1^{2r}}{n^{\theta_2}}\right)\right). \tag{6.46}$$

By a similar analysis we can deduce from Lemma 6.4(c) and (6.35) that

$$\bar{\zeta}_{k,l} \le A\lambda_0^{2r}\omega^{2s-1}n^{r/s-\theta_2}. \tag{6.47}$$

We can now go back to (6.33), which implies

$$a_{k-1,l} \ge a_{k,l}\frac{ks\left(N-m-\bar{\xi}_{k,l}\right)}{m\bar{\xi}_{k-1,l}}. \tag{6.48}$$

But clearly $\bar{\xi}_{k-1,l} \le n^r$ and so, using (6.34), $\pi_{k-1,l} \ge e^{-\lambda_0/10}$. With this lower bound we can repeat the arguments above and prove (6.46) and (6.47) with k replaced by $k-1$. Furthermore, where $r \ge s$, this lower bound and Lemma 6.3(c) shows

$$\bar{\zeta}''_{k-1,l} \le 2sr!\lambda_0^{r-1}. \tag{6.49}$$

But using these estimates now in (6.33) gives

$$\frac{a_{k,l}}{a_{k-1,l}} = \frac{\lambda}{k}(1+\beta_{k,l}), \tag{6.50}$$

where, $|\beta_{k,l}| = o(\lambda_0^{-1})$ provided

$$\theta < \frac{\theta_2}{(2r^2+1)s}. \tag{6.51}$$

Note that (6.50) = (6.6) and that this completes the inductive step in the proof of (6.33) for $k \le k_0$. For $k > k_0$ the only thing that changes is that we replace (6.48) by

$$a_{k+1,l} \ge \frac{\left(m - s(k+l) - \bar{\zeta}'_{k,l}\right)\left(\bar{\xi}_{k,l} - s(k+l) - 2\bar{\zeta}_{k,l} - \bar{\zeta}''_{k,l}\right)}{ks(N-m)}a_{k,l},$$

which enables us to use (6.46), (6.47), and (6.49) with k replaced by $k+1$. The rest is as before. This completes the proof of (6.6) and the theorem. □

Remark. We have identified five upper bounds: (6.13), (6.15), (6.22), (6.23), and (6.51). It turns out however that (6.50) dominates the others.

Acknowledgement. It is a pleasure to acknowledge the help given by Andrzej Ruciński. He pointed out many errors in an earlier draft and provided a proof for an inequality in Lemma 4.4.

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