Perfect matchings in random bipartite graphs with minimal degree at least 2

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Abstract

We show that a random bipartite graph with $n+n$ vertices and cn random edges, and minimum degree at least two, has a perfect matching **whp**.

1 Introduction

To quote from Lovász [15], "the problem of the existence of 1- factors (perfect matchings), the solution of which (the Kőnig-Hall theorem for bipartite graphs and Tutte's theorem for the general case) is an outstanding result making this probably the most developed field of graph theory". Erdős and Rényi ([8],[9]) found a way to use these results for a surprisingly sharp study of existence of perfect matchings in random graphs. For $B_{n,m}$, a random bipartite graph with $n+n$ vertices and $m = n(\ln n + c_n)$ random edges, they proved [8] that

$$\lim_{n\to\infty} \mathbf{Pr}(B_{n,m} \text{ has a perfect matching}) = \lim_{n\to\infty} \mathbf{Pr}(\delta(B_{n,m}) \geq 1) = \begin{cases} 0 & c_n \to -\infty, \\ e^{-2e^{-c}}, & c_n \to c, \\ 1 & c_n \to \infty, \end{cases}$$

where δ denotes minimum degree. Of course minimum degree at least one is a trivial necessary condition for the existence of a perfect matching. The Hall theorem turned out to be perfectly tailored for use in combination with probabilistic techniques, pioneered by the authors several years earlier, [8]. Even though Tutte's theorem for the non-bipartite case is considerably more involved, in [9] they managed to extend the analysis to the random graph $G_{n,m}$, a random

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general graph with n vertices and $m = \frac{n}{2}(\ln n + c_n)$ edges, showing that

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \mathbf{Pr}(G_{n,m} \text{ has a perfect matching}) = \lim_{n\to\infty} \mathbf{Pr}(\delta(G_{n,m}) \geq 1) = \begin{cases} 0 & c_n \to -\infty, \\ e^{-e^{-c}} & c_n \to c, \\ 1 & c_n \to \infty. \end{cases}$$

In both cases a perfect matching becomes likely as soon as one has sufficiently many random edges for the minimum degree to be at least one with high probability (**whp**[1]). This has led researchers to consider the existence of perfect matchings in models of a random graph in which the minimum degree requirement is always satisfied. Perhaps the first result along these lines is due to Walkup [19]. He considered a κ-out model $B_{\kappa-\text{out}}$ of a random bipartite graph, again with $n+n$ vertices $V_1 + V_2$. Here each vertex $v \in V_i$ "chooses" κ random neighbours in its complementary class V_{3-i}. Walkup showed that

$$\lim_{n\to\infty} \mathbf{Pr}(B_{\kappa-out} \text{ has a perfect matching}) = \begin{cases} 0 & \kappa = 1 \\ 1 & \kappa \geq 2 \end{cases}$$

Frieze [10] proved a non-bipartite version of this result, the argument being based on Tutte's theorem and considerably harder. Very recently Karoński and Pittel [13] have proven **whp** existence of a perfect matching in what they called the $B_{(1+e^{-1})-out}$ graph, a subgraph of B_{2-out}, obtained from B_{1-out} by letting each of its degree 1 vertices select another random neighbor in the complementary class. Observe that in all of these results [19], [10] and [13] the number of random edges depends linearly on the number of vertices, and the minimum degree has been raised to 2, in a sharp contrast with the case m being of order $n \log n$. Here is why. When there are order $n \ln n$ random edges, there are few vertices of degree 1 and they are far apart. In sparser models, with minimum degree 1, **whp** there will be a linear (in n) number of vertices of degree 1, and some two vertices of degree 1 will have a common neighbor, which rules out a perfect matching. In the case of random regular graphs it turns out that minimum degree 3 is required, Bollobás [3]: Let G_r denote a random r-regular graph on vertex set $[n]$, n even. Then

$$\lim_{n\to\infty} \mathbf{Pr}(G_r \text{ has a perfect matching}) = \begin{cases} 0 & r = 2, \\ 1 & r = 1 \text{ or } r \geq 3. \end{cases}$$

The case $r = 1$ is trivial since then G_r is itself a perfect matching of $[n]$. G_2 is **whp** a collection of $O(\ln n)$ disjoint cycles and they will all have to be even for G_2 to have a perfect matching. The meat of the result is therefore in the case $r \geq 3$ and this follows from r-connectivity and Tutte's theorem.

Another approach was considered by Bollobás and Frieze [6]. Let $\mathcal{G}_{n,m}^{\delta\geq\kappa}$ denote the set of graphs with vertex set $[n]$, m edges and minimum degree at least κ. Let $G_{n,m}^{\delta\geq\kappa}$ be sampled uniformly from $\mathcal{G}_{n,m}^{\delta\geq\kappa}$. By conditioning on minimum degree 1, say, we will need fewer random edges to get a perfect matching **whp**: Let $m = \frac{n}{4}(\ln n + 2\ln\ln n + c_n)$.

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \mathbf{Pr}(G_{n,m}^{\delta\geq 1} \text{ has a perfect matching}) = \begin{cases} 0 & c_n \to \infty \textit{ sufficiently slowly}, \\ e^{-\frac{1}{8}e^{-c}} & c_n \to c, \\ 1 & c_n \to \infty. \end{cases} \quad (1)$$

The restriction "sufficiently slowly" may seem out of place, but bear in mind that if $m = n/2$ then the probability of a perfect matching is 1. The precise threshold between $n/2$ and $\frac{1}{4}n \ln n$

[1] A sequence of events $\mathcal{E}_n$ occurs *with high probability* (**whp**) if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

for the non-existence of a perfect matching was not determined. (Using the approach developed in the present paper for the bipartite case, we have found that "sufficiently slowly" in (1) can be replaced simply by $m > n/2$.) This work was extended in Bollobás, Fenner and Frieze [4] who considered the probability that $G_{n,m}^{\delta\geq k}$ has $\lfloor k/2 \rfloor$ disjoint Hamilton cycles plus a further disjoint perfect matching if k is odd.

In this paper and [12] we continue this line of research. In [12] we considered the bipartite version of (1). Let $\mathcal{B}_{n,m}^{\delta\geq\kappa}$ denote the set of bipartite graphs with vertex set $[n],[n]$, m edges and minimum degree at least κ. Let $B_{n,m}^{\delta\geq\kappa}$ be sampled uniformly from $\mathcal{B}_{n,m}^{\delta\geq\kappa}$.

Theorem 1. *Let $m = \frac{n}{2}(\ln n + 2\ln\ln n + c_n)$. then*

$$\lim_{n\to\infty} \mathbf{Pr}(B_{n,m}^{\delta\geq 1} \text{ has a perfect matching}) = \begin{cases} 0 & c_n \to -\infty,\ m > n, \\ e^{-\frac{1}{4}e^{-c}} & c_n \to c, \\ 1 & c_n \to \infty. \end{cases} \tag{2}$$

The probability on the RHS of (2) is the limiting probability that a pair of vertices of degree 1 have a common neighbor. Thus, the probability that a perfect matching exists is (close to) 1 when either $m = n/2$ or c_n is large, and the probability is very small for m everywhere in between, except c_n not far to the left from 0.

If we consider $G_{n,cn}^{\delta\geq 2}$ then we have to allow for the existence of small components which are isolated *odd* cycles i.e. we will not have a "probability one" result. Also, for a change, we will allow n to assume odd values as well. For a graph $G = (V, E)$, let $\mu^*(G)$ denote the maximum matching number. Slightly stretching, we say that G has a perfect matching if $\mu^*(G) = [|V|/2]$. Let $X(G)$ stand for the total number of odd isolated cycles in G. Clearly

$$\mu^*(G) \leq \nu(G) := \left[\frac{|V| - X(G)}{2}\right].$$

Let μ_n^*, X_n, ν_n stand for μ^*, X, ν computed at $G = G_{n,cn}^{\delta\geq 2}$. It was shown in [12] that

Theorem 2. *Let $c > 1$ be an absolute constant. Then*

$$\lim_{n\to\infty} \mathbf{Pr}(\mu_n^* = \nu_n) = 1,$$

and X_n is, in the limit, Poisson (λ),

$$\lambda = \frac{1}{4}\log\frac{1+\sigma}{1-\sigma} - \frac{\sigma}{2}, \quad \sigma := \frac{\rho}{e^\rho - 1},$$

and ρ satisfies

$$\frac{\rho(e^\rho - 1)}{e^\rho - 1 - \rho} = 2c.$$

In particular, as $n \to \infty$,

$$\mathbf{Pr}(G_{n,cn}^{\delta\geq 2} \text{ has a perfect matching}) = o(1) + \begin{cases} e^{-\lambda}, & \text{if } n \text{ even}, \\ e^{-\lambda} + \lambda e^{-\lambda}, & \text{if } n \text{ odd}. \end{cases} \tag{3}$$

Notice that $c = 1$ corresponds to the random 2-regular (nonbipartite) graph, which typically has $\Theta(\log n)$ odd (isolated) cycles. Sure enough, the explicit term in the RHS of (3) approaches zero as $c \downarrow 1$.

It was shown in Aronson, Frieze and Pittel [1] that **whp** a simple greedy algorithm of Karp and Sipser [14] found a matching that was within $\tilde{O}(n^{1/5})$ of optimal. Theorem 2 shows that the Karp-Sipser algorithm is **whp** also $\Omega(n^{1/5})$ from optimal.

For integer $k \geq 2$ let graph G have property $\mathcal{A}_k$ if G contains $\lfloor k/2 \rfloor$ edge disjoint Hamilton cycles, and, if k is odd, a further edge disjoint matching of size $\lfloor n/2 \rfloor$. Bollobás, Cooper, Fenner and Frieze [5] show that for $k \geq 2$, there exists a constant $c_k \leq 2(k+2)^3$ such that if $c \geq c_k$, $G_{n,cn}^{\delta \geq k+1}$ has property $\mathcal{A}_k$. Thus the current paper deals with the property $\mathcal{A}_1$ and proves a sharp result. It is reasonable to conjecture that the true value for c_k, $k \geq 2$ is $(k+1)/2$. Note that if $c = (k+1)/2$ and cn is integer then $G_{n,cn}^{\delta \geq k+1}$ is a random $(k+1)$-regular graph and this is known to have $\mathcal{A}_k$ **whp**, Robinson and Wormald [18].

Now we come to the result of this paper: The next natural question in this line of research is: How many random edges are needed if we constrain the minimum degree of a bipartite graph to be at least 2, so ruling out the possibility of two vertices of degree 1 having a common neighbour.

Theorem 3. *Let $c \geq 2$ be an absolute constant. Then*

$$\lim_{n \to \infty} \mathbf{Pr}(B_{n,cn}^{\delta \geq 2} \text{ has a perfect matching}) = 1.$$

If $c = 2$ then we are dealing with 2-regular bipartite graphs and all such graphs have a perfect matching. Thus the content of the theorem lies in the case $c > 2$.

Remark 1. *The theorem does not indicate what happens when $c = 2 + o(1)$ and so it is an open question to determine the smallest $\omega = o(n)$ such that $B_{n,2n+\omega}^{\delta \geq 2}$ has a perfect matching* **whp**.

2 Proof of Theorem 3

We will use Hall's necessary and sufficient condition for the existence of a perfect matching in a bipartite graph.

The random graph $B_{n,m}^{\delta \geq 2}$ has no perfect matching iff for some $k \geq 2$ there exists a *k-witness*. Let R, C (rows,columns) be disjoint copies of $[n]$. A k-witness is a pair of sets $K \subseteq R, L \subseteq C$, or $K \subseteq R, L \subseteq C$, such that $|K| = k, |L| = k-1$ and $N(K) \subseteq L$. Here $N(K)$ denotes the set of neighbours of vertices in K. A k-witness is *minimal* if there does not exist $K' \subset K, L' \subset L$ such that (K', L') is a k'-witness, where $k' < k$. It is straightforward that if (K, L) is a minimal k-witness then every member of L has degree at least two in $B_n(K \cup L)$, the subgraph of B_n induced by $K \cup L$. Therefore the subgraph has at least $2(k-1)$ edges. We can restrict our attention to $k \leq n/2$ since for $k > n/2$ we can consider $C \setminus L, R \setminus K$. For $2 \leq k \leq n/2$, let $W_{n,k,\mu}$ denote the random number of minimal k-witnesses, such that $B_n(K \cup L)$ has μ edges, $\mu \geq 2(k-1)$. Actually, since $k \leq n/2$, we also have $\mu \leq m - n$.

Let now $m = cn$ where $c > 2$ is a fixed constant and let $B = B_{n.m}^{\delta \geq 2}$. A direct application of Hall's theorem has resisted our efforts. Along these lines we can only manage

Lemma 1. *There exists an $\epsilon = \epsilon(c)$ such that,*

$$\mathbf{Pr}(\exists K \subseteq R : \ |K| \leq \epsilon n \text{ and } |N(K)| < |K|) = O(n^{-1}).$$

However this lemma can be used in the proof of the following: Let $\mu^*(G)$ denote the size of a maximum matching in G.

Lemma 2. *For $t \geq 1$, $m \sim cn$, $\epsilon_1 = \epsilon_1(n) > 0$ and $\omega = K \log n$ for some sufficiently large $K = K(c)$,*

If $\mathbf{Pr}(\mu^*(B^{\delta\geq 2}_{n,m-\omega}) \geq n - t) \geq 1 - \epsilon_1$ *then*

$$\mathbf{Pr}(\mu^*(B^{\delta\geq 2}_{n,m}) \geq n - t + 1) \geq 1 - \epsilon_1 + \frac{(\log n)^3}{n^{1/2}}.$$

We also prove

Lemma 3. *If $m \sim cn$ then*

$$\mathbf{Pr}(\mu^*(B^{\delta\geq 2}_{n,m}) \leq n - O(n^{.49})) \leq n^{-4}.$$

With these two preceding lemmas we can easily prove Theorem 3. Let $m \sim cn$ and let $m_r = m - r\omega$, $r = 0, 1, \dots, An^{.49}$. Then

$$\mathbf{Pr}(\mu^*(B^{\delta\geq 2}_{n,m_r}) \leq n - r) \leq n^{-1} + (O(n^{.49}) - r)\frac{(\log n)^3}{n^{1/2}}. \tag{4}$$

This is proved by downwards induction on r with the base case $r = An^{.49}$ (for some constant $A > 0$) being verified by Lemma 3 and Lemma 2 providing the inductive step. Theorem 3 is the case $r = 0$ of (4).

3 A result from [12]

We define functions

$$f_t(z) = \sum_{\ell \geq t} \frac{z^\ell}{\ell!} = e^z - \sum_{\ell < t} \frac{z^\ell}{\ell!} \tag{5}$$

for $t \geq 1$.

Let the ν_1-tuple $\mathbf{c} = (c_1, \dots, c_{\nu_1})$ and the ν_2-tuple $\mathbf{d} = (d_1, \dots, d_{\nu_2})$ of nonnegative integers be given. Introduce $N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu}, \mu)$, $\boldsymbol{\nu} = (\nu_1, \nu_2)$, the total number of bipartite graphs with μ edges, and degree sequences $a_i, i \in [\nu_1]$, $b_j, j \in [\nu_2]$ such that $a_i \geq c_i$, $(i \in [\nu_1])$, and $b_j \geq d_j$, $(j \in [\nu_2])$. Of course, $N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu}, \mu) = 0$ if $\mu < \sum_i c_i$, or $\mu < \sum_j d_j$. So we assume that $\mu \geq \max\{\sum_i c_i, \sum_j d_j\}$.

Then let

$$G_{\mathbf{c}}(x) = \prod_{i \in [\nu_1]} f_{c_i}(x) \text{ and } H_{\mathbf{d}}(y) = \prod_{j \in [\nu_2]} f_{d_j}(y). \tag{6}$$

We introduce the following notation: we write $A \leq_b B$ in place of $A = O(B)$ when the expression B is long. We believe that it enhances readability.

Lemma 4.

(a) *Suppose that $\nu_1, \nu_2 = O(\mu)$ and $\mu = O(\nu_i \log \nu_i)$, $i = 1, 2$. Suppose that $\mu^{-1} \leq_b r_1, r_2 = O(\log \mu)$. Then*

$$N_{\mathbf{c},\mathbf{d}}(\boldsymbol{\nu}, \mu) \leq_b \frac{\mu!}{(\nu_1 \nu_2 r_1 r_2)^{1/2}} \frac{G_{\mathbf{c}}(r_1) H_{\mathbf{d}}(r_2)}{(r_1 r_2)^\mu}. \tag{7}$$

(b)

$$|\mathcal{B}^{\delta\geq 2}_{n,m}| \sim m! \left(\frac{f_2(\rho)^n}{\rho^m n^{1/2}}\right)^2 F(c) \tag{8}$$

where ρ satisfies $\rho f_1(\rho)/f_2(\rho) = c$ and F is an explicitly given function.

3.1 Proof of Lemma 1

We show that

$$\sum_{\substack{2\le k\le \epsilon m\\ 2k\le \mu<m}} E_{n,k,\mu} \to 0, \tag{9}$$

for some $\epsilon > 0$. Here $E_{n,k,\mu}$ is the expected number of the minimal k-witnesses (K, L), $|K| = k, |L| = k-1$, $N(K) = L$, with μ edges. We know that every vertex from L has at least two neighbors from K.

Picking $z \in (0, \rho)$, we use (7) with $r_1 = r_2 = z$ to bound N_1, the number of such bipartite graphs on $K + L$. We use (7) with $r_1 = r_2 = \rho$ to bound N_2, the total number of feasible bipartite graphs that remain after deletion of all vertices belonging to K.

Using (8), we then obtain

$$E_{n,k,\mu} \le_b E^*_{n,k,\mu} = \frac{\binom{n}{k}\binom{n}{k-1}}{kz\binom{m}{\mu}} \cdot \frac{f_2(z)^{2k-1}}{z^{2\mu}} \cdot \frac{\rho^{2\mu}e^{(k-1)\rho}}{f_2(\rho)^{2k-1}}. \tag{10}$$

Consequently

$$\frac{E^*_{n,k,\mu+1}}{E^*_{n,k,\mu}} = \frac{\mu+1}{m-(\mu+1)} \cdot \frac{\rho^2}{z^2} \le \begin{cases} \frac{1}{2} & \mu < \overline{\mu} := \lfloor \frac{z^2}{3\rho^2} m \rfloor \\ \frac{2}{3} & \mu \le \lfloor 1.1\overline{\mu} \rfloor \end{cases} \tag{11}$$

Therefore, if $k \le 0.5\overline{\mu}$,

$$\sum_{\mu=2k}^{\max\{\overline{\mu},2.2k\}} E^*_{n,k,\mu} \le_b E^*_{n,k,2k}. \tag{12}$$

Furthermore, using (10) for the k in question and $\mu = 2k$, $mf_2(\rho)/n = \rho(e^\rho - 1)$,

$$\begin{aligned}
\frac{E^*_{n,k+1,2(k+1)}}{E^*_{n,k,2k}} &\le 4\frac{(n-k)_2}{(m-2k)_2} \cdot \frac{f_2(z)^2}{z^4} \cdot \frac{\rho^4 e^\rho}{f_2(\rho)^2} \\
&= (1+O(z^2)) \left(\frac{f_2(z)}{z^2/2}\right)^2 \left(\frac{\rho/2}{\sinh \rho/2}\right)^2 \\
&\le \frac{1+O(z)}{1+\rho^2/12} \\
&< 1,
\end{aligned}$$

provided that z is chosen sufficiently small. So

$$\sum_{k=2}^{\overline{\mu}/2} E^*_{n,k,2k} \le_b E^*_{n,2,4} = O(n^3 m^{-4}) = O(n^{-1}),$$

and, invoking (12),

$$\sum_{k=2}^{\overline{\mu}/2} \sum_{\mu=2k}^{\max\{\overline{\mu},2,2k\}} E_{n,k,\mu} = O(n^{-1}). \tag{13}$$

Let us now bound $E_{n,k,\mu}$ for the same k's, but $\mu > \max\{\overline{\mu}, 2, 2k\}$. To bound N_1 this time, we choose $r_1 = r_2 = \mu/k$. In particular, $r_i \ge 2.2$, thus bounded away from zero, just like the optimal r, the root of $rf_1(r)/f_2(r) = \mu/k$. (For $\mu = 2k$, the root would be zero!) Using $f_2(\mu/k) \le e^{\mu/k}$ and the notation $\le_p$ to hide a polynomial factor in n, we get

$$E_{n,k,\mu} \le_p \exp(nJ(x,y)), \quad x = k/n,\ y = \mu/n, \tag{14}$$

where

$$J(x,y) = 2H(x) - cH(y/c) + 2y\log\frac{x\rho e}{y} + x\log\frac{e^{\rho}}{f_2(\rho)^2},$$

and $(x,y) \in D$,

$$\begin{aligned} D &:= \{(x,y) |\, x \le \overline{x},\, y \ge \max\{\overline{y}, 2.2x\}\}, \\ \overline{x} &:= \frac{cz^2}{6\rho^2}, \quad \overline{y} := \frac{cz^2}{3\rho^2}. \end{aligned}$$

Notice at once that, for $(x,y) \in D$,

$$\begin{aligned} J_x(x,y) &= 2(-\log x + \log(1-x)) + \frac{2y}{x} + \log\frac{e^{\rho}}{f_2(\rho)^2} \\ &\ge 4\log z^{-1} + O(1) > 0, \end{aligned}$$

if z is small enough. For such a z, $J(x,y)$ increases with x for every y, as long as $(x,y) \in D$. In addition, the equation

$$J_y(x,y) = \ln\left(\frac{x^2\rho^2}{(c-y)y}\right) = 0$$

has two roots $y_{\pm}(x) > 0$,

$$y_{\pm} = \frac{1}{2}(c \pm \sqrt{c^2 - 4\rho^2 x^2}),$$

as $x \le 1/2$ and $\rho < c$, and $y_- = O(x^2) = O(z^4)$, $y_+ = c - O(z^4)$. In particular, $y^*(x) := \max\{\overline{y}, 2.2x\} \in (y_-, y_+)$, if z is sufficiently small. Furthermore, as a function of y, $J(x,y)$ decreases on $[y_-, y_+]$, and increases on $[y_+, c]$. Therefore,

$$\max\{J(x,y) : \, (x,y) \in D\} = \max_{x \le \overline{x}}\{J(x, y^*(x)), J(x,c)\}.$$

If $y^*(x) = 2.2x$, then

$$\begin{aligned} J(x,y) &= 2x\log(1/x) - 2.2x\log(1/x) + O(x) \\ &= -0.2x\log(1/x) + O(x) \\ &\le -0.05\frac{cz^2}{\rho^2}\log(1/z). \end{aligned}$$

If $y^*(x) = \overline{y}$, so that $x \le \overline{y}/2.2$, then

$$\begin{aligned} J(x,y) &= 2x\log(1/x) - \overline{y}\log(1/\overline{y}) + O(\overline{y}) \\ &\le \left(\frac{2}{2.2} - 1\right)\overline{y}\log(1/\overline{y}) + O(\overline{y}) \\ &\le -0.05\frac{cz^2}{\rho^2}\log(1/z). \end{aligned}$$

Finally

$$J(x,c) = -2(c-x)\log(1/x) + O(1).$$

Therefore, for z sufficiently small,

$$\max\{J(x,y) : \, (x,y) \in D\} \le -0.05\frac{cz^2}{\rho^2}\log\frac{1}{z} < 0.$$

From (14) we get then

$$E_{n,k,\mu} \le e^{-\alpha n}, \quad \alpha = -0.04\frac{cz^2}{\rho^2},$$

so

$$\sum_{k=2}^{\overline{\mu}/2} \sum_{\mu \ge \max\{\overline{\mu},2,2k\}} E_{n,k,\mu} \le e^{-\alpha^* n}, \quad \alpha^* = 0.03\frac{cz^2}{\rho^2}\log\frac{1}{z}.$$

Combining this with (13) and the definition of $\overline{\mu}$ in (11), we prove (9) with $\epsilon = \frac{cz^2}{6\rho^2}$. □

3.2 Bipartite Model

We now describe the graph model we will use for the remainder of our analysis. It is a bipartite version of the "random sequence model" considered in Section 2 of [14]. Let $\mu \ge 1$ and the disjoint sets R, C be given. R and C have meaning of a row set and a column set respectively, and μ a number of edges. For $\mathbf{x} \in R^\mu$ and $\mathbf{y} \in C^\mu$, we define a multi-bipartite graph $G_{\mathbf{x},\mathbf{y}}$ as having a vertex set $R + C$, and the edge set $E(G_{\mathbf{x},\mathbf{y}}) = \{(x_\ell, y_\ell);\ 1 \le \ell \le \mu\}$. Then the *degree* of $i \in R$ ($j \in C$ resp.) in $G_{\mathbf{x},\mathbf{y}}$ equals $d_{\mathbf{x}}(i) = |\{\ell \in [\mu] :\ x_\ell = i\}|$ ($d_{\mathbf{x}}(j) = |\{\ell \in [\mu] :\ y_\ell = j\}|$ resp.). Define

$$R^\mu_{\ge 1} = \{\mathbf{x} \in R^\mu :\ d_{\mathbf{x}}(i) \ge 1,\ i \in R\}, \quad C^\mu_{\ge 1} = \{\mathbf{y} \in C^\mu :\ d_{\mathbf{y}}(j) \ge 1,\ j \in C\}.$$

That is $R^\mu_{\ge 1}$, say, is a set of all $\mathbf{x}$ such that every $i \in R$ has positive degree in $\mathbf{x}$. For $\boldsymbol{\nu} = (v_{1,R}, v_{1,C}, v_R, v_C, \mu)$, let

$$\begin{aligned} \mathcal{B}_{R,C}(\boldsymbol{\nu}) = \{(\mathbf{x},\mathbf{y}) \in R^\mu_{\ge 1} \times C^\mu_{\ge 1} :& \\ &\text{there are } v_{1,R} \text{ indices of degree 1 in } R \\ &\text{there are } v_R \text{ indices of degree} \ge 2 \text{ in } R \\ &\text{there are } v_{1,C} \text{ indices of degree 1 in } C \\ &\text{there are } v_C \text{ indices of degree} \ge 2 \text{ in } C\} \end{aligned}$$

Thus $\mathcal{B}_{R,C}(\boldsymbol{\nu})$ is the set of all multi-bipartite graphs $G_{\mathbf{x},\mathbf{y}}$ without isolated vertices, and with the specified numbers of (*light*) vertices of degree 1, and of (*heavy*) vertices of degree 2 at least, separately among the row vertices and the column vertices.

We let

$$v_1 = v_{1,R} + v_{1,C} \text{ and } v = v_R + v_C.$$

Thus we consider $\mathcal{B}^{\delta \ge 2}_{n,m}$ to be the simple graphs among the collection $\mathcal{B}_{R,C}(\boldsymbol{\nu}^{(m)})$ where $|R| = |C| = n$ and $\boldsymbol{\nu}^{(m)} = (0,0,n,n,m)$.

We now discuss the distribution of the degree sequence of $G_{\mathbf{x},\mathbf{y}}$. Fix $\mathbf{x}, \mathbf{y}$ and let $R_1 = \{i \in R :\ d_{\mathbf{x}}(i) = 1\}$, $C_1 = \{j \in C :\ d_{\mathbf{y}}(j) = 1\}$.

Lemma 5. *Suppose that $(\mathbf{x},\mathbf{y})$ is chosen uniformly at random from $\mathcal{B}_{R,C}(\boldsymbol{\nu})$. Then $\vec{x} = \{d_{\mathbf{x}}(i) :\ i \in R \setminus R_1\}, \vec{y} = \{d_{\mathbf{y}}(j) :\ j \in C\}$ is distributed as*

$$(\vec{Z}, \vec{Z}') = (\{Z_i :\ i \in R \setminus R_1\}, \{Z'_j :\ j \in C \setminus C_1\}).$$

Here Z_i are independent copies of $Po(\lambda; \ge 2)$ conditioned on $\sum_{i \in R \setminus R_1} Z_i = \mu - v_{1,R}$, Z'_j are independent copies of $Po(\lambda'; \ge 2)$ conditioned on $\sum_{j \in C \setminus C_1} Z'_j = \mu - v_{1,C}$, and $\vec{Z}$, $\vec{Z}'$ are mutually independent. The parameters $\lambda > 0$, λ' are arbitrary.

Proof Since $\mathbf{x}$, $\mathbf{y}$ are mutually independent, it suffices to consider $\mathbf{x}$ only. Assume without loss of generality that $R \setminus R_1 = [v_R]$. Let $s = \mu - v_{1,R}$ and

$$S = \left\{ \vec{x} \in [s]^{v_R} : \sum_{1 \le i \le v_R} x_i = s \text{ and } \forall i,\, x_i \ge 2 \right\}.$$

Fix $\vec{\xi} \in S$. Then, by the definition of $\mathbf{x}$ and $\vec{Z} = \{Z_i\}_{i \in [v_R]}$,

$$\mathbf{Pr}(\vec{x} = \vec{\xi}) = \left(\frac{s!}{\xi_1!\xi_2!\dots\xi_v!} \right) \Big/ \left(\sum_{\vec{x} \in S} \frac{s!}{x_1!x_2!\dots x_v!} \right).$$

On the other hand,

$$\begin{aligned}
\mathbf{Pr}\left(\vec{Z} = \vec{\xi} \,\middle|\, \sum_{1 \le i \le v_R} Z_i = s \right) &= \left(\prod_{1 \le i \le v_R} \frac{z^{\xi_i}}{(e^z - 1 - z)\xi_i!} \right) \Big/ \left(\sum_{\vec{x} \in S} \prod_{1 \le j \le v_R} \frac{z^{x_i}}{(e^z - 1 - z)x_i!} \right) \\
&= \left(\frac{(e^z - 1 - z)^{-v_R} z^s}{\xi_1!\xi_2!\dots\xi_{v_R}!} \right) \Big/ \left(\sum_{\vec{x} \in S} \frac{(e^z - 1 - z)^{-v} z^s}{x_1!x_2!\dots x_{v_R}!} \right) \\
&= \mathbf{Pr}(\mathbf{x} = \vec{\xi}).
\end{aligned}$$

□

To make the most out of this underlying independence, we set $\lambda = z_R$, $\lambda' = z_C$, where

$$\frac{\mu - v_{1,R}}{v_R} = \frac{z_R(e^{z_R} - 1)}{f_R}, \quad \frac{\mu - v_{1,C}}{v_C} = \frac{z_C(e^{z_C} - 1)}{f_C},$$

where $f(x) = f_2(x) = e^x - 1 - x$ and we have abbreviated $f_R = f(z_R)$, $f_C = f_2(z_C)$. We will also use z, the root of

$$\frac{\mu - v_1}{v} = \frac{z(e^z - 1)}{f(z)}.$$

Lemma 6. *Suppose that $v_R z_R \to \infty$ and a is such that $a^2(v_R z_R)^{-1} \to 0$. Then*

$$\mathbf{Pr}\left(\sum_{i \in R \setminus R_1} Z_i = \mu - v_{1,R} - a \right) = \frac{1 + O((1 + a^2)(v_R z_R)^{-1})}{(2\pi v_R \mathbf{Var}(Z))^{1/2}},$$

where $Z = Po(z_R; \ge 2)$. An analogous estimate holds for the column set C.

Proof This follows immediately from Lemma 21 in the appendix. □

Notice that $\mathbf{Var}(Z) = \Theta(z_R)$. Using Lemma 6 with $a = 0$, we see that the probability of the conditioning event is of order $(v_R z_R)^{-1/2} \ge \Theta(n^{-1/2})$. So, a $\mathbf{qs}^2$ event expressed in terms of $\vec{Z}$ remains a $\mathbf{qs}$ event when $\vec{Z}$ is replaced by $\vec{x}$. The same relation holds between $\vec{Z}'$ and $\vec{y}$.

In particular, since

$$\mathbf{Pr}(Po(z_R; \ge 2) \ge \log n) = O(n^{-\Omega(\log\log n)})$$

we have that for $(\mathbf{x}, \mathbf{y})$ chosen uniformly from $\mathcal{B}_{R,C}(\boldsymbol{\nu}^{(m)})$,

$$\mathbf{Pr}(\max\{\Delta_{\mathbf{x}}, \Delta_{\mathbf{y}}\} > \log n) = O(n^{-\Omega(\log\log n)}), \tag{15}$$

where $\Delta_{\mathbf{x}} = \max_{j \in R}\{d_{\mathbf{x}}(j)\}$ and $\Delta_{\mathbf{y}} = \max_{j \in C}\{d_{\mathbf{y}}(j)\}$.

[2] A sequence of events $\mathcal{E}_n$ is said to occur *quite surely* (**qs**) if $\mathbf{Pr}(\mathcal{E}_n) = 1 - O(n^{-K})$ for any constant $K > 0$.

Lemma 7. *Suppose* $(\mathbf{x},\mathbf{y})$ *is chosen randomly from* $\mathcal{B}_{R,C}(\boldsymbol{\nu}^{(m)})$ *where* $R = C = [n], m = cn$.

(a) *Conditional on being simple,* $G_{\mathbf{x},\mathbf{y}}$ *is distributed as* $B^{\delta\geq 2}_{n,cn}$.

(b) $\mathbf{Pr}(G_{\mathbf{x},\mathbf{y}}$ *is simple*$) \sim \exp\left(-\frac{\rho^2}{2(1-e^{-\rho})^2}\right)$ *where* $\frac{\rho(e^\rho-1)}{e^\rho-1-\rho} = c$.

Proof
(a) If $G_{\mathbf{x},\mathbf{y}}$ is simple then it has vertex set $[n]+[n]$ and m edges. Also, there are $(m!)^2$ distinct equally likely values of $(\mathbf{x},\mathbf{y})$ which yield the same graph.

(b) If we condition on the degree sequence $d_{\mathbf{x}}, d_{\mathbf{y}}$ then given $\max\{\Delta_{\mathbf{x}},\Delta_{\mathbf{y}}\} \leq \log n$,

$$\mathbf{Pr}(G_{\mathbf{x},\mathbf{y}} \text{ is simple}) \sim e^{-\lambda(d_{\mathbf{x}})\lambda(d_{\mathbf{y}})/2}$$

where if $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ then $\lambda(\mathbf{a}) = \frac{1}{m}\sum_{i=1}^n a_i(a_i-1)$, see for example [16]. Now

$$\mathbf{E}(n\lambda(d_{\mathbf{x}})) = \mathbf{E}(n\lambda(d_{\mathbf{y}})) \sim n\frac{\rho}{1-e^{-\rho}}$$

and this is true conditional on the **qs** event $\max\{\Delta_{\mathbf{x}},\Delta_{\mathbf{y}}\} \leq \log n$.

Now the random variable $n\lambda(d_{\mathbf{x}})$ is the sum of independent random variables and it is easy to show concentration around its mean. Thus for example $\mathbf{Pr}\left(\left|\lambda(d_{\mathbf{x}}) - \frac{\rho}{1-e^{-\rho}}\right| \geq n^{-1/3}\right) \leq n^{-a}$ for any constant $a > 0$ and the lemma follows. □

3.3 Proof of Lemma 2

Let $\omega = \lceil K \log n \rceil$ for some large constant $K > 0$. Consider the bipartite graph Γ with vertex set $\mathcal{B}^{\delta\geq 2}_{n,m-\omega} + \mathcal{B}^{\delta\geq 2}_{n,m}$ and an edge (G,H) iff

$$E(G) \subseteq E(H) \text{ and } E(G) \setminus E(H) \text{ is a matching.}$$

Consider the following experiment SAMPLE:

- Choose G randomly from $\mathcal{B}^{\delta\geq 2}_{n,m-\omega}$
- Add a random matching M, disjoint from $E(G)$ of size ω to obtain $H \in \mathcal{B}^{\delta\geq 2}_{n,m}$.

This induces a probability measure $\mathbf{Q}$ on $\mathcal{B}^{\delta\geq 2}_{n,m}$. Let d_Γ denote degree in Γ.

Lemma 8.

$$G \in \mathcal{B}^{\delta\geq 2}_{n,m-\omega} \textit{ implies } \qquad \frac{(n^2-m-2\omega n)^\omega}{\omega!} \leq d_\Gamma(G) \leq \binom{n^2}{\omega}.$$

Proof The RHS is obvious. For the LHS let us bound from below the number of *ordered* sequences $e_1, e_2, \ldots, e_\omega$ of ω edges which are disjoint from $E(G)$ and form a matching. Observe that after choosing $e_1, e_2, \ldots, e_i$ we rule out at most $m - \omega + 2in$ choices for e_{i+1}. (The $m-\omega$ edges of G plus the further $\leq 2in$ edges incident with $e_1, e_2, \ldots, e_i$). Thus there are always at least $n^2 - m - 2\omega n$ choices for e_{i+1}. Dividing by $\omega!$ accounts for removing the ordering. □

Thus for n large and $G, G' \in \mathcal{B}^{\delta\geq 2}_{n,m-\omega}$,

$$\left|\frac{d_\Gamma(G)}{d_\Gamma(G')} - 1\right| \leq \frac{4\omega^2}{n}. \tag{16}$$

We now consider the degrees $d_\Gamma(H)$ for $H \in \mathcal{B}^{\delta\geq 2}_{n,m}$.

For $H \in \mathcal{B}^{\delta\geq 2}_{n,m}$ let $E_>(H)$ be the edges of H joining vertices of degree at least 3. If $e \in E(H)\setminus E(G)$ then other edges of H incident to e must already be in $E(G)$. So, if (G,H) is an edge of Γ then $E(H) \setminus E(G) \subseteq E_>(H)$.

Lemma 9. *Let*

$$\theta = c^{-1}\rho^2, \text{ where } \frac{\rho(e^\rho - 1)}{e^\rho - 1 - \rho} = c.$$

If H is chosen uniformly at random from $\mathcal{B}^{\delta\geq 2}_{n,m}$ then **qs**

(a)

$$\Delta(H) \leq \log n.$$

(b)

$$|E_>(H) - \theta n| = O(n^{1/2}\log n).$$

Proof Let $\mathbf{x}, \mathbf{y}$ be chosen uniformly from $\mathcal{B}_{R,C}(\boldsymbol{\nu}^{(m)})$. Part (a) follows from (15) and Lemma 7. Now to part (b). Let W be the number of pairs (x_i, y_i), $i \leq m$ such that $d_\mathbf{x}(x_i)$, $d_\mathbf{y}(y_i) \geq 3$. We know that, conditioned on simplicity, $W = E_>(H)$. We see that

$$\mathbf{E}(W) = \frac{m_{\mathbf{x},3}m_{\mathbf{y},3}}{m}, \tag{17}$$

where

$$m_{\mathbf{x},3} = m - 2|\{j \in R:\ d_\mathbf{x}(j) = 2\}| \text{ and } m_{\mathbf{y},3} = m - 2|\{j \in C:\ d_\mathbf{y}(j) = 2\}|$$

Now, in the notation of Lemma 5,

$$\mathbf{E}\left(\sum_{i=1}^n Z_i - 2|\{i:\ Z_i = 2\}|\right) = n\rho, \tag{18}$$

The sum in (18) is of independent random variables and it is straightforward to show enough concentration around the mean to prove that

$$\left|\sum_{i=1}^n Z_i - 2|\{i:\ Z_i = 2\}| - n\rho\right| \leq n^{1/2}\log n \qquad \mathbf{qs}.$$

It then follows from Lemmas 6, 7 that

$$|m_{\mathbf{x},3} - n\rho| \leq n^{1/2}\log n \text{ and similarly } |m_{\mathbf{y},3} - n\rho| \leq n^{1/2}\log n \qquad \mathbf{qs}. \tag{19}$$

Suppose now the condition (19) holds, which we call the event $\mathcal{E}_1$. Then $\mathcal{E}_1$ holds **qs**. It follows from (17) that

$$\mathbf{E}(W \mid \mathcal{E}_1) = \theta n + O(n^{1/2}\log n), \quad \theta := c^{-1}\rho^2. \tag{20}$$

Assume $\mathcal{E}_1$ holds and fix $\mathbf{x}$ completely and fix $\mathbf{y}$ up to a random permutation. Call the conditional probability space Ψ. We appeal to the Azuma-Hoeffding inequality to show that in Ψ, W is tightly concentrated around its mean. The A-H inequality applies since transposing any two elements of a permutation of $\mathbf{y}$ may change W by at most 2, see Appendix C. So, for every $u > 0$,

$$\mathbf{Pr}_\Psi(|W - \mathbf{E}_\Psi(W)| \geq u) \leq 2e^{-u^2/(8cn)}.$$

Removing the conditioning on $\mathcal{E}_1$ we obtain

$$\mathbf{Pr}(|W - \mathbf{E}(W \mid \mathcal{E}_1)| \geq u) \leq \mathbf{Pr}(\mathcal{E}_1^c) + 2e^{-u^2/(8cn)}.$$

So, substituting $u = n^{1/2}\log n$ and using (20), we see that

$$|W - \theta n| \leq An^{1/2}\log n \qquad \mathbf{qs},$$

if the constant A is sufficiently large. Recalling that $W = E_>(H)$ on the event $\mathcal{E}_0$, and that $\mathbf{Pr}(\mathcal{E}_0)$ is of order n^{-1}, we have proved the part (b). □

Now let $\tilde{\mathcal{B}}$ be the set of $H \in \mathcal{B}_{n,m}^{\delta\geq 2}$ satisfying the conditions of the above lemma i.e.

- The number of edges joining two vertices of degree ≥ 3 is in the range $\theta n \pm An^{1/2}\log n$ for some constant $A > 0$.
- The maximum degree $\Delta(H) \leq \log n$.

According to the lemma

$$|\mathcal{B}_{n,m}^{\delta\geq 2} \setminus \tilde{\mathcal{B}}| \leq |\tilde{\mathcal{B}}|n^{-K}, \quad \forall K > 0. \tag{21}$$

Note next that

Lemma 10.

$H \in \tilde{\mathcal{B}}$ implies

$$\frac{(\theta n - An^{1/2}\log n - 2\omega\log n)^\omega}{\omega!} \leq d_\Gamma(H) \leq \binom{\theta n + An^{1/2}\log n}{\omega}.$$

Proof The upper bound is obvious. As in Lemma 8, for the LHS let us bound from below the number of *ordered* sequences $e_1, e_2, \ldots, e_\omega$ of ω edges which are contained in $E_>(H)$ and form a matching. Observe that after choosing $e_1, e_2, \ldots, e_i$ we rule out at most $2i\Delta$ choices for e_{i+1}. Thus there are always at least $\theta n - An^{1/2}\log n - 2\omega\Delta$ choices for e_{i+1}. Dividing by $\omega!$ accounts for removing the the ordering. □

So for $H, H' \in \tilde{\mathcal{B}}$,

$$\left|\frac{d_\Gamma(H)}{d_\Gamma(H')} - 1\right| \leq \frac{2A\omega\log n}{\theta n^{1/2}}. \tag{22}$$

Finally, for $H \in \mathcal{B}_{n,m}^{\delta\geq 2} \setminus \tilde{\mathcal{B}}$, $H' \in \tilde{\mathcal{B}}$,

$$\frac{d_\Gamma(H)}{d_\Gamma(H')} \leq \frac{\binom{m}{\omega}}{\frac{(\theta n - An^{1/2}\log n - 2\omega\log n)^\omega}{\omega!}} \leq \left(\frac{2c}{\theta}\right)^\omega, \tag{23}$$

as the total number of ways to delete a matching of size ω from $H \in \mathcal{B}_{n,m}^{\delta\geq 2}$ is $\binom{m}{\omega}$ at most.

Let $G_0 \in \mathcal{B}_{n,m-\omega}^{\delta\geq 2}$ be fixed. By (16), if $H \in \mathcal{B}_{n,m}^{\delta\geq 2}$ then

$$\begin{aligned}
\mathbf{Q}(H) &= \mathbf{Pr}(\textsc{sample chooses } H) \\
&= \frac{1}{|\mathcal{B}_{n,m-\omega}^{\delta\geq 2}|} \times \sum_{(G,H)\in E(\Gamma)} \frac{1}{d_\Gamma(G)} \\
&= \frac{1 + O(\omega^2/n)}{|\mathcal{B}_{n,m-\omega}^{\delta\geq 2}|} \cdot \frac{d_\Gamma(H)}{d_\Gamma(G_0)}.
\end{aligned} \tag{24}$$

From this relation, (22), and (23), it follows that

$$H, H' \in \tilde{\mathcal{B}} \quad \text{implies} \quad \left| \frac{\mathbf{Q}(H)}{\mathbf{Q}(H')} - 1 \right| \leq \frac{3A\omega \log n}{\theta n^{1/2}}, \tag{25}$$

$$H \in \mathcal{B}_{n,m}^{\delta \geq 2} \setminus \tilde{\mathcal{B}},\ H' \in \tilde{\mathcal{B}} \quad \text{implies} \quad \frac{\mathbf{Q}(H)}{\mathbf{Q}(H')} \leq \left(\frac{3c}{\theta}\right)^{\omega}. \tag{26}$$

Furthermore, invoking also

$$\sum_{G \in \mathcal{B}_{n,m-\omega}^{\delta \geq 2}} d_\Gamma(G) = \sum_{H \in \mathcal{B}_{n,m}^{\delta \geq 2}} d_\Gamma(H),$$

and picking $H' \in \tilde{\mathcal{B}}$, we obtain (see (16), (22)):

$$\frac{d_\Gamma(H')}{d_\Gamma(G_0)} \leq \left(1 + \frac{3A\omega \log n}{\theta n^{1/2}}\right) \frac{|\mathcal{B}_{n,m-\omega}^{\delta \geq 2}|}{|\tilde{\mathcal{B}}|}. \tag{27}$$

Combining (21), (24), (26), and (27), we get: for every $K > 0$,

$$\begin{aligned} \mathbf{Q}(\mathcal{B}_{n,m}^{\delta \geq 2} \setminus \tilde{\mathcal{B}}) &\leq \mathbf{Q}(H') \left(\frac{3c}{\theta}\right)^{\omega} \cdot n^{-2K} |\tilde{\mathcal{B}}| \\ &= \frac{1 + O(\omega^2/n)}{|\mathcal{B}_{n,m-\omega}^{\delta \geq 2}|} \cdot \frac{d_\Gamma(H')}{d_\Gamma(G_0)} \left(\frac{3c}{\theta}\right)^{\omega} n^{-2K} |\tilde{\mathcal{B}}| \\ &= O\big((3c/\theta)^{\omega} n^{-2K}\big) \\ &\leq n^{-K}. \end{aligned} \tag{28}$$

Since $\mathbf{Q}(\mathcal{B}_{n,m}^{\delta \geq 2}) = 1$, from (28) and (25) we deduce that, for $H \in \tilde{\mathcal{B}}$,

$$\left| \mathbf{Q}(H) - \frac{1}{|\mathcal{B}_{n,m}^{\delta \geq 2}|} \right| \leq \frac{1}{|\mathcal{B}_{n,m}^{\delta \geq 2}|} \times \frac{4A\omega \log n}{\theta n^{1/2}} \times \frac{1}{1 - n^{-K}} \leq \frac{1}{|\mathcal{B}_{n,m}^{\delta \geq 2}|} \times \frac{5A\omega \log n}{\theta n^{1/2}}. \tag{29}$$

This means that on the graph set $\tilde{\mathcal{B}}$ the probability measure $\mathbf{Q}$ is almost uniform. It is worth repeating that $\mathbf{Q}$ is induced by picking a random $G \in \mathcal{B}_{n,m-\omega}^{\delta \geq 2}$, and adding to G a random matching M of cardinality ω which is disjoint from $E(G)$.

Now let $\mathbf{Pr}_M$ denote probability w.r.t. a graph chosen uniformly from $\mathcal{B}_{n,M}^{\delta \geq 2}$ and let $\mu^*(G)$ denote the size of the largest matching in G. We want to prove, using the near uniformity of $\mathbf{Q}$, that $\mathbf{Pr}_m(\mu^*(G) = n) \to 1$, if $m = cn$ and $c > 2$.

From the previous part we know that there exists $\alpha = \alpha(c)$ such that

$$\mathbf{Pr}_{m-\omega}(|N(S)| \geq |S| : \forall S \subset [n],\ 0 < |S| \leq \alpha n) \geq 1 - \frac{\gamma}{n} \tag{30}$$

for some $\gamma = \gamma(c)$. (Here S is a set rows, or a set of columns.)

Now, given $G \in \mathcal{B}_{n,m-\omega}^{\delta \geq 2}$ such that $\mu^*(G) \in [n-t, n)$ and the event in (30), fix some matching M of size $n - t$ and let x be a row vertex and y be a column vertex not covered by M.

Suppose G does not contain a matching of size $n - t + 1$, i.e. $\mu^*(G) = n - t$. Let A be the set of row vertices reachable from x by an alternating path w.r.t. M, and let B, the set of column vertices, be defined analogously for y. (Of course, the sets A and B depend on the choice of a maximum matching M. To achieve uniqueness, we assume that M is the lexicographically first among all maximum matchings.) Each such path is of even length, and we include x into A, and y into B, as corresponding to the paths of zero length. There does not exist an edge connecting A and B, since otherwise we could use the resulting path between x and y to get, in a standard way, a larger matching. (Therefore if any of the ω edges added to G in SAMPLE join A to B, $\mu^*(G') > \mu^*(G)$ for the new graph G'.) Furthermore, for every row vertex in A, all its column neighbors must be covered by M, since otherwise there would exist an alternating path connecting x and an uncovered column vertex, and there would exist a larger matching M'. This implies that $N(A)$ consists of all column vertices on the paths from x, so that $|N(A)| = |A| - 1$, as x is the only vertex in A not covered by M. Similarly, $|N(B)| = |B| - 1$. Then necessarily $|A| \geq \alpha n$, $|B| \geq \alpha n$. So if G is such that the event in (30) holds, then—conditioned on G—the probability that none of the ω added edges of SAMPLE join A to B is at most

$$\left(1 - \frac{(\alpha n - \omega)^2}{n^2}\right)^\omega \leq \left(1 - \frac{\alpha^2}{2}\right)^\omega \leq \frac{\gamma}{n},$$

if we pick K in $\omega = \lceil K \log n \rceil$ sufficiently large. Therefore, if $H \in \mathcal{B}_{n,m}^{\delta \geq 2}$ then

$$\begin{aligned} \mathbf{Q}(\mu^*(H) < n - t + 1) &\leq \mathbf{Pr}_{m-\omega}(\mu^*(G) < n - t) + \frac{\gamma}{n} + (1 - \alpha^2/2)^\omega \leq \xi(m - \omega, t) + \frac{2\gamma}{n}; \\ \xi(m - \omega, t) : &= \mathbf{Pr}_{m-\omega}(\mu^*(G) < n - t) = 1 - \mathbf{Pr}_{m-\omega}(\mu^*(G) \geq n - t). \end{aligned}$$

So

$$\mathbf{Q}(\{\mu^*(H) < n - t + 1\} \wedge \{H \in \tilde{\mathcal{B}}\}) \leq \xi(m - \omega, t) + \frac{2\gamma}{n}$$

and then, using (29),

$$\mathbf{Pr}_m(\{\mu^*(H) < n - t + 1\} \wedge \{H \in \tilde{\mathcal{B}}\}) \leq \left(\xi(m - \omega, t) + \frac{2\gamma}{n}\right)\left(1 + \frac{5A\omega \log n}{\theta n^{1/2}}\right)$$

and

$$\begin{aligned} \mathbf{Pr}_m(\mu^*(H) < n - t + 1) &\leq \left(\xi(m - \omega, t) + \frac{2\gamma}{n}\right)\left(1 + \frac{5A\omega \log n}{\theta n^{1/2}}\right) + n^{-K} \\ &\leq \xi(m - \omega, t) + \frac{6A\omega \log n}{\theta n^{1/2}}, \end{aligned}$$

(where n^{-K} bounds $\mathbf{Pr}_m(H \notin \tilde{\mathcal{B}})$). □

3.4 Proof of Lemma 3

We now go back to our analysis of the graph $G_{\mathbf{x},\mathbf{y}}$.

In our analysis below we will only need to consider graphs for which

$$v_1 \leq_b n^{.32} < n^{.66} \leq_b v_R, v_C \leq n. \tag{31}$$

$$z_R, z_C \leq 3c. \tag{32}$$

We next look at the expected number of vertices of a given degree in $G_{\mathbf{x},\mathbf{y}}$. We use the notation $V_{k,X}$, $X = R, C$ to denote the set of vertices of degree of degree k in X and $v_{k,X} = |V_{k,X}|$.

Lemma 11. *For vertices $i \in R \setminus R_1, j \in C \setminus C_1$, and $2 \leq k, l \leq \log n$,*

$$\mathbf{Pr}(d_{\mathbf{x}}(i) = k) = \frac{z_R^k}{k! f_R}\left(1 + O\left(\frac{(\log v)^2}{v_R z_R}\right)\right) \tag{33}$$

$$\mathbf{Pr}(d_{\mathbf{x}}(i) = k, d_{\mathbf{y}}(j) = \ell) = \frac{z_R^k}{k! f_R}\frac{z_C^\ell}{\ell! f_C}\left(1 + O\left(\frac{(\log v)^2}{v_R z_R} + \frac{(\log v)^2}{v_C z_C}\right)\right). \tag{34}$$

Proof Since $\mathbf{x}$,$\mathbf{y}$ are independent, it is enough to prove (33). Using Lemma 6,

$$\begin{aligned}
\mathbf{Pr}(d_{\mathbf{x}}(i)=k) &= \frac{\mathbf{Pr}\left(Y_i=k \text{ and } \sum_{j\neq i} Y_j = v_R - v_{1,R} - k\right)}{\mathbf{Pr}\left(Y_i=k \text{ and } \sum_{j} Y_j = v_R - v_{1,R}\right)} \\
&= \frac{\frac{z_R^k}{f_R k!}\frac{1+O(k^2(v_R z_R)^{-1})}{(2\pi(v_R-1)\mathbf{Var}(Y))^{1/2}}}{\frac{1+O((v_R z_R)^{-1})}{(2\pi v_R \mathbf{Var}(Y))^{1/2}}} \\
&= \frac{z_R^k}{k! f_R}\left(1+O\left(\frac{(\log v)^2}{v_R z_R}\right)\right)
\end{aligned}$$

□

Thus we can write that for $2 \leq k \leq \log n$,

$$\mathbf{E}(v_{k,X}) = \frac{v_X z_X^k}{k! f_X} + O\left(\frac{(\log v)^2}{z_X}\right) \tag{35}$$

$$\mathbf{E}(v_{k,R} v_{\ell,C}) = \frac{v_R v_C z_R^k z_C^\ell}{k!\ell! f_R f_C} + O\left(\frac{v(\log v)^2}{z_R} + \frac{v(\log v)^2}{z_C}\right) \tag{36}$$

We prove Lemma 3 by considering the following algorithm for finding a matching in a graph G. It is a technical modification of one described first by Karp and Sipser [14]. We apply the algorithm to the bipartite multigraph $G = G_{\mathbf{x},\mathbf{y}}$ where $(\mathbf{x},\mathbf{y})$ is chosen randomly from $\mathcal{B}_{R,C}(\boldsymbol{\nu}^{(m)})$ i.e. $R = C = [n], v_1 = 0, v = 2n, m = cn$. In the light of Lemma 7, we need only show that the following algorithm KSGREEDY finds a matching of size $n - O(n^{.49})$ with sufficiently high probability.

KSGREEDY

begin
  $M \leftarrow \emptyset$;
  while $E(G) \neq \emptyset$ **do**
  begin
    ***A*1**: If G has vertices of degree one in R and C, choose one, x say, randomly from R if $v_R \leq v_C$ and randomly from C otherwise.
    If $v_{1,R} = 0$ or $v_{1,C} = 0$ and $v_1 > 0$ choose x randomly from the set with vertices of degree 1.
    Let $e = \{x, y\}$ be the unique edge of G incident with x;
    ***A*2**: Otherwise, (no vertices of degree one) choose
    $e = \{x, y\} \in E$ randomly
    $G \leftarrow G \setminus \{x, y\}$;
    $M \leftarrow M \cup \{e\}$
  end;
  Output M
end

The reason for choosing a vertex of R when $v_R \leq v_C$ in Step A1 is that we must try to ensure that $|v_R - v_C|$ does not grow too large. This is because $|v_R - v_C|$ is a lower bound on the number of isolated vertices that will be created from now on. The choice of R in this case reduces $|v_R - v_C|$, in expectation.

KSGREEDY is defined on graphs. Formally, we need to define its action on pairs of sequences $\mathbf{x}, \mathbf{y}$. As in [1] we use $\star$'s to denote deleted edges i.e. if $x_i = a, y_i = b$ and the algorithm requires the removal of edge (a,b) then we make the assignments $x_i = y_i = \star$. Thus at a general step of the algorithm we are left with a pair of sequences $\mathbf{x}, \mathbf{y}$ from $([n] \cup \{\star\})^m$ which satisfy $x_i = \star$ iff $y_i = \star$ for $i \in [m]$. The sets R, C are defined by $R = \{j \in [n] : \exists i \text{ such that } x_i = j\}$ and $C = \{j \in [n] : \exists i \text{ such that } y_i = j\}$. The edges of this extended definition of $G_{\mathbf{x},\mathbf{y}}$ are simply $\{(x_i, y_i) : x_i \neq \star\}$. The next step in analogy to the argument from [1] is relate the evolution of $G_{\mathbf{x},\mathbf{y}}$ to a Markov chain on $\boldsymbol{\nu}$. So let $\boldsymbol{\nu}(0) = (0, 0, n, n, cn)$ and let $\boldsymbol{\nu}(t), t \geq 0$ be the sequence of states seen during KSGREEDY. The following lemma can be justified by arguments similar to those used for Lemma 3 of [1].

Lemma 12. *The random sequence $\boldsymbol{\nu}(t)$, $t = 0, 1, 2, \ldots$, is a Markov chain.*

Proof See Appendix A. □

We shall for convenience introduce a stopping time $\mathcal{S}$ where

$$\mathcal{S} = \begin{cases} \min\{t > 0 : \ |\boldsymbol{\nu}(t) - \boldsymbol{\nu}(t-1)| \geq \log n) & \text{if such } t \text{ exist} \\ n & \text{otherwise} \end{cases}$$

Note that

$$\mathbf{Pr}(\exists t : \ |\boldsymbol{\nu}(t) - \boldsymbol{\nu}(t-1)| \geq \log n) = O(n^{-K}) \tag{37}$$

for any constant $K > 0$. This follows from (15).

Note that $t < \mathcal{S}$ implies that

$$|z(t+1) - z(t)| = O\left(\frac{\log n}{v}\right). \tag{38}$$

$\mathcal{S}$ is our generic stopping time for the first occurrence of one of some unlikely events. As we proceed, we will find other unlikely events and we will update $\mathcal{S}$ accordingly, with just a remark. Also, in the next section we work under the asumption that $t < \mathcal{S}$.

3.4.1 One step parameter changes

We now consider the the expected change in $\boldsymbol{\nu}$ due to one step of KSGREEDY.

Notice that

$$z \leq \frac{2\mu - v_1}{v},$$

the fraction being the average degree of a heavy vertex. Now a simple calculation shows that with probabilty $1 - O(n^{-4})$

$$\text{no vertex subset of } B_{n,m}^{\delta \geq 2} \text{ has average degree more than } 3m/n. \tag{39}$$

Given d, the property $P = \{G : \text{no vertex subset of } G \text{ has average degree more than } d\}$ is monotone increasing. Let two states $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ be such that the transition probability $p(\boldsymbol{\nu}'|\boldsymbol{\nu})$ is positive. Let G be chosen uniformly among all $G_{\mathbf{x},\mathbf{y}}$ such that $\boldsymbol{\nu}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\nu}$. One step of KSGREEDY applied to G produces a subgraph $G' = G_{\mathbf{x}',\mathbf{y}'}$. We know that $\mathbf{Pr}(\boldsymbol{\nu}(G') = \boldsymbol{\nu}'|G) = p(\boldsymbol{\nu}'|\boldsymbol{\nu}) > 0$, and that, conditioned on $\boldsymbol{\nu}(G') = \boldsymbol{\nu}'$, the graph G' is distributed unformly. Thus, for $p(\boldsymbol{\nu}'|\boldsymbol{\nu}) > 0$, we can couple two random graphs G and G', distributed uniformly on the set of all graphs with $\boldsymbol{\nu}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\nu}$ and $\boldsymbol{\nu}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\nu}'$ respectively, and such that $G' \subset G$. This means that $\mathbf{Pr}(G(t) \in P) \geq \mathbf{Pr}(G(t+1) \in P)$, $t \geq 0$. Using (39)), we get: for every $t \geq 0$, with probability $1 - O(n^{-4})$

$$\text{no vertex subset of } G(t) \text{ has average degree more than } 3m/n.$$

Hence, with probability $1 - O(n^{-3})$ the last event holds for all t simultaneously. So we will proceed assuming that

$$z \leq 3c. \tag{40}$$

Next let

$$\gamma = \gamma(\boldsymbol{\nu}) = |v_R - v_C| + v_1 \text{ and } \theta_v = \frac{\gamma}{v}.$$

A simple estimation, under the assumption that $\theta_v = o(1)$, yields

$$\left| \frac{2m - v_1}{v} - \frac{m - v_{1,R}}{v_R} \right| = \left| \frac{m(v_R - v_C)}{vv_R} - \frac{v_1}{v} + \frac{v_{1,R}}{v_R} \right| \leq 8c\theta_v.$$

Let $g(x) = \frac{x(e^x-1)}{e^x-1-x}$. We know that g assumes values $(\mu - v_{1,X})/v_X$ at z_X, $X = R, C$, and $(2\mu - v_1)/v$ at z. Then calculations yield that $g'(x) = \frac{x^2e^x+2e^x-1-e^{2x}}{(e^x-1-x)^2} \in [1, 3]$. It follows immediately that there exists a constant c_1 such that

$$|z_R - z|, \quad |z_C - z| \leq c_1\theta_v.$$

Thus with $f = f_2(z)$ we can replace (35) and (36) by

$$\mathbf{E}(v_{k,X}) = \frac{vz^k}{2k!f} + O\left(v\theta_v \log v + \frac{(\log v)^2}{z}\right) \tag{41}$$

$$\mathbf{E}(v_{k,R}v_{\ell,C}) = \frac{v^2z^{k+\ell}}{4k!\ell!f^2} + O\left(v^2\theta_v \log v + \frac{v(\log v)^2}{z}\right) \tag{42}$$

In the following we will abbreviate the error terms to

$$\Theta = O\left(\theta_v \log v + \frac{(\log v)^2}{vz}\right).$$

In the analysis we will be able to concentrate on cases where

$$z \geq n^{-.17},\ v = \Omega(nz^2) \text{ and } \gamma = O(n^{.28}(\log n)^8). \tag{43}$$

Thus

$$\Theta = o(z^2) \quad \textit{throughout}. \tag{44}$$

Now we go through the steps of the algorithm and compute the expected changes in the parameters.

Case 1: Deleting a vertex x of degree 1 and its neighbour y. Assume that $x \in X$ and $\overline{X} = \{R, C\} \setminus X$.

Let $\boldsymbol{\nu}'$ refer to the state after one step.

Lemma 13. *Assume that* $\log n = O((vz)^{1/2})$ *and the conditions (31), (32) hold. Let* $\mathcal{I}$ *be the event that KSGREEDY removes an isolated edge in this step. Then* **qs**

(a) *If* $\mathcal{I}$ *occurs then*

$$v'_{1,R} = v_{1,R} - 1,\ v'_{1,C} = v_{1,C} - 1,\ m' = m - 1,\ v'_R = v_R,\ v'_C = v_C.$$

(b) *If $\mathcal{I}$ does not occur then*

$$\begin{aligned}
\mathbf{E}(v'_{1,X} \mid \boldsymbol{\nu}) &= v_{1,X} - 1 + \frac{z^2 e^z}{(e^z-1)^2} + \Theta \\
v'_{1,\overline{X}} &= v_{1,\overline{X}} && (45)\\
\mathbf{E}(v'_1 \mid \boldsymbol{\nu}) &= v_1 - 1 + \frac{z^2 e^z}{(e^z-1)^2} + \Theta && (46)\\
\mathbf{E}(v'_X \mid \boldsymbol{\nu}) &= v_X - \frac{z^2 e^z}{(e^z-1)^2} + \Theta && (47)\\
\mathbf{E}(v'_{\overline{X}} \mid \boldsymbol{\nu}) &= v_{\overline{X}} - 1 + \Theta && (48)\\
\mathbf{E}(v' \mid \boldsymbol{\nu}) &= v - 1 - \frac{z^2 e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v'_X - v'_{\overline{X}} \mid \boldsymbol{\nu}) &= v_X - v_{\overline{X}} + 1 - \frac{z^2 e^z}{(e^z-1)^2} + \Theta && (49)\\
\mathbf{E}(\mu' \mid \boldsymbol{\nu}) &= \mu - 1 - \frac{z e^z}{e^z - 1} + \Theta \\
\mathbf{E}(v_{0,X} \mid \boldsymbol{\nu}) &= O\left(\frac{v_1 + (\log n)^2}{\mu}\right) \\
\mathbf{E}(v_{0,\overline{X}} \mid \boldsymbol{\nu}) &= 0.
\end{aligned}$$

Here $v_{0,X}$ is the number of isolated vertices in $X(= R, C)$ that are created by the step and we will let $v_0 = v_{0,R} + v_{0,C}$.

Furthermore, if $|v_X - v_{\overline{X}}| \geq \log n$ then

$$\begin{aligned}
&v_X > v_{\overline{X}} \textit{ implies } \gamma' = \gamma - v_0. && (50)\\
&v_X \leq v_{\overline{X}} \textit{ implies } \gamma' = \gamma + 2(v'_1 - v_1) + v_0 && (51)\\
&v_X \leq v_{\overline{X}} \textit{ implies } \mathbf{E}(\gamma' \mid \boldsymbol{\nu}) = \gamma - 2\left(1 - \frac{z^2 e^z}{(e^z-1)^2}\right) + \Theta. && (52)
\end{aligned}$$

Proof Assume that $X = R$. We begin by conditioning on the degree sequence of $\mathbf{x}, \mathbf{y}$, assuming that it meets the conditions (31), (32). As we know, under these conditions, **qs** the maximum degree is at most $\log n$, (15). So we proceed assuming that this maximum degree condition holds. Now $\mathbf{x}$,$\mathbf{y}$ are just *mutually independent* random permutations of the multi-sets $\prod_{i \in R} i^{d_{\mathbf{x}}(i)}$ and $\prod_{j \in C} j^{d_{\mathbf{y}}(j)}$, respectively. Suppose we delete $x \in R$ of degree 1 and its neighbour $y \in C$.

If $\mathcal{I}$ occurs, i.e. $y \in C_1$ then $v'_{1,X} = v_{1,X} - 1, v'_X = v_X$ $X = R, C$ and $\mu' = \mu - 1$. We note that

$$\mathbf{Pr}(y \in C_1 \mid \mathbf{x}, \mathbf{y}) = \frac{v_{1,C}}{\mu} = \Theta,$$

so this case only contributes to the error term. Note also that (50), (51) hold in this case.

Let $\mathcal{I}$ not occur, so that $d_{\mathbf{y}}(y) \geq 2$. Now for $2 \leq k \leq \log n$ we have

$$\mathbf{Pr}(d_{\mathbf{y}}(y) = k \mid \mathbf{x}, \mathbf{y}) = \frac{k v_{k,C}}{\mu}. \tag{53}$$

Add the event $\{d_{\mathbf{y}}(y) = k\}$ to the conditioning on $\mathbf{x}, \mathbf{y}$, denoting the resulting conditioning by

$\mathcal{H}$. Suppose y has $k_{i,j}$ neighbours (excluding x) of degree i that are joined to y by j edges. Then

$$\begin{aligned}
\mu' &= \mu - k && (54)\\
v'_{1,C} &= v_{1,C} && (55)\\
v'_{1,R} &= v_{1,R} - 1 - k_{1,1} + \sum_{i\geq 2} k_{i,i-1} && (56)\\
v'_C &= v_C - 1 && (57)\\
v'_R &= v_R - \sum_{i\geq 2} k_{i,i-1} - \sum_{i\geq 2} k_{i,i} && (58)\\
v'_{0,R} &= \sum_{j\geq 1} k_{j,j} && (59)\\
v'_{0,C} &= 0 && (60)
\end{aligned}$$

Then we have

$$v_0 = \sum_{i\geq 1} k_{i,i}$$

and

$$v'_1 - v_1 = -1 - k_{1,1} + \sum_{i\geq 2} k_{i,i-1}.$$

Assuming $|v_R - v_C| > \log n$ we have

$$\gamma' - \gamma = \begin{cases} v'_1 - v_1 - 1 + \sum_{i\geq 2} k_{i,i-1} + \sum_{i\geq 2} k_{i,i} & v_R \leq v_C \\ v'_1 - v_1 + 1 - \sum_{i\geq 2} k_{i,i-1} - \sum_{i\geq 2} k_{i,i} & v_R \leq v_C \end{cases}$$

and (50), (51) follow.

Note next that

$$\mathbf{E}(k_{1,1} \mid \mathcal{H}) = \frac{(k-1)(v_{1,R}-1)}{\mu - 1}. \qquad (61)$$

Further, for $\max(2,j) \leq i \leq \log n$,

$$\mathbf{E}(k_{i,j} \mid \mathcal{H}) = v_{i,R}\binom{k-1}{j}\frac{(i)_j(\mu-1-i)_{k-1-j}}{(\mu-1)_{k-1}} = \begin{cases} \frac{(k-1)iv_{i,R}}{\mu}\left(1 + O\left(\frac{(\log n)^2}{\mu}\right)\right) & j=1, \\ O\left(\frac{(\log n)^{2j}}{\mu^{j-1}}\right) & j \geq 2. \end{cases} \qquad (62)$$

Thus from (53), (56) and (62) we get

$$\begin{aligned}
\mathbf{E}(v'_{1,R} \mid \mathbf{x}, \mathbf{y}) &= v_{1,R} - 1 - \mathbf{E}(k_{1,1} \mid \mathbf{x}, \mathbf{y}) + \mathbf{E}\left(\sum_{i\geq 2} k_{i,i-1} \mid \mathbf{x}, \mathbf{y}\right) \\
&= v_{1,R} - 1 + \sum_{k\geq 2} \frac{kv_{k,C}}{\mu}\frac{(k-1)2v_{2,R}}{\mu} + \Theta.
\end{aligned}$$

Removing the conditioning on vertex degrees we get

$$\begin{aligned}
E(v'_{1,R} \mid \boldsymbol{\nu}) &= v_{1,R} - 1 + \sum_{k\geq 2} \frac{kz_C^k v_C}{k! f_C \mu} \cdot (k-1) \cdot \frac{z_R^2 v_R}{f_R \mu} + \Theta \\
&= v_{1,R} - 1 + \frac{z_R^2 z_C^2 v_R v_C e^{z_C}}{f_R f_C \mu^2} + \Theta \\
&= v_{1,R} - 1 + \frac{z^4 v^2 e^z}{4f^2\mu^2} + \Theta \\
&= v_{1,R} - 1 + \frac{z^2 e^z}{(e^z - 1)^2} + \Theta.
\end{aligned}$$

The remaining quantities can now be filled in the same way using (53) and (54) – (58).

$$\begin{aligned}
\mathbf{E}(v'_{1,C} \mid \boldsymbol{\nu}) &= v_{1,C} + \Theta \\
\mathbf{E}(v'_{R} \mid \boldsymbol{\nu}) &= v_R - (\mathbf{E}(v'_{1,R} \mid \boldsymbol{\nu}) - (v_{1,R} - 1)) + \Theta \\
&= v_R - \frac{z^2 e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v'_{C} \mid \boldsymbol{\nu}) &= v_C - 1 + \Theta \\
\mathbf{E}(v'_{0,R} \mid \boldsymbol{\nu}) &= \sum_{k \geq 2} \frac{k\mathbf{E}(v_{k,C} \mid \boldsymbol{\nu})}{\mu} \sum_{j \geq 1} \mathbf{E}(k_{j,j} \mid \boldsymbol{\nu}) \\
&= \sum_{k \geq 2} \frac{k\mathbf{E}(v_{k,C} \mid \boldsymbol{\nu})}{\mu} \left(\frac{(k-1)(v_{1,R}-1)}{\mu - 1} + O\left(\frac{(\log n)^2}{\mu} \right) \right) \\
&= O\left(\frac{v_1 + (\log n)^2}{\mu} \right) \\
v'_{0,C} &= 0 \\
\mathbf{E}(\mu' \mid \boldsymbol{\nu}) &= \mu - 1 - \sum_{k \geq 2} \frac{k\mathbf{E}(v_{k,C} \mid \boldsymbol{\nu})}{\mu - v_{1,C}} + \Theta \\
&= \mu - 1 - \sum_{k \geq 2} \frac{k z_C^k v_C}{k! f_C \mu} \cdot (k-1) + \Theta \\
&= \mu - 1 - \frac{z_C^2 v_C e^{z_C}}{\mu f_C} + \Theta \\
&= \mu - 1 - \frac{z^2 v e^z}{2\mu f} + \Theta \\
&= \mu - 1 - \frac{z e^z}{e^z - 1} + \Theta
\end{aligned}$$

Finally note that since $|v' - v| = o(\log n)$ **qs**, then we see that if $|v_C - v_R| \geq \log n$ then $v'_C - v'_R$ has the same sign as $v_C - v_R$ and we can use the equations (46) – (49) to get (52). □

Note that

$$\begin{aligned}
1 - \frac{z^2 e^z}{(e^z-1)^2} &= 1 - \left(\frac{z}{e^{z/2} - e^{-z/2}} \right)^2 \\
&= 1 - \left(\frac{1}{\sum_{j \geq 0} \frac{z^{2j}}{2^{2j}(2j+1)!}} \right)^2 \\
&\geq \min\left\{ \frac{z^2}{48}, \frac{1}{2} \right\},
\end{aligned} \tag{63}$$

cf. Corollary 3 of [1].

So in Case 1, for $\boldsymbol{\nu}(t) \in W_1$, we have

$$\mathbf{E}(v'_1 \mid \boldsymbol{\nu}) \leq v_1 - \min\left\{ \frac{z^2}{48}, \frac{1}{2} \right\} + \Theta \leq v_1 - \min\left\{ \frac{z^2}{50}, \frac{1}{2} \right\}, \tag{64}$$

on using (44).

Remark 2. *In what follows there is a proliferation of large related constants and the reader may find it difficult to check where these constants come from. We will adopt the convention that the* *subscript of such constants is defined by the equation number where they are first used. In this spirit $C_{64} = 50$.*

Furthermore, introduce $M = 2\mu - v_1$. Lemma 13 implies that

$$\frac{\mathbf{E}(M' - M \mid \boldsymbol{\nu})}{\mathbf{E}(v' - v \mid \boldsymbol{\nu})} = \frac{1 + \frac{2ze^z}{e^z-1} + \frac{z^2e^z}{(e^z-1)^2}}{1 + \frac{z^2e^z}{(e^z-1)^2}} + \Theta. \tag{65}$$

Case 2: Deleting a random edge when $v_1 = 0$.

Lemma 14. *Assume that* $\log n = O((vz)^{1/2})$ *and* $v_1 = 0$*. Then*

$$\begin{aligned}
\mathbf{E}(v'_{1,R} \mid \boldsymbol{\nu}) &= \frac{z^2e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v'_{1,C} \mid \boldsymbol{\nu}) &= \frac{z^2e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v'_1 \mid \boldsymbol{\nu}) &= 2\frac{z^2e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v'_R \mid \boldsymbol{\nu}) &= v_R - 1 - \frac{z^2e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v'_C \mid \boldsymbol{\nu}) &= v_C - 1 - \frac{z^2e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v' \mid \boldsymbol{\nu}) &= v - 2 - 2\frac{z^2e^z}{(e^z-1)^2} + \Theta \\
\mathbf{E}(v'_R - v'_C \mid \boldsymbol{\nu}) &= v_R - v_C + \Theta \qquad (66) \\
\mathbf{E}(\mu' \mid \boldsymbol{\nu}) &= \mu - 1 - 2\frac{ze^z}{e^z-1} + \Theta \\
\mathbf{E}(v_{0,R} \mid \boldsymbol{\nu}) &= O\left(\frac{\log^2 n}{\mu}\right) \qquad (67) \\
\mathbf{E}(v_{0,C} \mid \boldsymbol{\nu}) &= O\left(\frac{\log^2 n}{\mu}\right) \qquad (68)
\end{aligned}$$

Furthermore

$$\begin{aligned}
\mathbf{Pr}(v'_{1,R} > 0 \mid \boldsymbol{\nu}) &= \frac{z}{e^z-1} + \Theta = 1 - \frac{z}{2} + O(z^2) + \Theta. \qquad (69) \\
\mathbf{Pr}(v'_{1,R} > 0 \textit{ or } v'_{1,C} > 0 \mid \boldsymbol{\nu}) &= 1 - \left(1 - \frac{z}{e^z-1}\right)^2 + \Theta = 1 - \frac{z^2}{4} + O(z^3) + \Theta. \qquad (70) \\
\mathbf{Pr}(v'_{1,R} > 0 \textit{ and } v'_{1,C} = 0 \mid \boldsymbol{\nu}) &= \frac{z}{e^z-1}\left(1 - \frac{z}{e^z-1}\right) + \Theta = \frac{z}{2} + O(z^2) + \Theta \qquad (71)
\end{aligned}$$

Proof We again condition on the degree sequence $\mathbf{x}, \mathbf{y}$. Choosing a random edge means choosing a random x from $\mathbf{x}$, and then an x's random neighbour y from $\mathbf{y}$. Given $k\epsilon 2$, $\ell \geq 2$, let us add the event $\{d_{\mathbf{y}}(y) = k, d_{\mathbf{x}}(x) = \ell\}$ to the conditioning on $\mathbf{x}, \mathbf{y}$, denoting the resulting conditioning by $\mathcal{H}$. Besides the numbers $\{k_{i,j}\}$ (see the proof of Lemma 13), let let x have $\ell_{i,j}$

neighbours (excluding y) of degree i that are joined to x by $j \leq i$ edges. Then

$$\begin{aligned}
\mu' &= \mu - k - \ell + 1 & (72)\\
v'_{1,C} &= \sum_{i\geq 2} \ell_{i,i-1} & (73)\\
v'_{1,R} &= \sum_{i\geq 2} k_{i,i-1} & (74)\\
v'_C &= v_C - 1 - \sum_{i\geq 2} \ell_{i,i-1} - \sum_{i\geq 2} \ell_{i,i} & (75)\\
v'_R &= v_R - 1 - \sum_{i\geq 2} k_{i,i-1} - \sum_{i\geq 2} k_{i,i} & (76)\\
v_{0,R} &= \sum_{j\geq 1} k_{j,j} & (77)\\
v_{0,C} &= \sum_{j\geq 1} \ell_{j,j} & (78)
\end{aligned}$$

Now (62) still holds and there is an analogous expression for the $\ell_{i,j}$.

The rest of proof of the lemma follows the same pattern as that for Lemma 13 and is left to the reader, who should notice how close the claim is to two applications of the previous case. □

Observe that (65) holds in this case too.

Note also that in both cases

$$\mathbf{E}(v_0 \mid \boldsymbol{\nu}) = O\left(\frac{v_1 + (\log n)^2}{\mu}\right). \tag{79}$$

(The hidden constant here is denoted C_{79}.)

Remarkably, the equations involving $v_1, v'_1, v, v', \mu, \mu'$ are up to the error terms, identical to those given in Lemmas 6 and 7 of [1].

3.4.2 Multi-step parameter changes

At this point let us try to summarise the (conditional) expected changes in the parameters v_1, γ, γ_1 as the algorithm proceeds. Here

$$\gamma_1 = |v_R - v_C|.$$

We need to show that v_1 does not grow large as v_1 determines the rate at which isolated vertices are created. We also need to show that γ_1 does not grow large so that we can for one thing approximate z_R, z_C by z.

Let us first consider the case where $v_1 > 0$. There is a *preferred* side from which to choose the vertex x of degree 1. This is the side with fewest vertices of degree 2 or more. v_1 is well behaved. When $v_1 > 0$ the expected change in v_1 is negative and so it is relatively easy to show that it is unlikely to get too large. γ_1 is not so well behaved. It has a positive expected change when we are forced to choose x on the less preferred side. However, this expected increase is compensated by the expected decrease in v_1. This is why we introduce the parameter γ which is much better behaved than γ_1. Indeed, if x is chosen on the preferred side then the expected change in γ is negative and if x is chosen on the less prefered side then the change is non-positive (deterministically).

Now consider what happens if $v_1 = 0$. There is only a small chance that $v_1' = 0$ too and the expected increase in γ_1 while v_1 remains zero turns out to be negligible. If $v_1' > 0$ then we see a rise in γ (due to the rise in v_1). One more step will show an expected decrease i γ_1. The rise in v_1 is handled by looking forward to the next time that $v_1 = 0$, see Lemma 18.

The main analysis of the chain $(\boldsymbol{\nu}(t))$ is restricted to the times when $\boldsymbol{\nu}(t) \in W_0$ or $\boldsymbol{\nu}(t) \in W_1$. where for $\sigma = 0, 1$,

$$W_\sigma = \{\boldsymbol{\nu} : \ z \geq n^{-\alpha_i},\ v \geq A_{80} n z^2,\ v_1 \leq B_{80} n^{2\alpha_\sigma} (\log n)^3,\ \gamma \leq C_{80} n^{2\alpha_\sigma} (\log n)^4\}. \tag{80}$$

Here $\alpha_0 = .14$, $\alpha_1 = .17$ and A_{80}, B_{80}, C_{80} are large constants and $W_0 \subseteq W_1$.

Note that (43) holds for $\boldsymbol{\nu}(t) \in W_1$.

For $\sigma = 0, 1$ we introduce stopping times

$$\mathcal{T}_\sigma = \begin{cases} \min\{t \leq \mathcal{S} : \ \boldsymbol{\nu}(\tau) \notin W_\sigma\} \\ n \qquad \text{if no such } t \text{ exist} \end{cases}$$

Now it follows from (70) that we can find $K > 0$ such that with probability $1 - O(n^{-5})$ there is no sequence $t, t+1, \ldots, t + K \log n < n$ such that $v_1(\tau) = 0$ for $t \in [t, t + K \log n]$. So we introduce another stopping time

$$\mathcal{S}_1 = \begin{cases} \min\{t \leq \mathcal{S} : \ v_1(\tau) = 0, \tau \in [t - K \log n, t]\}. \\ n \qquad \text{if no such } t \text{ exist} \end{cases}$$

Now $\mathcal{S} < \mathcal{S}_1$ with probability $1 - O(n^{-5})$ and so we replace $\mathcal{S}$ by

$$\mathcal{S} := \min\{\mathcal{S}, \mathcal{S}_1\}.$$

We will allow $\mathcal{T}_0, \mathcal{T}_1$ to use this new definition of $\mathcal{S}$ in their definition.

Lemma 15. *Let t be such that*

$$\boldsymbol{\nu}(t) \in W_1 \text{ and } v_1(t) = 0$$

and

$$\gamma_1(t) = v_C(t) - v_R(t) \gg (\log n)^3.$$

Let

$$t' = \begin{cases} \min\{t < \tau \leq \mathcal{T}_1 : \ v_1(\tau) > 0\} & \text{if such } \tau \text{ exist} \\ \mathcal{T}_1 & \text{otherwise} \end{cases}$$

so that $v_1(\tau) = 0$ for $t \leq \tau < t'$.

Then

$$\mathbf{E}(\gamma_1(t'+1) - \gamma_1(t) \mid \boldsymbol{\nu}(t)) \leq -\beta(t)$$

where

$$\beta(t) = -\min\left\{\frac{z(t)^2}{C_{81}}, C_{81a}\right\}, \tag{81}$$

where $C_{81} = 2C_{64}$ and C_{81a} depends on c.

Proof We first observe that either $\mathcal{S}_1 \leq t + K \log n$ or $t' < t + K \log n$ and then $v_C(\tau) - v_R(\tau) \gg (\log n)^3$ for $t \leq \tau \leq t' + 1$.

We next assemble the following facts, that hold for $0 < \tau - t \leq K \log n$.

Claim 1

$$\mathbf{Pr}(t'=\tau \mid t'>\tau-1,\boldsymbol{\nu}(t)) = 1-\left(1-\frac{z}{e^z-1}\right)^2+\Theta \stackrel{def}{=} \pi_1+\Theta$$

where we can take $z=z(t)$.

We write

$$\mathbf{Pr}(t'=\tau \mid t'>\tau-1,\boldsymbol{\nu}(t)) = \sum_{\boldsymbol{\nu}\in S}\mathbf{Pr}(t'=\tau \mid t'>\tau-1,\boldsymbol{\nu}(\tau-1)=\boldsymbol{\nu})\mathbf{Pr}(\boldsymbol{\nu}(\tau-1)=\boldsymbol{\nu} \mid \boldsymbol{\nu}(t)) \tag{82}$$

where $S=\{\boldsymbol{\nu}:\ |\boldsymbol{\nu}-\boldsymbol{\nu}(t)|\le K(\log n)^2\}$.

Now if $\boldsymbol{\nu}\in S$ then

$$\mathbf{Pr}(t'=\tau \mid t'>\tau-1,\boldsymbol{\nu}(\tau)) = 1-\left(1-\frac{z}{e^z-1}\right)^2+\Theta \tag{83}$$

which follows directly from (70), since we are assuming that $v_1(\tau)=0$ and since $|z(\tau)-z(t)|=O\left(\frac{(\log n)^2}{v}\right)$ (see (38)) and this quantity is $o(z^3)$. The claim follows from (82) and (83).

Claim 2

$$\mu_2=\mathbf{E}(\gamma_1(\tau+1)-\gamma_1(\tau)) \mid t'>\tau,\boldsymbol{\nu}(t)))=\Theta.$$

This again follows from (66) since we are again assuming that $v_1(\tau-1)=0$ and we know that $z(\tau)$ and $z(t)$ are very close.

Claim 3

$$\begin{aligned}\mathbf{Pr}(v_{1,R}(t')>0 \mid \boldsymbol{\nu}(t)) &= \frac{\frac{z}{e^z-1}}{1-\left(1-\frac{z}{e^z-1}\right)^2}+\Theta \\ &\stackrel{def}{=} \pi_3+\Theta \\ &= 1-\frac{z}{2}+O(z^2).\end{aligned}$$

This follows directly from (69) and (70) and the fact that $z(t')$ and $z(t)$ are very close.

Claim 4

$$\mathbf{E}(\gamma_1(t'+1)-\gamma_1(t') \mid v_{1,R}(t')>0,\boldsymbol{\nu}(t)) = -1+\frac{z^2e^z}{(e^z-1)^2}+\Theta \stackrel{def}{=} \mu_4+\Theta.$$

This follows from (49) with $X=R$, since $v_C(t')-v_R(t')\gg(\log n)^3$.

Claim 5

$$\mathbf{E}(\gamma_1(t'+1)-\gamma_1(t') \mid v_{1,R}(t')=0,\boldsymbol{\nu}(t)) = 1-\frac{z^2e^z}{(e^z-1)^2}+\Theta=-\mu_4+\Theta.$$

This follows from (49) with $X=C$, since $v_C(t')-v_R(t')\gg(\log n)^3$.

Putting these facts together we get

$$\mathbf{E}(\gamma_1(t'+1)-\gamma_1(t)\mid \boldsymbol{\nu}(t)) = O(\Theta\log n)+\pi_3\mu_4+(1-\pi_3)(-\mu_4)+\Theta. \tag{84}$$

Explanation of (84): The first term accounts for the expected increase in γ_1 between t and t'. Indeed from Claim 2 we have

$$\begin{aligned}\mathbf{E}(\gamma_1(t')-\gamma_)1(t)\mid \boldsymbol{\nu}(t)) &\leq \sum_{\tau=t}^{t+K\log n}\mathbf{E}(|\gamma_1(\tau+1)-\gamma_1(\tau)1_{t'>\tau}\mid \boldsymbol{\nu}(t)) \\ &= O(\Theta\log n).\end{aligned} \tag{85}$$

The term $\pi_3\mu_4$ accounts for the expected increase $\gamma_1(t'+1)-\gamma_1(t')$ when $v_{1,R}(t')>0$ (this is negative) and the term $(1-\pi_3)(-\mu_4)$ accounts for the expected increase $\gamma_1(t'+1)-\gamma_1(t')$ when $v_{1,R}(t')=0$.

Writing $p=\frac{z}{e^z-1}<1$ we see that $\pi_3=\frac{1}{2-p}>\frac{1}{2}$

$$\mathbf{E}(\gamma_1(t'+1)-\gamma_1(t)\mid \boldsymbol{\nu}(t)) = (1-2\pi_3)\mu_4+O(\Theta\log n)\leq -\min\left\{\frac{z^2}{2C_{64}},C_{86}\right\}. \tag{86}$$

for some $C_{86}>0$ which depends on c. The lemma follows. □

We put these ideas to work in the next few sections.

3.4.3 Number of vertices left isolated

Let $\alpha_\sigma, W_\sigma, \mathcal{T}_\sigma, i=0,1$ be as in (80).

The analysis is in two parts, $t=1\ldots,\mathcal{T}_0$ and $t=\mathcal{T}_0+1\ldots\mathcal{T}_1$. The reason for this split will not become apparant until the middle of proof of Lemma 19 and so the reader will have to take the need for a split on trust.

Note that if $\boldsymbol{\nu}\in W_1$ then $vz^2\geq A_{80}nz^4\gg(\log v)^2$ and so the conclusions of Lemmas 13 and 14 are valid.

Fix $\sigma=0$ or 1 Now let X_t, $t=0,1,\ldots,\mathcal{T}_\sigma-1$ be the number of isolated vertices created at time t and let $X_t=0$ for $t\geq\mathcal{T}_\sigma$. Let also set $\boldsymbol{\nu}(t)\equiv\boldsymbol{\nu}(\mathcal{T}_\sigma-1)$ for $t\geq\mathcal{T}_\sigma$. Then the random variables X_t satisfy

$$X_t\geq 0;\ X_t\leq\log n;\ \mathbf{E}(X_t\mid\cdot)\leq\frac{B_{80}C_{79}n^{2\alpha_\sigma}(\log n)^3}{\mu(t)},$$

where $\mid\cdot$ denotes conditioning with respect to $\{\boldsymbol{\nu}(\tau)\}_{0\leq\tau<t}$, and C_{79} is the hidden constant in (79). Putting $\lambda=(\log n)^{-2}$ and using $\lambda X_t\leq(\log n)^{-1}$, $e^x\leq 1+1.5x$, $x\downarrow 0$, we see that

$$\mathbf{E}(e^{\lambda X_t}\mid\cdot)\leq 1+1.5\lambda\mathbf{E}(X_t\mid\cdot)\leq e^{1.5\lambda\mathbf{E}(X_t|\cdot)}.$$

Therefore, introducing $X=\sum_{t\geq 0}X_t$, and using the bound for $\mathbf{E}(X_t\mid\cdot)$ together with $\sum_t\mathbf{E}(m^{-1}(t))\leq\log n$, we have

$$\mathbf{E}(e^{\lambda X})\leq\exp(2B_{80}C_{79}n^{2\alpha_\sigma}(\log n)^2).$$

Applying the Markov inequality, we obtain

$$\mathbf{Pr}(X\geq 3B_{80}C_{79}n^{2\alpha_\sigma}(\log n)^4)\leq e^{-3B_{80}C_{79}n^{2\alpha_\sigma}(\log n)^4\lambda}\mathbf{E}e^{\lambda X}\leq e^{-B_{80}C_{79}n^{2\alpha_\sigma}(\log n)^3}. \tag{87}$$

This proves:

Lemma 16.

$$\textit{At most } C_{88} n^{2\alpha_\sigma} (\log n)^4 \textit{ isolated vertices are created up to time } \mathcal{T}_\sigma, \textbf{ qs} \tag{88}$$

where $C_{88} = 3B_{80}C_{79}$.

Our next task is to get a good estimate of $\boldsymbol{\nu}(\mathcal{T}_\sigma)$ at the stopping time $\mathcal{T}_\sigma$.

Lemma 17. *With probability* $1 - O(n^{-2})$

$$v_1(t) \le C_{89} n^{2\alpha_\sigma} (\log n)^3 \qquad \forall t \in [1, \mathcal{T}_\sigma], \tag{89}$$

where $C_{89} = 8C_{64}$ *(and hence we can take* $B_{80} = 8C_{64}$*).*

Proof Let $\Delta_\sigma = C_{89} n^{2\alpha_\sigma} (\log n)^3$. First of all, by (15), **qs** the conditions $v_1(t-1) = 0$, $v_1(t) > 0$ and $t \ge 1$ imply that $v_1(t) \le \log n$. In view of this, for $t_1 < t_2$, define the event

$$\mathcal{E}_1(t_1, t_2) = \{v_1(t_2) - v_1(t_1) > C_{89} n^{2\alpha_\sigma} (\log n)^3\} \cap \{\forall\, t \in [t_1, t_2),\, v_1(t) > 0\}.$$

Clearly then it suffices to prove that

$$\mathbf{Pr}\left(\bigcup_{t_1 < t_2 \le \mathcal{T}_\sigma} \mathcal{E}_1(t_1, t_2) \right) = O(n^{-2}).$$

Define

$$Y_t = \begin{cases} v_1(t+1) - v_1(t) & \text{if } v_1(t) > 0 \text{ and } t < \mathcal{T}_\sigma, \\ -\beta_{90} & \text{otherwise,} \end{cases}$$

where,

$$\beta_{90} = \frac{1}{C_{64} n^{2\alpha_\sigma}}. \tag{90}$$

We notice upfront that, for $t_1 < t_2 \le \mathcal{T}_\sigma$, and $v_1(t) > 0$ for $t \in [t_1, t_2)$,

$$v_1(t_2) - v_1(t_1) = \sum_{t=t_1}^{t_2 - 1} Y_t.$$

Now, by the definition of $\mathcal{T}$, $|Y_t| \le \log n$ for all t. Furthermore, if $\lambda = \frac{\beta_{90}}{2(\log n)^2}$ then

$$\begin{aligned} \mathbf{E}(e^{\lambda Y_u} \mid \boldsymbol{\nu}(\tau), \tau < t+u) &\le 1 + \mathbf{E}(\lambda Y_u) + \sum_{i \ge 2} \frac{\lambda^i (\log n)^i}{i!} \\ &\le 1 - \lambda\beta_{90} + \lambda^2 (\log n)^2 \\ &\le 1. \end{aligned} \tag{91}$$

For $t < \mathcal{T}_\sigma$, (91) follows from the definition of Y_t, β_{90}, (64), and the definition of the stopping time $\mathcal{T}_\sigma$. For $t \ge \mathcal{T}_\sigma$, (91) holds trivially. Thus, the occurrence of the event $\mathcal{E}_1(t_1, t_2) \cap \{t_2 \le \mathcal{T}_\sigma\}$ implies

$$\sum_{t=t_1}^{t_2} Y_t > \Delta_\sigma. \tag{92}$$

Now from (91),

$$\mathbf{Pr}\left(\sum_{t=t_1}^{t_2} Y_t > \Delta_\sigma \mid \boldsymbol{\nu}(\tau), \tau \leq t_1\right) \leq e^{-\lambda\Delta_\sigma}\mathbf{E}\left(\prod_{t=t_1}^{t_2} e^{\lambda Y_t} \mid \boldsymbol{\nu}(\tau), \tau \leq t_1\right) \leq e^{-\lambda\Delta_\sigma}. \tag{93}$$

Since the number of pairs (t_1, t_2) is $\binom{n}{2} = O(n^2)$, the statement follows. □

To account for the unlikely failure of (89) we introduce a stopping time

$$\mathcal{S}_2 = \begin{cases} \min\{t \leq \mathcal{T}_1 : \ v_1(t) > C_{89} n^{2\alpha_\sigma}(\log n)^3\} \\ n \qquad \text{if no such } t \text{ exist} \end{cases}$$

and let

$$\mathcal{S} := \min\{\mathcal{S}, \mathcal{S}_2\}.$$

Now to deal with γ.

Lemma 18. *With probability* $1 - O(n^{-2})$

$$\gamma(t) \leq 2C_{88} n^{2\alpha_\sigma}(\log n)^4 \qquad \forall t \in [1, \mathcal{T}_\sigma]. \tag{94}$$

(Thus we can take $C_{80} = 2C_{88}$*).*

Proof For $t_0 < t_2$ let

$$\mathcal{E}_2(t_0, t_2) = \{\gamma(t_2) - \gamma(t_0) > 2C_{88} n^{2\alpha_\sigma}(\log n)^4\}$$

and also for $t_0 < t_1 < t_2$ let

$$\mathcal{E}_3(t_0, t_1, t_2) = \{\gamma(t_1) - \gamma(t_0) < C_{88} n^{2\alpha_\sigma}(\log n)^4 \text{ and } \gamma(t) - \gamma(t_0) \geq C_{88} n^{2\alpha_\sigma}(\log n)^4,\ t_1 < t < t_2 \text{ and } \gamma(t_2) - \gamma(t_0) > 2C_{88} n^{2\alpha_\sigma}(\log n)^4\}.$$

Note that if $t_2 < \mathcal{T}_\sigma$ and $\mathcal{E}_2(t_0, t_2)$ occurs then $\mathcal{E}_3(t_0, t_1, t_2)$ occurs for some $t_1 > t_0$.

$$\mathbf{Pr}\left(\bigcup_{0 \leq t_0 < t_1 < t_2 \leq \mathcal{T}_\sigma} \mathcal{E}_3(t_0, t_1, t_2)\right) = O(n^{-2}). \tag{95}$$

Fix $t_2 > t_1 > t_0$. If $\mathcal{E}_3(t_0, t_1, t_2)$ occurs for some $t_0 \leq \mathcal{T}_\sigma$ then the sign of $v_R - v_C$ does not change between t_1 and t_2. (If $v_R - v_C$ drops to zero then we will have $\gamma = v_1 \leq C_{89} n^{2\alpha_\sigma}(\log n)^3$). We will assume that $v_R(t_1) - v_C(t_1) < 0$ and introduce the stopping time

$$t_I = \begin{cases} \min\{t_1 \leq t \leq t_2 : \ v_R(t) \geq v_C(t) \\ t_2 \qquad \text{if no such } t \text{ exist} \end{cases}$$

We define a sequence of times $\tau_0 = t_1 \leq \tau_1 < \ldots < \tau_r \leq \tau_{r+1} = t_I$ as follows: $\tau_1 = \min\{t_1 \leq \tau < \min\{t_2, \mathcal{T}_3\} : \ v_1(\tau) = 0\}$. If such a τ does not exist then we take $\tau_1 = t_I$. Assume that we have defined τ_i with $v_1(\tau_i) = 0$. Define $\tau_i' = 1 + \min\{\tau_i < \tau \leq t_I : \ v_1(\tau) > 0\}$. If τ_i' does not exist then $r = i$. If τ_i' does exist then let $\tau_i^* = \min\{\tau_i \leq \tau \leq t_I : \ v_1(\tau) = 0\}$. If τ_i^* exists then $\tau_{i+1} = \tau_i^*$, otherwise $r = i$. We now bound the change in γ over these intervals.

(a) We first consider $\gamma(\tau_1) - \gamma(\tau_0)$. This is zero if $\tau_1 = \tau_0$ and so assume that $v_1(t_0) > 0$. If $v_{1,R}(t_1) = 0$ then (45) implies that $v_{1,R}(\tau) = 0$ for $t_0 \leq \tau \leq t_I\}$ and then (50) implies that $\gamma(\tau_1) \leq \gamma(\tau_0)$.

So assume that $v_{1,R}(t_1) > 0$. For $1 \leq u \leq n$ define

$$Y_u = \begin{cases} \gamma_1(t_1 + u) - \gamma_1(t_1 + u - 1) & t_0 + u \leq \tau_1 \\ -\beta_{97} & otherwise \end{cases} \tag{96}$$

where

$$\beta_{97} = \frac{1}{C_{81} n^{2\alpha_\sigma}}. \tag{97}$$

Now we have $|Y_u| \leq \log n$ and (81) implies that

$$\mathbf{E}(Y_u \mid \boldsymbol{\nu}(\tau), \tau < t_1 + u) \leq -\beta_{97}.$$

So with $\lambda = \frac{\beta}{2 \log n}$ we can argue as in (91) that

$$\mathbf{E}(e^{\lambda Y_u} \mid \boldsymbol{\nu}(\tau), \tau < t_1 + u) \leq 1.$$

We then argue as in (93) that for all $0 < T \leq n$,

$$\mathbf{Pr}\left(\sum_{u=1}^{T} Y_u \geq 10(\log n)^3 \beta_{97}^{-1}\right) \leq n^{-5}.$$

It follows that

$$\mathbf{Pr}(\gamma(\tau_1) - \gamma(\tau_0) \geq 10(\log n)^3 \beta_{97}^{-1}) \leq n^{-4}. \tag{98}$$

(b) We now consider the random variables $\gamma(\tau_{i+1}) - \gamma(\tau_i)$ for $i \geq 1$. Fix $1 \leq i \leq r$. (When $i = r$ parts of the argument may have to be omitted or modified in a trivial way). It follows from Lemma 15 that

$$\mathbf{E}(\gamma_1(\tau_i') - \gamma_1(\tau_i) \mid \boldsymbol{\nu}(\tau), \tau \leq \tau_i)) \leq -\beta_{99} = -\frac{1}{C_{81} n^{2\alpha_\sigma}}. \tag{99}$$

Suppose that $v_{1,R}(\tau_i') = k$ and $v_{1,C}(\tau_i') = \ell$. Thus

$$\gamma(\tau_i') - \gamma(\tau_i) = \gamma_1(\tau_i') - \gamma_1(\tau_i) + k + \ell. \tag{100}$$

Next let $\tau_i'' = \min\{\tau \geq \tau_i' :\ v_{1,R} = 0\}$. It follows from (51) that

$$-2k - 2\ell \leq \gamma(\tau_i'') - \gamma(\tau_i') \leq -2k + Z_i$$

where

$$Z_i = \sum_{\tau = \tau_i'}^{\tau_i''} v_0(\tau).$$

This is because $v_R < v_C$ and $v_{1,R}(\tau_i'') = 0$ and $v_{1,C}(\tau_i'') \leq v_{1,C}(\tau_i')$ (by (45)). (The $-2k$ in the lower bound accounts for the possibility of Case a in Lemma 13).

Now (50) implies that

$$\gamma(\tau_{i+1}) = \gamma(\tau_i'') - Z_i'$$

where

$$Z_i' = \sum_{\tau = \tau_i''}^{\tau_{i+1}} v_0(\tau).$$

So if

$$\Gamma_i = \gamma(\tau_{i+1}) - \gamma(\tau_i) - Z_i + Z_i'$$

then

$$\Gamma_i \leq \gamma_1(\tau_i') - \gamma_1(\tau_i) + \ell - k \text{ and } |\Gamma_i| \leq 5\log n.$$

So

$$\mathbf{E}(\Gamma_i \mid \boldsymbol{\nu}(\tau_i)) \leq -\beta_{99} + \Theta \leq -\beta_{99}/2.$$

Putting $\lambda = \frac{\beta_{99}}{25(\log n)^2}$ we can argue as in (91) that $\mathbf{E}(e^{\lambda\Gamma_i} \mid \boldsymbol{\nu}(\tau_i)) \leq 1$. Putting $\Gamma_i = 0$ for $i = r+2, \ldots, n$ we see that

$$\mathbf{Pr}\left(\sum_{i=1}^{r+1} \Gamma_i \geq 125(\log n)^3 \beta_{99}^{-1}\right) = \mathbf{Pr}\left(\sum_{i=1}^{n} \Gamma_i \geq 125(\log n)^3 \beta_{99}^{-1}\right) \leq e^{-125\lambda(\log n)^3\beta_{99}^{-1}} = n^{-5}. \tag{101}$$

But

$$\gamma(\tau_{r+1}) - \gamma(\tau_1) = \sum_{i=1}^{r+1} \Gamma_i + \sum_{i=1}^{r+1}(Z_i - Z_i') \leq \sum_{i=1}^{r+1} \Gamma_i + \sum_{i=1}^{r+1} Z_i$$

and (88) implies that $\sum_{i=1}^{r+1} Z_i \leq C_{88} n^{2\alpha_\sigma}(\log n)^4$ **qs**. Therefore,

$$\mathbf{Pr}(\gamma(\tau_{r+1}) - \gamma(\tau_1) \geq 2C_{88} n^{2\alpha_\sigma}(\log n)^4) = O(n^{-5}). \tag{102}$$

The lemma follows from (98) and (102). □

We now check the second condition of W_σ, $v \geq A_{80} n z^2$.

Going back to (65), we are left to consider the differential equation,

$$\frac{dM}{dv} = \frac{1 + \frac{vz^2e^z}{mf} + \frac{v^2z^4e^z}{4m^2f^2}}{1 + \frac{v^2z^4e^z}{4m^2f^2}}.$$

The solution of this was obtained in [1]: Here z^* is the value of z at $t = 0$ and $M^* = 2m^* = 2cn$.

$$M = \frac{M^*(e^z - 1)z}{z^*(e^{z^*} - 1)} \exp\left\{-\int_z^{z^*} \frac{\xi e^\xi}{e^\xi(1+\xi) - 1} d\xi\right\}. \tag{103}$$

Then (up to a v_1 error term)

$$\begin{aligned} v &= \frac{2mf(z)}{z(e^z - 1)} \\ &= \frac{2m^* f(z)}{z^*(e^{z^*} - 1)} \exp\left\{-\int_z^{z^*} \frac{\xi e^\xi}{e^\xi(1+\xi) - 1} d\xi\right\}. \end{aligned} \tag{104}$$

So we define

$$J_1 = \frac{v}{nf(z)} \exp\left\{\int_z^{z^*} \frac{\xi e^\xi}{e^\xi(1+\xi) - 1} d\xi\right\}$$

and

$$J_2 = \frac{m}{nz(e^z - 1)} \exp\left\{\int_z^{z^*} \frac{\xi e^\xi}{e^\xi(1+\xi) - 1} d\xi\right\}.$$

Lemma 19. *Let $\mathcal{T}_{-1} = 0$. Then for $\sigma = 0, 1$,*

$$\mathbf{Pr}\left(\max_{\tau \in [\mathcal{T}_{\sigma-1}, \mathcal{T}_\sigma]} |J_i(\boldsymbol{\nu}(\tau)) - J_i(\boldsymbol{\nu}(\mathcal{T}_{\sigma-1}))| > n^{-\alpha_\sigma/4}\right) = O(n^{-4}), \qquad i = 1, 2.$$

Proof Now fix $i = 1$ or 2 and let $J(t) = J_i(\boldsymbol{\nu}(t))$. Let now $K = (\log n)^2$ and define $Q(t) = \exp\{K(J(t) - J(\mathcal{T}_{\sigma-1}))\}$ for $\mathcal{T}_{\sigma-1}) \le t < \mathcal{T}_\sigma$. Let $Q(t) = 0$ for $t \ge \mathcal{T}_\sigma$.

We consider only $i = 1$ since the other case is very similar. For $t > \mathcal{T}_\sigma$, we obviously have $Q(t) = Q(t-1) = 0$. For $\mathcal{T}_{\sigma-1}) \le t \le \mathcal{T}_\sigma$ we can write

$$\mathbf{E}(Q(t) \mid \{\boldsymbol{\nu}(s)\}_{s<t}) \le Q(t-1)\mathbf{E}\left\{\mathbf{1}_{|\boldsymbol{\nu}(t)-\boldsymbol{\nu}(t-1)|\le \log n} \exp\left[K(J(t) - J(t-1))\right] \middle| \boldsymbol{\nu}(t-1)\right\}. \tag{105}$$

Since $\boldsymbol{\nu}(t-1) \in W_\sigma$, each of $m(t-1)$ and $v(t-1)$ is of order $n^{1-2\alpha_\sigma}$ at least. The same holds then for $\boldsymbol{\nu}(t) \in B(\boldsymbol{\nu}(t-1), \log n) = \{\boldsymbol{\nu} : \ |\boldsymbol{\nu} - \boldsymbol{\nu}(t-1)| \le \log n\}$. Consequently $v(t)z(t)$ is of order $n^{1-3\alpha_\sigma}$ at least. Moreover, it can be easily verified that, uniformly for such $\boldsymbol{\nu}$ and $i = 1, 2$, $x, y = v, m$,

$$\frac{\partial J}{\partial x} = O\left(\frac{1}{vz}\right), \tag{106}$$

$$\frac{\partial^2 J}{\partial x \partial y} = O\left(\frac{1}{v^2 z^2}\right). \tag{107}$$

Let $\hat{\boldsymbol{\nu}} = (v, m)$ i.e. drop all other parameters. Assuming $\boldsymbol{\nu}(t) \in B(\boldsymbol{\nu}(t-1), \log n)$, expanding the exponential function, and viewing $J(t)$ as a function of v, m only,

$$\exp\{K(J(t) - J(t-1))\} = [1 + K\nabla J(t)^T(\hat{\boldsymbol{\nu}}(t) - \hat{\boldsymbol{\nu}}(t-1)) + O(K^2(\log n)^2/(vz)^2)], \tag{108}$$

since

$$K \log n = o(vz). \tag{109}$$

Consequently, equation (105) becomes

$$\begin{aligned}\mathbf{E}(Q(t)|\{\boldsymbol{\nu}(s)\}_{s<t}) \le\ & Q(t-1)\left\{1 + K\nabla J(t)^T \mathbf{E}[\hat{\boldsymbol{\nu}}(t) - \hat{\boldsymbol{\nu}}(t-1)|\boldsymbol{\nu}(t-1)]\right\} \\ & + O(Q(t-1)K^2(\log n)^2/(vz)^2).\end{aligned} \tag{110}$$

Putting

$$\mathbf{F}(\hat{\boldsymbol{\nu}}) = \begin{bmatrix} 1 + \frac{vz^2e^z}{mf} + \frac{v^2z^4e^z}{4m^2f^2} \\ 1 + \frac{v^2z^4e^z}{4m^2f^2} \end{bmatrix}$$

and using Lemmas 13 and 14,

$$\begin{aligned}\nabla J(t)^T \mathbf{E}[\hat{\boldsymbol{\nu}}(t) - \hat{\boldsymbol{\nu}}(t-1)|\boldsymbol{\nu}(t-1)] &= \nabla J(t-1)^T\left[\mathbf{F}(\hat{\boldsymbol{\nu}}(t-1)) + \Theta\right] \\ &= O\big(\|\nabla J(t-1)\|\Theta\big) \\ &= O\left(\frac{\log n}{vz}\Theta\right).\end{aligned} \tag{111}$$

($\nabla J(\hat{\boldsymbol{\nu}}) \perp \mathbf{F}(\hat{\boldsymbol{\nu}})$ since $J(\hat{\boldsymbol{\nu}})$ is constant along the trajectory of $d\hat{\boldsymbol{\nu}}/dt = \mathbf{F}(\hat{\boldsymbol{\nu}})$!)

Therefore, for $t - 1 < \mathcal{T}_\sigma$ and hence for all $t \ge \mathcal{T}_{\sigma-1}$,

$$\mathbf{E}(Q(t) \mid \{\boldsymbol{\nu}(s)\}_{s<t}) \le Q(t-1)(1 + O(K(\log n)\Theta/(vz))) = Q(t-1)(1 + O(n^{7\alpha_\sigma - 2}(\log n)^4).$$

So for any positive ϵ, the random sequence

$$\{R(t)\} := \{(1+n^{7\alpha_0+\epsilon-2})^{-t}Q(t)\}$$

is a *supermartingale.*

Introduce a stopping time

$$\mathcal{T}'_\sigma = \begin{cases} \min\{\mathcal{T}_{\sigma-1} \le t < \mathcal{T}_\sigma : J(t) - J(\mathcal{T}_{\sigma-1}) > n^{-\alpha_\sigma/4}/2\}, & \text{if such } t \text{ exist,} \\ \mathcal{T}_\sigma, & \text{otherwise.} \end{cases}$$

For the reminder of the proof of the lemma, we take $\sigma = 0$. We will continue to use the subscript σ because we will return and finish the case $\sigma = 1$ later. Let $n_\sigma = \mathcal{T}_\sigma - \mathcal{T}_{\sigma-1}$.

Now, applying the Optional Sampling Theorem (Durrett [7]) to the supermartingale $\{R(t)\}$ and the stopping time $\mathcal{T}'_0$ we get

$$\begin{aligned} \mathbf{E}[Q(\mathcal{T}'_\sigma)] &\le \mathbf{E}(1+n^{7\alpha_\sigma+\epsilon-2})^{n_\sigma} \cdot \mathbf{E}[Q(\mathcal{T}_{\sigma-1})] & (112) \\ &\le \mathbf{E}(1+n^{7\alpha_\sigma+\epsilon-2})^{n} \cdot \mathbf{E}[Q(\mathcal{T}_{\sigma-1})] \\ &= (1+n^{7\alpha_\sigma+\epsilon-2})^{n} \\ &= 1+o(1), \text{ as } n \to \infty, & (113) \end{aligned}$$

for ϵ sufficiently small.

Remark 3. *Note that in the case $\sigma = 1$ we have $7\alpha_1 - 2 > -1$ and we cannot argue that $(1+n^{7\alpha_1+\epsilon-2})^n = 1+o(1)$. We will have to argue instead that* **whp** *$n_1 = O(n^{1-2\alpha_0})$ and then all we need is that $7\alpha_1 - 2 < 2\alpha_0 - 1$.*

Since

$$\mathbf{E}[Q(\mathcal{T}'_\sigma)] \ge e^{n^{\alpha_\sigma/4}/2} \cdot \mathbf{Pr}\{\mathcal{T}'_\sigma < \mathcal{T}_\sigma\},$$

we have

$$\mathbf{Pr}\{\max_{\mathcal{T}_{\sigma-1} \le t < \mathcal{T}_\sigma}[J(t) - J(\mathcal{T}_{\sigma-1})] > n^{-\alpha_\sigma/4}/2\} = \mathbf{Pr}\{\mathcal{T}'_\sigma < \mathcal{T}_\sigma\} = O(e^{-n^{\alpha_\sigma/4}/2}).$$

Analogously,

$$\mathbf{Pr}\{\min_{\mathcal{T}_{\sigma-1} \le t < \mathcal{T}_\sigma}[J(t) - J(\mathcal{T}_{\sigma-1})] < -n^{-\alpha_\sigma/4}/2\} = O(e^{-n^{\alpha_\sigma/4}/2}).$$

So **qs**

$$\max_{\mathcal{T}_{\sigma-1} \le t < \mathcal{T}_\sigma} |J(t) - J(\mathcal{T}_{\sigma-1})| \le n^{-\alpha_\sigma/4}/2.$$

It only remains to note the equation (37).

This completes the proof of the lemma for $\sigma = 0$. □

At time $\mathcal{T}_0$ either (i) $z \le n^{-\alpha_0}$, (ii) $z > 3c$, (iii) $v < A_{80}nz^2$ or (iv) $v_1 > B_{80}n^{2\alpha_0}(\log n)^3$ or (v) $\gamma > C_{80}n^{2\alpha_0}(\log n)^4$.

Possibility (ii) is ruled out by (39). (104) and Lemma 19 show that for $t \in [0, \mathcal{T}_0]$, $v(t) \approx A_2 nz^2$ where

$$A = \frac{m^*}{nz^*(e^{z^*}-1)} \exp\left\{-\int_0^{z^*} \frac{\xi e^\xi}{e^\xi(1+\xi)-1} d\xi\right\}.$$

This rules out possibility (iii) if we take

$$A_{80} = A/2. \tag{114}$$

Possibility (iv) is ruled out by Lemma 17 and possibility (v) is ruled out by Lemma 18. So we can assume that at time $\mathcal{T}_0$,

$$z \approx n^{-\alpha_0} \tag{115}$$
$$v \approx Anz^2 \tag{116}$$
$$m \approx Anz^2 \tag{117}$$

$$z(\mathcal{T}_0) - z(\mathcal{T}_0 - 1) = O((\log n)/v) \tag{118}$$

is the justification for (115), $m \approx v$ comes from $z = o(1)$ and

$$2 \leq \frac{2m - v_1}{v} = \frac{z(e^z - 1)}{f(z)} = 2\left(1 + \frac{z}{6} + O(z^2)\right). \tag{119}$$

So if we condition on (115)–(117) then we can go back to the proof of Lemma 19 at equation (112) and take $n_1 = O(n^{1-2\alpha_0})$ and now find that (113) holds. We have fulfilled the condition laid out in Remark 3 and finish the proof of the lemma for the case $\sigma = 1$.

We can therefore argue that at time $\mathcal{T}_1$, (115,116,115) hold with α_0 replaced by α_1. The number of isolated vertices that are created from $\mathcal{T}_1$ onwards is bounded by the sum of (i) $\sum_{k\geq 3} kv_k(\mathcal{T}_1)$ and (ii) the number κ_1 of components of $G(\mathcal{T}_1)$ which are paths of odd length. Now **qs**

$$\sum_{k\geq 3} kv_k(\mathcal{T}_1) \approx v(\mathcal{T}_1)z(\mathcal{T}_1)/3 = O(n^{1-3\alpha-1}) = O(n^{.49}).$$

The number of paths of odd length is bounded by the number of vertices of degree 1 which is $O(n^{2\alpha_1+o(1)})$.

This completes the proof of Lemma 3.

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A Proof of Lemma 12

For $\boldsymbol{\nu} = (v_{1,R}, v_{1,C}, v_R, v_C, m)$ let $Z_{\boldsymbol{\nu}} = \{(\mathbf{x}, \mathbf{y}) \in ([n] \cup \{\star\})^{2cn}$ with m pairs $x_i, y_i \neq \star$ etc..

Lemma 20. *Each* $(\mathbf{x}', \mathbf{y}') \in Z_{\boldsymbol{\nu}'}$ *arises by a transition of KSGREEDY from the same number* $D(\boldsymbol{\nu}, \boldsymbol{\nu}')$ *of* $(\mathbf{x}', \mathbf{y}') \in Z_{\boldsymbol{\nu}}$.

Proof **Case 1:** $v_1 > 0$ and an R-vertex x of degree 1 is selected and its neighbour y is of degree at least 2.

Let y be the C-neighbour of x in $G_{\mathbf{x},\mathbf{y}}$ and suppose y has $k_{i,j}$ neighbours of degree i that are incident $j \le i$ times with y (multiple edges). Then

$$\begin{aligned}
m' &= m - 1 - \sum_{i,j} j k_{i,j} \\
v'_{1,C} &= v_{1,C} \\
v'_{1,R} &= v_{1,R} - 1 - k_{1,1} + \sum_{i \ge 2} k_{i,i-1} \\
v'_C &= v_C - 1 \\
v'_R &= v_R - \sum_{i \ge 2} k_{i,i-1} - \sum_{i \ge 2} k_{i,i}
\end{aligned}$$

Given $\mathbf{x}', \mathbf{y}'$, our choices for $\mathbf{x}, \mathbf{y}$ are determined as follows: Observe that always the number of choices depends only on $\boldsymbol{\nu}, \boldsymbol{\nu}'$. Also let $n_R = n - v'_{1,R} - v'_R, n_C = n - v'_{1,C} - v'_C$ be the number of vertex labels missing from $\mathbf{x}', \mathbf{y}'$. Each quantity below should be multiplied by the number of choices of where to replace $\star$'s by vertex labels.

- Choose x, y in $n_R n_C$ ways.
- Choose the sequence $k_{i,j}$ such that the above equations hold.
- Choose the labels for the $\sum_i k_{i,i}$ new isolated R vertices in $\binom{n_R - 1}{k_{1,1}, k_{2,2}, \ldots}$ ways.
- Choose the labels for the $\sum_{i \ge 2} k_{i,i-1}$ vertices which become degree 1 in $\binom{v'_{1,R}}{k_{2,1}, k_{3,2}, \ldots}$ ways.
- Now let

  $$\mu = \sum_{i,j} j k_{i,j} - \sum_{i \ge 1} k_{i,i} - \sum_{i \ge 2} k_{i,i-1} = m - m' - 1 - \sum_{i \ge 1} i k_{i,i} - \sum_{i \ge 2} (i-1) k_{i,i-1}$$

  be the number of unaccounted for edges. These edges join y to vertices which remain of degree at least 2. Assign labels to these edges in $(v'_R)^\mu$ ways.

Case 2: $v_1 > 0$ and an R-vertex x of degree 1 is selected and its neighbour y is of degree 1.

Here there are $n_R n_C$ times the number of choices of where to replace $\star$'s by vertex labels.

Case 3: $v_1 = 0$

We add the parameters $\ell_{i,j}$ for the number of neighbours of x, other than y, which are of degree i and are incident $j \le i$ times with x. We then have

$$\begin{aligned}
m' &= m - 1 - \sum_{i,j} j k_{i,j} \\
v'_{1,C} &= \sum_{i \ge 2} \ell_{i,i-1} \\
v'_{1,R} &= \sum_{i \ge 2} k_{i,i-1} \\
v'_C &= v_C - \sum_{i \ge 2} \ell_{i,i-1} - \sum_{i \ge 2} \ell_{i,i} \\
v'_R &= v_R - \sum_{i \ge 2} k_{i,i-1} - \sum_{i \ge 2} k_{i,i}
\end{aligned}$$

The number of choices for $\mathbf{x}, \mathbf{y}$ can now be enumerated:

- Choose x, y in $n_R n_C$ ways.
- Choose the sequences $k_{i,j}, \ell_{i,j}$ such that the above equations hold.
- Choose the labels for the $\sum_i k_{i,i}$ new isolated R vertices in $\binom{n_R-1}{k_{1,1},k_{2,2},\ldots}$ ways.
- Choose the labels for the $\sum_i \ell_{i,i}$ new isolated C vertices in $\binom{n_C-1}{k_{1,1},k_{2,2},\ldots}$ ways.
- Choose the labels for the $\sum_{i\geq 2} k_{i,i-1}$ R vertices which become degree 1 in $\binom{v'_{1,R}}{k_{2,1},k_{3,2},\ldots}$ ways.
- Choose the labels for the $\sum_{i\geq 2} \ell_{i,i-1}$ C vertices which become degree 1 in $\binom{v'_{1,C}}{k_{2,1},k_{3,2},\ldots}$ ways.
- Now let

$$\mu = m - m' - \rho - \sum_{i\geq 2} ik_{i,i} - \sum_{i\geq 2}(i-1)k_{i,i-1} - \sum_{i\geq 2} i\ell_{i,i} - \sum_{i\geq 2}(i-1)\ell_{i,i-1}$$

  be the number of unaccounted for edges. Let $\mu_R = \sum_{j\leq i-2} jk_{i,j}$ and $\mu_C = \mu - \mu_R$. Assign labels to these edges in $(v'_R)^{\mu_R}(v'_C)^{\mu_C}$ ways.

□

The rest of the proof of Lemma 12 is essentially identical to that of Lemma 3 of [1]. □

B Local Limit Theorem

Lemma 21. *If $\sum_i \mathbf{Var}(Y_i) \to \infty$ and the c_i are uniformly bounded, then*

$$\mathbf{Pr}\left(\sum_{i=1}^{\nu_1} Y_i = \mu + a\right) = \frac{1 + O(a^2(\nu_1\rho_1)^{-1})}{(2\pi\sum_i \mathbf{Var}(Y_i))^{1/2}},$$

if $a^2(\nu_1\rho_1)^{-1} \to 0$. An analogous formula holds for $\mathbf{Pr}\left(\sum_j Z_j = \mu + a\right)$.

Proof Let $W = \sum_\ell Y_\ell$. As usual, we start with the inversion formula

$$\begin{aligned}\mathbf{Pr}\,(W = \tau) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\tau x}\mathbf{E}\left(e^{ix\sum_\ell Y_\ell}\right)dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\tau x}\prod_{\ell=1}^{\nu_1}\mathbf{E}(e^{ixY_\ell})^v\,dx, \end{aligned} \quad (120)$$

where $\tau = \mu + a$. Let

$$\Sigma_1 = \sum_\ell \frac{\rho_1}{c_\ell + 1} = \Theta(\nu_1\rho_1)$$

and consider first $|x| \geq \Sigma_1^{-5/12}$. Using an inequality (Pittel [17])

$$|f_t(z)| \leq f_t(|z|)\exp\left(-\frac{|z| - \text{Re } z}{t+1}\right), \quad (121)$$

we estimate

$$\begin{aligned}\frac{1}{2\pi}\int_{|x|\geq\Sigma_1^{-5/12}}\left|e^{-i\tau x}\prod_{\ell=1}^{\nu_1}\left(\frac{f_{c_\ell}(e^{ix}\rho_1)}{f_{c_\ell}(\rho_1)}\right)\right|dx &\leq \frac{1}{2\pi}\int_{|x|\geq\Sigma_1^{-5/12}}\prod_{\ell=1}^{\nu_1}e^{\rho_1(\cos x-1)/(c_\ell+1)}\,dx\\ &\leq e^{-\Sigma_1^{1/6}/3}. \qquad (122)\end{aligned}$$

For $|x|\leq\Sigma_1^{-5/12}$, putting $\eta=\rho_1e^{ix}$ and using $\sum_\ell\rho_1f'_{c_\ell}(\rho_1)/f_{c_\ell}(\rho_1)=\mu$, $d/dx=i\eta d/d\eta$ we expand as a Taylor series around $x=0$ to obtain

$$\begin{aligned}-i\tau x+\sum_\ell\log\left(\frac{f_{c_\ell}(e^{ix}\rho_1)}{f_{c\ell}(\rho_1)}\right) &= -iax-\frac{x^2}{2}\sum_\ell\mathcal{D}\left(\frac{\eta f'_{c_\ell}(\eta)}{f_{c_\ell}(\eta)}\right)\Bigg|_{\eta=\rho_1}\\ &-\frac{ix^3}{3!}\sum_\ell\mathcal{D}^2\left(\frac{\eta f'_{c_\ell}(\eta)}{f_{c_\ell}(\eta)}\right)\Bigg|_{\eta=\rho_1}\\ &+O\left[x^4\sum_\ell\mathcal{D}^3\left(\frac{\eta f'_{c_\ell}(\eta)}{f_{c_\ell}(\eta)}\right)\Bigg|_{\eta=\tilde{\eta}}\right]; \qquad (123)\end{aligned}$$

here $\tilde{\eta}=\rho_1e^{i\tilde{x}}$, with $\tilde{x}$ being between 0 and x, and $\mathcal{D}=\eta(d/d\eta)$. Now, the coefficients of $x^2/2$, $x^3/3!$ and x^4 are $\mathbf{Var}(W)$, $O\left(\mathbf{Var}(W)\right)$, $O\left(\mathbf{Var}(W)\right)$ respectively, and $\mathbf{Var}(W)$ is of order Σ_1. So the second and the third terms in (123) are $o(1)$ uniformly for $|x|\leq\Sigma_1^{-5/12}$. Therefore

$$\frac{1}{2\pi}\int_{|x|\leq\Sigma_1^{-5/12}}=\int_1+\int_2+\int_3, \qquad (124)$$

where

$$\begin{aligned}\int_1 &= \frac{1}{2\pi}\int_{|x|\leq\Sigma_1^{-5/12}}e^{-iax-\mathbf{Var}(W)x^2/2}\,dx\\ &= \frac{1}{\sqrt{2\pi\mathbf{Var}(W)}}+O\left(\frac{a^2+1}{\Sigma_1^{3/2}}\right), \qquad (125)\\ \int_2 &= O\left(\sum_\ell\mathcal{D}^2\left(\frac{\rho_1f'_{c_\ell}(\rho_1)}{f(\rho_1)}\right)\int_{|x|\leq\Sigma_1^{-5/12}}x^3e^{-\mathbf{Var}(W)x^2/2}\,dx\right)\\ &= O\left(\Sigma_1\int_{|x|\geq\Sigma_1^{-5/12}}|x|^3e^{-\mathbf{Var}(W)x^2/2}\,dx\right)\\ &= O(e^{-\alpha\Sigma_1^{1/6}}), \qquad (126)\end{aligned}$$

($\alpha>0$ is an absolute constant), and

$$\begin{aligned}\int_3 &= O\left[\Sigma_1\int_{|x|\leq\Sigma_1^{-5/12}}x^4e^{-\mathbf{Var}(W)x^2/2}\,dx\right]\\ &= O\left(\frac{1}{\Sigma_1^{3/2}}\right). \qquad (127)\end{aligned}$$

Using (120)-(127), we arrive at

$$\mathbf{Pr}\left(W=\tau\right)=\frac{1}{\sqrt{2\pi v\mathbf{Var}(W)}}\times\left[1+O\left(\frac{a^2+1}{\Sigma_1}\right)\right].$$

□

C Concentration of W.

We need to prove the following result.

Let S be a set with $|S| = N$. Let Ω be the set of $N!$ permutations of S. Let ω be chosen uniformly from Ω.

Let $Z = Z(\omega)$ be such that $|Z(\omega) - Z(\omega')| \leq 1$ when ω' is obtained from ω by interchanging two elements of the permutation.

Lemma 22.

$$\mathbf{Pr}(|Z - \mathbf{E}Z| \geq t) \leq 2e^{-2t^2/N}.$$

Proof For a fixed sequence permutation $(x_1, x_2, \ldots, x_n)$ and $0 \leq i \leq N$ let

$$Z_i(x_1, x_2, \ldots, x_i) = \mathbf{E}(Z \mid \omega_j = x_j, 1 \leq j \leq i).$$

Clearly the sequence $Z_0, Z_1, \ldots, Z_N$ is a martingale. To apply the Azuma -Hoeffding inequality, we need to show that

$$|Z_i(x_1, x_2, \ldots, x_i) - Z_i(x_1, x_2, \ldots, x_{i-1}, x_i')| \leq 1 \tag{128}$$

for all i-tuples $(x_1, x_2, \ldots, x_i)$ with distinct components, and $x_i' \neq x_1, \ldots, x_{i-1}$. (Indeed, the inequality (128) readily implies that $|Z_{i+1} - Z_i| \leq 1$.)

Consider

$$\Omega_1 = \{\omega \in \Omega : \ \omega_j = x_j, 1 \leq j \leq i\}$$

and

$$\Omega_1' = \{\omega \in \Omega : \ \omega_j = x_j, 1 \leq j \leq i-1, \omega_i = x_i'\}$$

and the map $f : \Omega_1 \to \Omega_1'$ defined as follows. If $\omega = x_1 x_2 \ldots x_{i-1} x_i y_{i+1} \ldots y_N$ and $y_j = x_i'$ then $f(\omega) = x_1 x_2 \ldots x_{i-1} x_i' y_{i+1} \ldots y_{j-1} x_i y_{j+1} \ldots y_N$. Observe that f is a bijection and that

$$|Z_i(x_1, x_2, \ldots, x_i) - Z_i(x_1, x_2, \ldots, x_i')| = \left| \frac{\sum_{y_{i+1}, \ldots, y_N} (Z(\omega) - Z(f(\omega)))}{(N-i)!} \right| \leq 1.$$

□