A NOTE ON THE LOCALIZATION NUMBER OF RANDOM GRAPHS: DIAMETER TWO CASE

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Abstract. We study the localization game on dense random graphs. In this game, a *cop* x tries to locate a *robber* y by asking for the graph distance of y from every vertex in a sequence of sets $W_1, W_2, \dots, W_\ell$. We prove high probability upper and lower bounds for the minimum size of each W_i that will guarantee that x will be able to locate y.

1. Introduction

In this paper we consider the following *Localization Game* related to the well studied *Cops and Robbers* game, see Bonato and Nowakowski [2] for a survey on this game. A robber is located at a vertex v of a graph G. In each round, a cop can ask for the graph distance between v and vertices $W = \{w_1, w_2, \dots, w_k\}$, where a new set of vertices W can be chosen at the start of each round. The cops win immediately if *the W-signature* of v, viz. the set of distances, $\text{dist}(v, w_i)$, $i = 1, 2, \dots, k$ is sufficient to determine v. Otherwise, the robber will move to a neighbor of v and the cop will try again with a (possibly) different *test set W*. Given G, the *localization number* $\lambda(G)$ is the minimum k so that the cop can eventually locate the robber. This game was introduced by Bosek et al. [3], who studied the localization game on geometric and planar graphs, and also independently, by Haslegrave et al. [6]. For some other related results see [4, 8, 9].

2. Results

The localization number is closely related to the *metric dimension* $\beta(G)$. This is the smallest integer k such that the cop can always win the game in *one* round. Clearly, $\lambda(G) \le \beta(G)$.

In this note we will study the localization number of the random graph $G_{n,p}$ with diameter two. Here and throughout the whole paper $\omega = \omega(n) = o(\log n)$ denotes a function tending arbitrarily slowly to infinity with n. We will also use the notation

$$q = 1 - p \text{ and } \rho = p^2 + q^2.$$

The metric dimension of $G_{n,p}$ was studied by Bollobás et al. [1]. If we specialize their result to large p then it can be expressed as:

Theorem 2.1 ([1]). *Suppose that*

$$\left(\frac{2\log n + \omega}{n}\right)^{1/2} \le p \le n\left(1 - \frac{3\log\log n}{\log n}\right).$$

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Then,

$$\frac{\log np}{\log 1/\rho} \lesssim \beta(G_{n,p}) \lesssim \frac{2\log n}{\log 1/\rho} \text{ a.a.s..} \tag{1}$$

(We write $A_n \lesssim B_n$ to mean that $A_n \le (1+o(1))B_n$ as n tends to infinity.) Note that the upper and lower bounds in (1) are asymptotically equal if $p \ge n^{-o(1)}$.

It is well-known (see, e.g., [5]) that if $np^2 \ge 2\log n + \omega$, then a.a.s. $\operatorname{diam}(G_{n,p}) \le 2$. We will condition on the diameter satisfying this. Graphs with diameter 2 enable some simplifications. Indeed, if a vertex v has W-signature $\{d_1, \dots, d_k\}$, where $W = \{w_1, \dots, w_k\}$, where $d_i = \operatorname{dist}(v, w_i)$, then

$$d_i = \begin{cases} 1 & \text{iff } \{v, w_i\} \in E \\ 2 & \text{iff } \{v, w_i\} \notin E. \end{cases}$$

Consequently, the probability that two vertices u and v in $G_{n,p}$ have the same W-signature, $W = \{w_1, \dots, w_k\}$, such that $u, v \notin W$ is equal to

$$\prod_{i=1}^{k} \mathbf{Pr}(u, v \in N(w_i) \text{ or } u, v \notin N(w_i)) = q^k.$$

The upper bound on p in the below theorem is determined by a result of [1] about the metric dimension of $G_{n,p}$.

Theorem 2.2. *Let*

$$\left(\frac{2\log n + \omega}{n}\right)^{1/2} \le p \le 1 - \frac{3\log\log n}{\log n} \quad \text{and} \quad \eta = \frac{\log(1/p)}{\log n}$$

and let c be a positive constant such that

$$0 < c < \min\left\{\frac{1}{2}\left(\frac{\log n - 3\log\log n}{\log 1/p} - 1\right), 1\right\}.$$

Then, a.a.s.

$$\left(1 - 2\eta - \frac{4\log\log n}{\log n}\right)\frac{2\log n}{\log 1/\rho} \le \lambda(G_{n,p}) \le (1 - c\eta)\frac{2\log n}{\log 1/\rho}.$$

2.1. **Observations about Theorem 2.2.**

First observe that if $p \ge \frac{\log n}{n^{1/3}}$, then

$$\frac{1}{2}\left(\frac{\log n - 3\log\log n}{\log 1/p} - 1\right) \ge 1$$

and so c can be any positive constant less than 1. Furthermore, for any $p \ge \left(\frac{2\log n + \omega}{n}\right)^{1/2}$ we have

$$\frac{1}{2}\left(\frac{\log n - 3\log\log n}{\log 1/p} - 1\right) \ge \frac{1}{2}\left(\frac{\log n - 3\log\log n}{\frac{1}{2}(\log n - \log(2\log n + \omega))} - 1\right) = \frac{1}{2} - o(1).$$

Hence, we can always take $c \ge \frac{1}{2} - o(1)$.

If $p = 1/n^\alpha$ for some constant $0 < \alpha < 1/2$, then,

$$\eta = \alpha \qquad \text{and} \qquad c \le \begin{cases} 1 - o(1) & \text{if } 0 < \alpha < \frac{1}{3} \\ \frac{1}{2\alpha} - \frac{1}{2} - o(1) & \text{otherwise.} \end{cases}$$

Moreover,

$$\rho = 1 - 2p + 2p^2 \text{ and so } \log 1/\rho = 2p + O(p^2) \approx \frac{2}{n^\alpha}.$$

Hence, Theorem 2.2 implies the following corollary.

Corollary 2.3. *Let $p = 1/n^\alpha$, where $0 < \alpha < 1/2$ is constant. Then, a.a.s.*

$$(1-2\alpha) n^\alpha \log n \lesssim \lambda(G_{n,p}) \lesssim \begin{cases} (1-\alpha) n^\alpha \log n & \text{if } 0 < \alpha < \frac{1}{3} \\ \left(\frac{1+\alpha}{2}\right) n^\alpha \log n & \text{otherwise.} \end{cases}$$

Also notice that the localization number here is always significantly smaller than the corresponding metric dimension [1].

Now observe that if $p = n^{-1/\omega}$, then

$$2\eta = \frac{2\log(1/p)}{\log n} = \frac{2}{\omega} = o(1).$$

Thus, Theorem 2.2 implies:

Corollary 2.4. *Let $p = n^{-1/\omega}$. Then,*

$$\lambda(G_{n,p}) \approx \frac{2\log n}{\log 1/\rho}.$$

Clearly, this also holds for any constant p. In particular, for $p = 1/2$, we get:

Corollary 2.5. *For almost all graphs G we have*

$$\lambda(G) \approx \frac{2\log n}{\log 2} = 2\log_2(n).$$

2.2. **Proof of Theorem 2.2 – lower bound.**

We will use the following form of Suen's inequality (see, e.g. [7]).

Suen's inequality. *Let $\theta_i, i \in I$ be indicator random variables which take value 1 with probability p_i. Let L be a dependency graph. Let $X = \sum_{i \in I} \theta_i$, and $\mu = \mathbf{E}(X) = \sum_{i \in I} p_i$. Moreover, write $i \sim j$ if $ij \in E(L)$, and let $\Delta = \frac{1}{2}\sum\sum_{i \sim j} \mathbf{E}(\theta_i\theta_j)$ and $\delta = \max_i \sum_{j \sim i} p_j$. Then,*

$$\mathbf{Pr}(X = 0) \le \exp\left\{ -\min\left\{ \frac{\mu^2}{8\Delta}, \frac{\mu}{2}, \frac{\mu}{6\delta} \right\} \right\}.$$

The lower bound in Theorem 2.2 will follow from the following result.

Lemma 2.6. *Let*

$$\left(\frac{2\log n + \omega}{n}\right)^{1/2} \le p \le 1 - \frac{1}{\log n} \qquad \text{and} \qquad \varepsilon = \frac{2\log\left(\frac{\log^2 n}{p}\right)}{\log n} \qquad \text{and} \qquad k = \frac{2(1-\varepsilon)\log n}{\log 1/\rho}.$$

Then a.a.s.,

$$\lambda(G_{n,p}) \ge k.$$

First observe that $\varepsilon = 2\eta + \frac{4\log\log n}{\log n}$ and so the lower bound in Theorem 2.2 holds.

Proof. For a fixed vertex u and k-set S let $X_{u,S}$ count the number of unordered pairs $w, v \in N(u)$ with the same signature induced by S. We prove that the probability that there is a vertex u and a k-set S such that $X_{u,S} = 0$ is $o(1)$. Consequently, this will imply that a.a.s. for every vertex u and k-set S there are at least two neighbors of u with the same signature in S. Hence, a.a.s. the localization number is at least k.
Clearly,

$$\mu = \mathbf{E}(X_{u,S}) = \binom{n-k-1}{2}\rho^k p^2 \geq \frac{p^2}{4}\exp\{k\log\rho + 2\log n\}$$
$$= \frac{p^2}{4}\exp\{-2(1-\varepsilon)\log n + 2\log n\} = \frac{p^2}{4}n^{2\varepsilon}$$

and

$$\begin{aligned}\Delta &\leq \binom{n}{3}(p^3+q^3)^k p^3 \\ &\leq \frac{p^3}{6}\exp\left\{k\log(p^3+q^3) + 3\log n\right\} \\ &= \frac{p^3}{6}\exp\left\{-2(1-\varepsilon)(\log n)\frac{\log(p^3+q^3)}{\log\rho} + 3\log n\right\}.\end{aligned}$$

Now, by Claim 2.7 below,

$$\Delta \leq \frac{p^3}{6}\exp\left\{-2(1-\varepsilon)(\log n)\cdot\frac{3}{2} + 3\log n\right\} = \frac{p^3}{6}n^{3\varepsilon}$$

and similarly

$$\delta \leq 2n(p^3+q^3)^k p^2 = 2p^2\exp\left(-3(1-\varepsilon)(\log n) + \log n\right) = 2p^2 n^{-2+3\varepsilon}.$$

Thus,

$$\frac{\mu^2}{8\Delta} \geq \frac{3}{64}pn^{\varepsilon}, \quad \frac{\mu}{2} \geq \frac{1}{8}(pn^{\varepsilon})^2 \quad \text{and} \quad \frac{\mu}{6\delta} \geq \frac{1}{48}n^{2-\varepsilon}.$$

Since $0 < \varepsilon < 1$ and $pn^{\varepsilon} \to \infty$ (due to our choice of ε) the lower bound in the first inequality is the smallest. Hence,

$$\mathbf{Pr}(X_{u,S} = 0) \leq \exp\left\{-\frac{3}{64}pn^{\varepsilon}\right\}.$$

Now we use the union bound to show that the probability that there is a vertex u and a k-set S such that $X_{u,S} = 0$ is $o(1)$. Indeed, this probability is at most

$$n\binom{n}{k}\exp\left\{-\frac{3}{64}pn^{\varepsilon}\right\} \leq \exp\left\{(k+1)\log n - \frac{3}{64}pn^{\varepsilon}\right\}. \tag{2}$$

Now observe that $\rho = (p+q)^2 - 2pq = 1 - 2pq$ and so

$$k = \frac{2(1-\varepsilon)\log n}{\log 1/\rho} = -\frac{2(1-\varepsilon)\log n}{\log(1-2pq)} \leq -\frac{2\log n}{\log(1-2pq)}.$$

Since $1 - x \geq e^{-2x}$ for any $0 \leq x \leq 1/2$ and $2pq \leq 1/2$ we get that

$$k \log n \leq \frac{(\log n)^2}{2pq}.$$

Furthermore, since by assumption $p \leq 1 - \frac{1}{\log n}$, we obtain $q \geq \frac{1}{\log n}$ and so

$$k \log n \leq \frac{(\log n)^3}{2p}.$$

Also

$$pn^{\varepsilon} = pe^{\varepsilon \log n} = \frac{(\log n)^4}{p}.$$

Thus, the exponent in (2) tends to $-\infty$. This completes the proof of Lemma 2.6. □

Claim 2.7. *Let $0 < p < 1$ and $p + q = 1$. Then,*

$$\frac{\log(p^3 + q^3)}{\log \rho} \geq \frac{3}{2}.$$

Proof. This inequality is equivalent to

$$\log(p^3 + q^3)^2 \leq \log(p^2 + q^2)^3$$

and so to

$$(p^3 + q^3)^2 \leq (p^2 + q^2)^3.$$

The latter is equivalent to

$$2p^3q^3 \leq 3p^4q^2 + 3p^2q^4 = 3p^2q^2(p^2 + q^2) = 3p^2q^2(1 - 2pq)$$

and consequently to

$$2pq \leq 3(1 - 2pq)$$

which is equivalent to

$$pq \leq \frac{3}{8}.$$

But the this is always true since $pq \leq \frac{1}{4}$. □

2.3. Proof of Theorem 2.2 – upper bound.

We will need the following auxiliary result:

Proposition 2.8. *Let*

$$\sqrt{\frac{2 \log n + \omega}{n}} \leq p \leq 1 - \frac{1}{\log n} \quad \textit{and} \quad \varepsilon = \frac{\log 1/p}{\log n} \quad \textit{and} \quad k = \frac{2(1 - c\varepsilon) \log n}{\log 1/\rho},$$

where

$$0 < c < \min\left\{\frac{1}{2}\left(\frac{\log n - 3 \log \log n}{\log 1/p} - 1\right), 1\right\}.$$

Let $G = G_{n,p} = (V, E)$ and let $U \subseteq V$ and $S \subseteq V$ be disjoint subsets such that $|U| = O(k)$ and $|S| = k$. Then, a.a.s. there is no *pair $u \in U$ and $v \in V \setminus S$ such that u and v have the same signature induced by S.*

Proof. Assume that ℓ is a positive constant and $|U| = \ell k$. The probability that there is a pair $u \in U$ and $v \in V \setminus S$ such that u and v have the same signature induced by S is at most

$$|U| \cdot |V| \cdot \rho^k \le \ell k \cdot \exp\{\log n + k \log \rho\} = \ell k \cdot n^{2c\varepsilon-1}.$$

But

$$k \le \frac{2\log n}{\log 1/\rho} = -\frac{2\log n}{\log(1-2pq)} \le \frac{\log n}{2pq},$$

since $1 - x \ge e^{-2x}$ for any $0 \le x \le 1/2$ and $2pq \le 1/2$. Furthermore, since $q \ge \frac{1}{\log n}$ we get

$$k \le \frac{(\log n)^2}{2p}.$$

Similarly,

$$n^{2c\varepsilon-1} = \frac{\exp\{2c\varepsilon \log n\}}{n} = \frac{1}{p^{2c}n}.$$

Thus,

$$\ell k \cdot n^{2c\varepsilon-1} \le \ell \frac{(\log n)^2}{2p} \cdot \frac{1}{p^{2c}n} = \frac{\ell(\log n)^2}{2p^{1+2c}n} \le \frac{\ell}{2\log n} = o(1),$$

where the latter inequality follows from the choice of c. □

Lemma 2.9.

(i) Let

$$e^{-\frac{\log n}{\omega}} \le p \le 1 - \frac{3\log\log n}{\log n}.$$

Then, a.a.s.

$$\lambda(G_{n,p}) \lesssim \frac{2\log n}{\log 1/\rho}.$$

(ii) Let

$$\left(\frac{2\log n + \omega}{n}\right)^{1/2} \le p \le e^{-\Omega(\log n)} \quad \text{and} \quad \eta = \frac{\log 1/p}{\log n} \quad \text{and} \quad k = \frac{2(1-c\eta)\log n}{\log 1/\rho},$$

where

$$0 < c < \min\left\{\frac{1}{2}\left(\frac{\log n - 3\log\log n}{\log 1/p} - 1\right), 1\right\}.$$

Then, a.a.s.

$$\lambda(G_{n,p}) \le k.$$

Proof. Part (i) follows immediately from Theorem 2.1.

Here we prove (ii). Let $S_1, \ldots, S_\ell$ be pairwise disjoint subsets of V such that $|S_i| = k$ and $\ell = O(1)$ and let $T_1 = V$. Now we reveal all edges between S_1 and $V \setminus S_1$. Let X_1 be the number of pairs with the same signature in S_1. Then,

$$\mathbf{E}(X_1) \le n^2 \rho^k = \exp\{2\log n - k\log\rho\} = n^{2c\eta}$$

and by the Markov inequality we have $X_1 \le \omega n^{2c\eta}$ a.a.s.. Thus, the set R of vertices with exactly the same signature in S as the robber is a.a.s. of size at most $\omega^{1/2} n^{c\eta}$. Also it is well known (see e.g. [5]) that each vertex a.a.s. has $\lesssim pn$ neighbors. Let T_2 consist

of R and the set of neighbors of R. The robber can move to somewhere in T_2. Clearly, $|T_2| \leq 2\omega^{1/2}n^{c\eta}pn$ a.a.s..
Now we start the second round by revealing the edges between S_2 and $V \setminus (S_1 \cup S_2)$. Let X_2 be the number of pairs with the same signature in S_2. By Proposition 2.8 we can assume that the only pairs with the same signature induced by S_2 are in $V \setminus (S_1 \cup S_2)$. Thus,

$$\mathbf{E}(X_2) \leq (2\omega^{1/2}n^{c\eta}pn)^2\rho^k = (2\omega^{1/2}p)^2 \exp((2+2c\eta)(\log n) - k\log\rho) = 4\omega p^2 n^{4c\eta}$$

and by the Markov inequality we get that a.a.s we have $X_2 \leq \omega^2 p^2 n^{4c\eta}$. Thus, the number of vertices with exactly the same signature as the robber in S_2 is at most $\omega p n^{2c\eta}$. Let T_3 consist of these vertices together with their neighbors. Clearly, $|T_3| \leq 2\omega p^2 n^{2c\eta+1}$.
We proceed inductively. Assume that $|T_i| \leq 2(\omega^{1/2}p)^{i-1}n^{(i-1)c\eta+1}$. Now

$$\mathbf{E}(X_{i+1}) \leq 2((\omega^{1/2}p)^{i-1}n^{(i-1)c\eta+1})^2\rho^k = 2(\omega^{1/2}p)^{2(i-1)}n^{2ic\eta}$$

and so by the Markov inequality,

$$X_{i+1} \leq \omega(\omega^{1/2}p)^{2(i-1)}n^{2ic\eta} \text{ a.a.s..} \tag{3}$$

Thus, the number of vertices with exactly the same signature in S_{i+1} is at most $\omega^{1/2}(\omega^{1/2}p)^{i-1}n^{ic\eta}$. Hence,

$$|T_{i+1}| \leq 2\omega^{1/2}(\omega^{1/2}p)^{i-1}n^{ic\eta}pn = 2(\omega^{1/2}p)^i n^{ic\eta+1},$$

completing the induction.
After ℓ rounds we get that with probability at least $1 - \ell\omega^{-1}$ we have, using (3),

$$\begin{aligned}|X_\ell| &\leq \omega(\omega^{1/2}p)^{2(\ell-2)}n^{2(\ell-1)c\eta} = \omega^{\ell-1}\exp\{2(\ell-2)\log p + 2(\ell-1)c\eta\log n)\} \\ &= \omega^{\ell-1}\exp\{-2(\ell-2-c(\ell-1))\log(1/p)\}.\end{aligned}$$

The latter is $o(1)$ for sufficiently large constant ℓ, since by assumption $\log(1/p) = \Omega(\log n)$. □

3. Summary

We have separated the localization value $\lambda(G_{n,p})$ from the metric dimension $\beta(G_{n,p})$ in the range where the diameter of $G_{n,p}$ is two a.a.s.. The same ideas should be applicable when p is smaller and it would be interesting to continue the analysis in this range. It would also be of interest to examine this problem on random regular graphs.

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