The height of random k-trees and related branching processes

Colin Cooper* Alan Frieze† Ryuhei Uehara ‡

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Abstract

We consider the height of random k-trees and k-Apollonian networks. These random graphs are not really trees, but instead have a tree-like structure. The height will be the maximum distance of a vertex from the root. We show that w.h.p. the height of random k-trees and k-Apollonian networks is asymptotic to $c \log t$, where t is the number of vertices, and $c = c(k)$ is given as the solution to a transcendental equation. The equations are slightly different for the two types of process. In the limit as $k \to \infty$ the height of both processes is asymptotic to $\log t/(k \log 2)$.

1 Introduction

We give a general method for obtaining the height of tree-like random processes, and illustrate the method by application to random k-trees and Apollonian networks.

The processes that we consider generate a sequence of graphs $G(t), t \geq 0$ where $G(t)$ is obtained from $G(t-1)$ by the addition of an extra vertex in some way. The initial structure is a k-clique with a distinguished vertex v, which we use as the root vertex. Of course, $G(t)$ is not necessarily a tree but it is convenient to adopt the terminology.

The height of a vertex u in $G(t)$ is its graph distance $d(v, u)$ from the root vertex v. The height $h(G(t))$ of $G(t)$ is the maximum height of one of its vertices. By considering the breadth first

*Department of Informatics, King's College, University of London, London WC2R 2LS, UK. Supported in part by EPSRC grant EP/J006300/1

†Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA15213, USA. Supported in part by NSF grant CCF0502793.

‡School of Information Science, JAIST, Asahidai 1-1, Nomi, Ishikawa 923-1292, Japan

search tree T_v rooted at v we can partition the vertices $u \in V$ into sets $L_i, i = 0, 1, ..., h_v$ based on the distance $i = d(v, u)$ from the root vertex v. We refer to sets L_i as the level sets of the BFS tree. The height of $G_k(t)$ is thus the height of T_v in the usual sense.

The general properties of random k-trees have been investigated by several authors including [4], [10], [17]. In particular, in an earlier study into the small world properties of random k-trees, Cooper and Uehara [4] found experimentally that the diameter of such trees was a rapidly decreasing function of k. The main result of this paper, given in Theorem 1, is that the diameter $D_k(t)$ of a random t vertex k-tree satisfies

$$\lim_{\substack{t\to\infty \\ k\to\infty}} \frac{k}{\log t}\, D_k(t) = \frac{2}{\log 2}.$$

The height of branching processes: Related work. The work in this area is so extensive it is impossible to summarize concisely. As our interest lies in the area of discrete random structures we must necessarily restrict our discussion to those authors who have had a direct influence on us, and on the techniques we use in this paper. Foremost among these are the works of Broutin and Devroye [3], Devroye [5, 6, 7], Kingman [15] and Pittel [18]. The formulation in these papers differs from the discrete context in which a new vertex is added at each step t, but the end product is the same. The basic model is a continuous time reproductive process in which the reproductive rate $\lambda(j)$ of the parent depends on the number of offspring j. Each child independently reproduces according to the same process. Such processes are known as a Crump-Mode-Jagers process (see Devroye [7]). The paper of Kingman [15] concerns the time B_N of the first birth in the N-th generation of an age dependent reproductive process of Crump-Mode type, with a proof that $B_N/N \to c$ as $N \to \infty$. To determine the constant c, the work uses the Cramér function of the process, which (crudely) is an optimization of the logarithm of the moment generating function of the distribution of reproduction waiting times. A full description of Cramér functions can be found in [3]. Pittel [18] applied Kingman's result to a branching process in which the number of children born to a parent within T steps is negative exponential with linear population dependent rate $\lambda(j) = aj + 1$ for some $a \geq 0$. This serves as a model of random recursive trees where a vertex chooses its parent v with probability proportional to $ad(v) + 1$, where $d(v)$ is the out-degree of v in the orientation of the tree away from the root. The cases of random and preferential attachment trees follow from setting $a = 0$, and $a = 1$ respectively. The general solution being that the height h_n of an n vertex tree satisfies $h_n \sim c \log n$, where $c = 1/((a + 1)\gamma)$ and γ is the positive root of $a\gamma + \log\gamma + 1 = 0$. (We use $A_n \sim B_n$ to denote $A_n = (1 + o(1))B_n$ as $n \to \infty$). In a sequence of papers, Devroye and Broutin and Devroye, develop a general approach in which the central structure is an infinite tree with branching factor b and a pair of independent random variables (Z, E) on the edges. The random variable Z measures increments in weighted height, and E measures the delay between birth of the parent and birth of the child. Typically E would be negative exponential rate 1. The height of a vertex u is the sum of the Z entries on the path from the vertex u to the root v. If h_T is the maximum height of the subtree at time T, then $h_T/T \to c$, where c is the maximum along a particular curve of an identity based on the

Cramér functions of Z and E, thus extending the original proof of Kingman [15]. A complete explanation of the technique, and a wide range of supporting examples are given in [3]. In general, for branchings based on the minimum of exponential waiting times with varying rate parameter, the time T and population size $n(T)$ are related by $T = \Theta(\log n)$. The need to obtain the explicit constant somewhat complicates the discussion.

The height of random k-trees. In the area of graph algorithms, k-trees form a well known graph class that generalize trees and play an important role in the study of graph minors (see [1, 2] for further details). One definition (among many) of k-trees is the following:

For any fixed positive integer $k \geq 2$, (i) A complete graph K_k of k vertices is a k-tree, (ii) For a k-tree G of t vertices, a new k-tree G' of $t+1$ vertices is obtained by adding a new vertex v incident to a clique of size $k-1$ in G. When $k = 2$, this process forms a tree, by extending a chosen vertex. When $k = 3$, the process forms a tree of triangles by extending the chosen edge, and so on.

The preferential attachment method for generating random tree processes extends the graph by attaching a new vertex to an existing vertex chosen with probability proportional to its degree. This is equivalent to choosing to attach to a random end point of a random edge, i.e. a random K_1 of a random K_2. The k-tree process described below generalizes this approach, in that we attach the new vertex to a random $(k-1)$-clique of a random k-clique.

A random k-tree $G_k(t)$, $t \geq k$, is obtained as follows. Start with $G_k(1)$, a k-clique C_1 with a distinguished vertex v_1. For $t > 1$, we obtain $G_k(t)$ from $G_k(t-1)$ by adding a vertex v_t and a set of $k-1$ edges from v_t to $G_k(t-1)$ chosen in the following way. Pick a k-clique C of $G_k(t-1)$ uniformly at random (*uar*), and choose a $(k-1)$-dimensional face F of C uar. Extend F to a k-clique C_t by the addition of edges from v_t to the vertices of F. Let the vertex set of C be $\{u_1, ..., u_k\}$. As $\{v_t, u_1, ..., u_k\}$ induces no k-cliques except C and C_t, the number of k-cliques in $G_k(t)$ is t. We are interested in the height of $G_k(t)$ above the distinguished vertex v_1, i.e. the maximum graph distance from v_1 to any vertex of $G_k(t)$.

A random 2-tree $G_2(t)$ is obtained by joining v_t to a random end point of a random edge; i.e. by preferential attachment. Thus the height of random 2-trees is given by the result of Pittel [18] discussed above. In the case of a random tree on t vertices generated by preferential attachment Pittel [18] established the w.h.p. result that the height $h(t)$ of the tree is asymptotic to $h(t) \sim c \log t$ where $c = 1/(2\gamma)$ and γ is the smallest positive solution to $1 + \gamma + \log \gamma = 0$. We include this result in our general statement as a special case. It follows naturally as a special case of our method and serves to check correctness of the base case.

Theorem 1. *For $k \geq 2$ let $h(t;k)$ be the height of a random k-tree on t vertices. Then w.h.p. $h(t;k) \sim c \log t$ where c is given as follows:*

Case $k = 2$ *[18], c is the solution of*

$$\frac{1}{2c}\exp\left\{1+\frac{1}{2c}\right\}=1.$$

Case $k \geq 3$ **constant**, *c is the solution of*

$$\frac{1}{c}=\sum_{\ell=0}^{k-2}\frac{k}{\ell+ak},$$

where the value of a is given by

$$\frac{\Gamma(k)\Gamma(ka)}{\Gamma(ka+k-1)}\ \exp\left(\sum_{\ell=0}^{k-2}\frac{ka+k-1}{\ell+ak}\right)=1,$$

and $\Gamma(\ell)$ *is the gamma function.*

Case $k \to \infty$

$$c \sim \frac{1}{k\log 2}.$$

In the above theorem, and in Theorem 2 we assume that k is constant or tends slowly to infinity with t. The bound on k used in the proofs is $k = o(\log^{1/3} t)$, but we do not attach any special significance to this value.

The table below compares asymptotic value of height (rounded up to the next integer) and results found by experiment for k-trees on $t = 2^{27}$ vertices.

Value of k	2	3	4	5	6	8	10	12	15	20
Height: Experimental result	16	10	8	7	5	4	4	3	3	3
Height: $\lceil \log t/(k\log 2)\rceil$	14	9	7	6	5	4	3	3	2	2

See Figure 1 in the appendix for a plot of the results obtained as a function of t, and Figure 2 for the fit to $\lceil \log t/(k\log 2)\rceil$.

The height of random k-Apollonian networks. An Apollonian network is the generalization of an Apollonian triangulation, which can be described as follows. Initially there is single triangle embedded in the plane. At the first step this triangle ABC is divided into three by insertion of a point D in the interior of the triangular face and adding lines DA, DB, DC. The triangles ABD, ACD, BCD replace the original triangle ABC in the embedded triangulation of the plane. At each subsequent step some triangular face is subdivided in the same manner.

A random k–Apollonian network $A_k(t)$, $t \geq 0$, is obtained as follows. Start with $A_k(0)$ a k-clique $C_0 = K_k$ with vertex set $\{c_1, ..., c_k\}$ embedded in $k-1$ dimensions. For $t > 0$, make $A_k(t)$ from $A_k(t-1)$ by adding a vertex v_t, and edges chosen as follows. Pick a k-clique C of $A_k(t-1)$ uar. Let the vertex set of C be $U = \{u_1, ..., u_k\}$. Insert v_t in the interior of C and join v_t to $u_i, i = 1, ..., k$ by an edge $u_i v_t,\ i = 1, ..., k$. This replaces C by k new embedded cliques with vertices $U + v_t - u_i,\ i = 1, ..., k$. (We will use the notation $U + v - w$ to mean $(U \cup \{w\}) \setminus \{v\}$.) The number of embedded k-cliques in $A_k(t)$ is $(k-1)t + 1$.

Theorem 2. *For $k \geq 3$ let $h(t; k)$ be the height of a random k–Apollonian network on t vertices. Then w.h.p. $h(t; k) \sim c \log t$ where c is given as follows:*

Case $k \geq 3$ constant *c is the solution of*

$$\frac{1}{c} = \sum_{\ell=0}^{k-1} \frac{k-1}{\ell + a(k-1)}, \tag{1}$$

where the value of a is given by

$$\frac{k!}{(a(k-1)) \cdots ((a+1)(k-1))} \exp\left(\sum_{\ell=0}^{k-1} \frac{(k-1)(a+1)-1}{\ell + a(k-1)}\right) = 1. \tag{2}$$

Case $k \to \infty$

$$c \sim \frac{1}{k \log 2}.$$

Recently, and independently of this work results for random Apollonian networks were obtained by Ebrahimzadeh et al [9] and Kolossváry [16]. The work of [9] adapted the results of Broutin and Devroye [3] to derive the height of random Apollonian triangulations. The value of $c = 0.8342...$ they obtained is the solution to (1), (2) with $k = 3$ and corresponds to a value of $a = 2.0683$. The work of [16] uses a different approach based on codes combined with general techniques for Markov processes. In an earlier work, Frieze and Tsourakis [13] bounded the height of random Apollonian triangulations from above by the height of a random 3-branching, using a result of [3].

General method The technique we describe is simple, bypasses the classic continuous time branching process results, (no prior knowledge needed) and works well for the complicated multi-type branching processes involved in k-trees, Apollonian networks etc. The main requirement is that the quantities $W_i(t)$ we estimate can be expressed as recurrences of the form

$$W_i(t+1) = W_i(t) + \frac{1}{t} \sum_{j \leq i} \alpha_{ij} W_j(t).$$

By partitioning the steps $t = 0, 1, 2...$ into small intervals, lower and upper bound approximations for $W_i(t)$ are obtained which can then be expressed via rational generating functions from which the coefficients can be extracted.

2 The height of random k-trees

2.1 Proof outline of the main theorems

In the construction of $G_k(t)$ we add a new vertex v_t at each step and extend this vertex to a unique k-clique. The number of of k-cliques in $G_k(t)$ is thus equal to the number of vertices t.

We use the parameters $\omega = \omega(t)$, s, and m given by

$$\omega = (1 + o(1)) \log^{2/3} t, \qquad s = \left\lceil t^{1/\omega} \right\rceil = o(t), \qquad m = \frac{\log t/s}{\log\left(1 + \frac{1}{\omega}\right)}, \tag{3}$$

where the $(1 + o(1))$ term is chosen to make m integer. The proof in Section 3 assumes $k = o(\sqrt{\omega})$, but our choice of ω is somewhat arbitrary anyway. The purpose of the parameter ω is to interpolate the interval $s, s+1, ..., t$ at steps $s_j = s(1 + 1/\omega)^j$, $j = 0, ..., m$.

We will prove Theorem 1 in the following way:

1. Let the height $h(G_k(t))$ be denoted by $h(t)$. We break our analysis of $h(t)$ into two parts.

   Let C be a fixed k-clique added at some step $1, ..., s$. If we consider only those k-cliques added at steps $s+1, ..., t$, then some (possibly empty) subset of these form a k-tree $G_C(t)$ rooted at C, i.e. at the lowest labeled vertex of C. This k-tree $G_C(t)$ is a subgraph of $G_k(t)$. Note that if C, C' are distinct k-cliques added at steps $1, ..., s$ the subgraphs $G_C(t)$ and $G_{C'}(t)$ have no k-cliques in common.

   Let the k-cliques added at steps $i = 1, ..., s$ be indexed $C_i : i = 1, ..., s$. The main problem is to obtain an asymptotic estimate for the maximum height of the subtrees $G_{C_i}(t),\ i = 1, .., s$. Let
   $$h_s(t) = \max_{C_i, i=1,...,s} \left\{ h(G_{C_i}(t)) \right\}. \tag{4}$$

   Informally, $h_s(t)$ is the height of $G_k(t)$ if we regard the first s of the k-cliques as rooted at level zero. The fact that C may be an ancestor of C' in $G_k(s)$ is not relevant to our estimate of $h_s(t)$.

   Let $h_0(s)$ be the height $h(s)$ of $G_k(s)$ rooted at v_1. The height $h(t)$ of $G_k(t)$ is bounded by
   $$h_s(t) \leq h(t) \leq h_0(s) + h_s(t). \tag{5}$$

2. In our description of the BFS tree T_v rooted at a distinguished vertex v. Let $N = 0, 1, ...$ denote the levels of T_v. For every k-clique C in $G_k(t)$, there is a level N, such that the vertices of C lie only in levels N and $N + 1$ of T_v (see Section 2.2). We use the notation $[N, (\ell, k - \ell)]$ to refer to those cliques with ℓ vertices in level N of the BFS tree T_v and $k - \ell$ vertices in level $N + 1$.

3. Let $W_{N,\ell}(t)$ be the expected number of $[N,(\ell,k-\ell)]$ configured cliques at step t rooted at any of the first s cliques. In Section 2.2 we obtain a recurrence for $W_{N,\ell}(t)$. In Sections 2.3 and 2.4 we obtain generating functions for lower and upper bounds $W^L_{N,\ell}(t) \leq W_{N,\ell}(t) \leq W^U_{N,\ell}(t)$.

4. Let $W_N(t) = W_{N,2}(t)$ be the expected number of $[N,(2,k-2)]$ configured cliques at step t rooted at any of the first s cliques. The height of these cliques above $G_k(s)$ is $N+1$. As the height depends on N but not ℓ, the value $\ell = 2$ was chosen for convenience in the proof.

5. In Section 3 we see how to extract the coefficients of the generating functions for $W^L_N(t), W^U_N(t)$.

6. Let $N = c\log(t/s)$, and let $N' = (1-\epsilon)N$, $N'' = (1+\epsilon)N$ for some $\epsilon \to 0$. In Sections 3.1 and 3.2 we find a value of N such that $W^U_{N''}(t) \to 0$ but $W^L_{N'}(t) \to \infty$. Thus w.h.p. $h_s(t) < N''$.

7. In Section 4 we prove that the height of a random k-tree at step t is at least N' w.h.p.

8. Let $h_0(s)$ be the height of $G_k(s)$ rooted at v_1. We argued above that

$$h_s(t) \leq h(t) \leq h_0(s) + h_s(t).$$

In Lemma 4 we prove that $h_0(s) = O(\log s)$ w.h.p., thus for s as given in (3), $\log s = (\log t)/\omega$. We have established that w.h.p. $h_s(t) \sim N = c\log(t/s)$, and thus

$$c\log t \leq h(t) \leq c\log t + O((\log t)/\omega).$$

As we assume that $k^2 = O(\omega)$, and from our proof $c = O(1/k)$, it follows that $G_k(t)$ has height $h(t) \sim c\log t$, w.h.p.

2.2 Recurrence for tree height

We will describe the structure of $G_k(t)$ in terms of the levels of vertices within each clique relative to the root vertex v_1. The following example using $k = 3$ is instructive of our labeling method. In $G_3(3)$ the initial clique $C_3 = K_3$ has v_1 at level $i = 0$ of the BFS tree and v_2, v_3 at level $i+1 = 1$. The index of the lowest level of C_3 is $i = 0$ and C_3 is oriented $(1,2)$ in that one vertex (v_1) is at level i and two vertices (v_2, v_3) are at level $i+1$. We will say that C_3 is configured $[0,(1,2)]$. Extending a face of C_3 gives rise to three possibilities. If face v_1v_2 or v_1v_3 is chosen, we obtain another $[0,(1,2)]$ configured clique C. If face v_2v_3 is chosen we obtain a $[1,(2,1)]$ configured clique $\{v_2, v_3, v_4\}$ between levels $i = 1$ and $i+1 = 2$.

We recall the inductive definition of a random k-tree. A random k-tree $G_k(t)$, $t \geq k$, is obtained as follows. Start with $G_k(1)$ a k-clique C_1 with a distinguished vertex v_1, i.e. with vertices $v_1, x_2, ..., x_k$, say. At subsequent steps $t > 1$, we obtain $G_k(t)$ from $G_k(t-1)$ by adding a k-clique C_t with distinguished vertex v_t and a set of $k-1$ edges from v_t to $G_k(t-1)$ chosen as follows. Pick a k-clique C of $G_k(t-1)$ uniformly at random. Let the vertex set of C be $\{u_1, ..., u_k\}$. Choose a $(k-1)$-dimensional face F of C uar. Suppose, for the purposes of description that the vertices of F are $\{u_1, ..., u_{k-1}\}$. Extend F to a k-clique $C_t = \{v_t, u_1, ..., u_{k-1}\}$ by the addition of edges from v_t to the vertices of F. As $\{v_t, u_1, ..., u_k\}$ induces no k-cliques except C and C_t, the number of k-cliques in $G_k(t)$ is t, and the number of vertices is $t + k - 1$. The precise k-cliques which have been added to form $G_k(t)$ can be found by choosing the k-clique containing the vertex with the highest label v_t, and deleting this vertex recursively.

In general we use the notation *clique* to refer to a k-clique which has been added according to our recursive process, and *face* to refer to a clique of dimension $k-1$. We regard $G_k(t)$ as rooted a vertex v_1. We are interested in the height of $G_k(t)$ rooted at v_1. The level sets of the vertices of the breadth first search tree rooted at v_1 form a convenient descriptive device. Inductively the vertices of each k-clique C lie in two adjacent levels i and $i+1$ of this BFS tree. The notation $[i, (\ell, k-\ell)]$ describes a k-clique C with ℓ vertices at level i and $k-\ell$ vertices at level $i+1$ relative to the BFS tree rooted at v_1. In this case we say C is $[i, (\ell, k-\ell)]$ configured. In this notation, the initial clique C_1 containing the foot vertex v_1 is $[0, (1, k-1)]$ configured.

Given that C is $[i, (\ell, k-\ell)]$ configured, the number and type of possible extensions of faces F of C to a new k-clique C' are obtained as follows. An extension of $C = \{u_1, ..., u_k\}$ consists of deleting a vertex u_j (to obtain a face F) and then inserting a vertex v to form $C' = \{u_1, ..., u_{j-1}, v, u_{j+1}, ..., u_k\}$. If the deleted u_j is chosen among the $k-\ell$ vertices at level $i+1$ then C' is configured $[i, (\ell, k-\ell)]$. If u_j is chosen among the ℓ vertices at level i, then provided $\ell > 1$, C' is configured $[i, (\ell-1, k-\ell+1)$. In the case that $\ell = 1$, so that C is configured $[i, (1, k-1)]$ then deleting u_1 results in a clique C' configured $[i+1, (k-1, 1)]$ with $k-1$ vertices at level $i+1$ and one vertex at level $i+2$.

Our first step is as follows. We first obtain bounds for $h_s(t)$ as defined in (4). Referring to (5) we will argue later that, for suitable choice of s, we have $h_0(s) = o(h_s(t))$, and hence $h(t) \sim h_s(t)$.

We will modify the notation $[i, (\ell, k-\ell)]$ to deal with our calculation of $h_s(t)$. Recall that $h_s(t)$ is the height of $G_k(t)$ if we regard the first s of the k-cliques as rooted at level zero. Let C' be a clique that was added at steps $s+1, ..., t$. Then C' is a descendant of one of the cliques added at the first s steps, i.e. $C^\star \in \{C_1, ..., C_s\}$. In this case we say that C' is *relatively* configured $[i, (\ell, k-\ell)]$ if C' is i levels higher than the level of $C^\star$ in $G_k(t)$.

Let $W_{i,\ell}(t)$ be the expected number of $[i, (\ell, k-\ell)]$ relatively configured cliques in $G_k(t)$. By assumption, at step s, $W_{0,1}(s) = s$. We have the following recurrences.

$$W_{0,1}(s) = s, \qquad W_{0,\ell}(t) = 0, \qquad \ell \geq 2, t \geq s. \tag{6}$$

Case $i = 0$: $[0, (1, k-1)]$ relatively configured cliques.

$$W_{0,1}(t+1) = W_{0,1}(t) + \frac{k-1}{k}\frac{W_{0,1}(t)}{t}. \tag{7}$$

Case $\ell = k-1, i \geq 1$: $[i, (k-1, 1)]$ relatively configured cliques.

$$W_{i,k-1}(t+1) = W_{i,k-1}(t) + \frac{1}{k}\frac{W_{i,k-1}(t)}{t} + \frac{1}{k}\frac{W_{i-1,1}(t)}{t}. \tag{8}$$

Case $\ell \neq k-1, i \geq 1$: $[i, (\ell, k-\ell)]$ relatively configured cliques .

$$W_{i,\ell}(t+1) = W_{i,\ell}(t) + \frac{k-\ell}{k}\frac{W_{i,\ell}(t)}{t} + \frac{\ell+1}{k}\frac{W_{i,\ell+1}(t)}{t}. \tag{9}$$

The recurrences (7)-(9) can be explained in the following way. To be specific, consider (9), and $t > s$. Let $\boldsymbol{W}_{i,\ell}(t)$ be a random variable giving the number of $[i, (\ell, k-\ell)]$ relatively configured cliques in $G_k(t)$. Then

$$\mathbf{E}(\boldsymbol{W}_{i,\ell}(t+1) \mid \boldsymbol{W}_{i,\ell}(t), \boldsymbol{W}_{i,\ell+1}(t)) = \boldsymbol{W}_{i,\ell}(t) + \frac{k-\ell}{k}\frac{\boldsymbol{W}_{i,\ell}(t)}{t} + \frac{\ell+1}{k}\frac{\boldsymbol{W}_{i,\ell+1}(t)}{t}.$$

The term $(k-\ell)/k$ is the probability to pick a face F with $k-\ell-1$ vertices at level $i+1$, from a $[i, (\ell, k-\ell)]$ relatively configured clique. Similarly, the term $(\ell+1)/k$ is the probability to pick a face F with ℓ vertices at level i from a $[i, (\ell+1, k-\ell-1)]$ relatively configured clique. Taking expectations again gives (9).

2.3 Lower bound for $W_{i,\ell}(t)$: Generating Function

As all of the $W_{i,\ell}(t)$ are monotone non-decreasing in t, replacing t by $t' \leq t$ in (6)-(9) gives a lower bound for the expected number of $[i, (\ell, k-\ell)]$ relatively configured cliques at step $t+1$. Recall from (3) that $s = \lceil t^{1/\omega} \rceil$. For $j \geq 0$ we will break the steps $s, s+1, ..., t$ into intervals $I_j = [s_j, s_{j+1}-1]$ where $s_0 = s$ and $s_j = \lceil s\lambda_0^j \rceil$. Here $\lambda_0 = 1 + 1/\omega$ where ω is given by (3). For fixed t we choose λ_0 to ensure $s_m = s\lambda_0^m = t$, so that

$$m = \frac{\log t/s}{\log \lambda_0}.$$

We can assume that ω is chosen so that m is an integer.

We now describe a sub-process which gives lower bounds $W^L \leq W$. Basically, to do this, for $\tau \in I_j$ we replace $W_{i,\ell}(\tau)$ by $W^L_{i,\ell}(s_j)$, so that only vertices which choose cliques from the lower bound sub-process added before s_j count towards the growth of the sub-process. Thus during I_j the equations corresponding to (6)-(9) for the sub-process can be replaced by the following.

$$W^L_{0,1}(s_{j+1}) = W^L_{0,1}(s_j) + \frac{k-1}{k} W^L_{0,1}(s_j) \sum_{\tau=s_j}^{s_{j+1}-1} \frac{1}{\tau} \tag{10}$$

$$W^L_{i,k-1}(s_{j+1}) = W^L_{i,k-1}(s_j) + \left(\frac{1}{k} W^L_{i,k-1}(s_j) + \frac{1}{k} W^L_{i-1,1}(s_j)\right) \sum_{\tau=s_j}^{s_{j+1}-1} \frac{1}{\tau} \tag{11}$$

$$W^L_{i,\ell}(s_{j+1}) = W^L_{i,\ell}(s_j) + \left(\frac{k-\ell}{k} W^L_{i,\ell}(s_j) + \frac{\ell+1}{k} W^L_{i,\ell+1}(s_j)\right) \sum_{\tau=s_j}^{s_{j+1}-1} \frac{1}{\tau}. \tag{12}$$

If $f(x)$ is monotone decreasing

$$f(a+1) + \cdots + f(b) \le \int_a^b f(x)dx \le f(a) + \cdots + f(b-1).$$

Thus

$$\sum_{\tau=s_j}^{s_{j+1}-1} \frac{1}{\tau} - \frac{1}{s_j} \le \int_{s_j}^{s_{j+1}} \frac{dx}{x} \le \sum_{\tau=s_j}^{s_{j+1}-1} \frac{1}{\tau}.$$

As $s_j = \lceil s\lambda_0^j \rceil$ it follows that

$$\begin{aligned} \sum_{\tau=s_j}^{s_{j+1}-1} \frac{1}{\tau} =& \frac{\theta_1}{s_j} + \log \frac{\lceil s\lambda_0^{j+1} \rceil}{\lceil s\lambda_0^j \rceil} && (13)\\ =& \log \lambda_0 (1+\delta_j), && (14) \end{aligned}$$

where $0 \le \theta_1 \le 1$ and $|\delta_j| \le 2/s_j$ provided $s \to \infty$.

Substitute (13)–(14) for the summation in (10)–(12). Let $\delta' = \max_j |\delta_j|$, thus $\delta' = o(1/\omega)$ (see (3)). Let

$$\lambda_1 = \lambda_0(1-\delta') = \lambda_0(1 - o(1/\omega)). \tag{15}$$

Replace λ_0 with $\lambda_1 = \lambda_0(1-\delta')$ to obtain a uniform lower bound on the recurrences for all j, and re-scale by dividing by s to obtain simplified recurrences $W^L_{i,\ell}(j) \le W_{i,\ell}(s_j)/s$. We obtain

$$\begin{aligned} W^L_{0,1}(0) =& 1, && (16)\\ W^L_{0,\ell}(0) =& 0 \qquad \ell \ge 2, && (17)\\ W^L_{0,1}(j+1) =& W^L_{0,1}(j)\left(1 + \frac{k-1}{k}\log\lambda_1\right), && (18)\\ W^L_{i,k-1}(j+1) =& W^L_{i,k-1}(j)\left(1 + \frac{1}{k}\log\lambda_1\right) + W^L_{i-1,1}(j)\frac{1}{k}\log\lambda_1, \quad i \ge 1, && (19)\\ W^L_{i,\ell}(j+1) =& W^L_{i,\ell}(j)\left(1 + \frac{k-\ell}{k}\log\lambda_1\right) + W^L_{i,\ell+1}(j)\frac{\ell+1}{k}\log\lambda_1, \quad i \ge 1, \ell \ne k-1. && (20) \end{aligned}$$

Let $G^L_{i,\ell}(z)$ be the generating function for $W^L_{i,\ell}(j),\ j \geq 0$, and let $\gamma_\ell = 1 + ((k-\ell)/k)\log\lambda_1$. It follows from (16), (18) that

$$G^L_{0,1}(z) = \frac{1}{1-\gamma_1 z}.$$

From (17), (19), (20), we obtain

$$G^L_{i,k-1}(z) = \gamma_{k-1} z\, G^L_{i,k-1}(z) + \left(\frac{1}{k}\log\lambda_1\right) z\, G^L_{i-1,1}(z),$$

$$G^L_{i,\ell}(z) = \gamma_\ell z\, G^L_{i,\ell}(z) + \left(\frac{\ell+1}{k}\log\lambda_1\right) z\, G^L_{i,\ell+1}(z), \quad i \geq 1, \ell \neq k-1.$$

Thus

$$\begin{aligned} G^L_{i,k-1}(z) =& \frac{1}{k}\frac{z\log\lambda_1}{1-\gamma_{k-1}z}\, G^L_{i-1,1}(z), \quad i \geq 1, \\ G^L_{i,\ell}(z) =& \frac{\ell+1}{k}\frac{z\log\lambda_1}{1-\gamma_\ell z}\, G^L_{i,\ell+1}(z) \quad i \geq 1, \ell \neq k-1. \end{aligned} \tag{21}$$

It follows inductively that

$$G^L_{i,1}(z) = \left(\frac{z^{k-1}k!(\log\lambda_1)^{k-1}}{k^k(1-\gamma_1 z)\cdots(1-\gamma_{k-1}z)}\right)^i \frac{1}{1-\gamma_1 z}, \tag{22}$$

and for $\ell = 2, ..., k-2$

$$G^L_{i,\ell}(z) = \frac{1}{k}\prod_{j=\ell}^{k-1}\frac{j+1}{1-\gamma_j z}\left(\frac{z\log\lambda_1}{k}\right)^{k-\ell} G^L_{i-1,1}(z). \tag{23}$$

2.4 Upper bound for $W_{i,\ell}(t)$: Generating Function

For simplicity of notation, put $\alpha_\ell = (k-\ell)/k$ and $\beta_\ell = (\ell+1)/k$. Then iterating the main variable backwards in recurrences (7) – (9), and recalling that $W_{i,\ell}(t)$ is non-decreasing in t

gives

$$\begin{aligned}
W_{0,1}(t+\sigma) &= W_{0,1}(t)\prod_{j=0}^{\sigma-1}\left(1+\frac{\alpha_1}{t+j}\right)\\
W_{i,k-1}(t+\sigma) &= W_{i,k-1}(t)\prod_{j=0}^{\sigma-1}\left(1+\frac{\alpha_{k-1}}{t+j}\right)+\alpha_{k-1}\sum_{j=0}^{\sigma-1}\frac{W_{i-1,1}(t+j)}{t+j}\prod_{i=j+1}^{\sigma-1}\left(1+\frac{\alpha_{k-1}}{t+i}\right)\\
&\le W_{i,k-1}(t)\prod_{j=0}^{\sigma-1}\left(1+\frac{\alpha_{k-1}}{t+j}\right)+\alpha_{k-1}\sum_{j=0}^{\sigma-1}\frac{W_{i-1,1}(t+\sigma)}{t+j}\prod_{i=j+1}^{\sigma-1}\left(1+\frac{\alpha_{k-1}}{t+i}\right)\\
W_{i,\ell}(t+\sigma) &= W_{i,\ell}(t)\prod_{j=0}^{\sigma-1}\left(1+\frac{\alpha_\ell}{t+j}\right)+\beta_\ell\sum_{j=0}^{\sigma-1}\frac{W_{i,\ell+1}(t+j)}{t+j}\prod_{i=j+1}^{\sigma-1}\left(1+\frac{\alpha_\ell}{t+i}\right)\\
&\le W_{i,\ell}(t)\prod_{j=0}^{\sigma-1}\left(1+\frac{\alpha_\ell}{t+j}\right)+\beta_\ell\sum_{j=0}^{\sigma-1}\frac{W_{i,\ell+1}(t+\sigma)}{t+j}\prod_{i=j+1}^{\sigma-1}\left(1+\frac{\alpha_\ell}{t+i}\right)
\end{aligned}$$

Let $t = s_j$, let $t+\sigma = s_{j+1}$ and let $W_{i,\ell}(j) = W_{i,\ell}(s_j)/s$ for all i, j, ℓ. Thus

$$\begin{aligned}
W_{0,1}(j+1) &= W_{0,1}(j)\prod_{t=s_j}^{s_{j+1}-1}\left(1+\frac{\alpha_1}{t}\right)\\
&\le \left(1+O(\tfrac{1}{s})\right)\lambda_0^{\alpha_1}W_{0,1}(j)\\
&\le \left(1+\frac{k-1}{k}\log\lambda'\right)W_{0,1}(j).
\end{aligned}$$

For $\lambda_0 = 1+1/\omega$ the value of $\lambda' = 1+1/\omega+O(1/\omega^2)$. To see this, for $a<1$ the function $f(x) = x^a-(1+a\log x)$ has a unique minimum at $x=1$, with $f(1) = f'(1) = 0$, so the Taylor expansion of $f(1+h) = O(h^2)$.

Similarly

$$\begin{aligned}
W_{i,k-1}(j+1) &\le W_{i,k-1}(j)\prod_{t=s_j}^{s_{j+1}-1}\left(1+\frac{\alpha_{k-1}}{t}\right)+\alpha_{k-1}W_{i-1,1}(j+1)\sum_{t=s_j}^{s_{j+1}-1}\frac{1}{t}\prod_{\tau=t+1}^{s_{j+1}-1}\left(1+\frac{\alpha_{k-1}}{\tau}\right)\\
&\le \left(1+O(\tfrac{1}{s})\right)\left(W_{i,k-1}(j)\lambda_0^{\alpha_{k-1}}+W_{i-1,1}(j+1)(\lambda_0^{\alpha_{k-1}}-1)\right)\\
&\le W_{i,k-1}(j)\left(1+\frac{1}{k}\log\lambda'\right)+W_{i-1,1}(j+1)\frac{1}{k}\log\lambda',
\end{aligned}$$

and

$$
\begin{aligned}
W_{i,\ell}(j+1) &\le W_{i,\ell}(j) \prod_{t=s_j}^{s_{j+1}-1} \left(1+\frac{\alpha_\ell}{t}\right) + \beta_\ell W_{i,\ell+1}(j+1) \sum_{t=s_j}^{s_{j+1}-1} \frac{1}{t} \prod_{\tau=t+1}^{s_{j+1}-1} \left(1+\frac{\alpha_\ell}{\tau}\right) \\
&\le \left(1+O(\tfrac{1}{s})\right) \left(W(j)\lambda_0^{\alpha_\ell} + \frac{\beta_\ell}{\alpha_\ell} W_{i,\ell+1}(j+1)(\lambda_0^{\alpha_\ell}-1) \right) \\
&\le W_{i,\ell}(j)(1+\alpha_\ell \log\lambda') + \beta_\ell W_{i,\ell+1}(j+1)\log\lambda'.
\end{aligned}
$$

We thus obtain the following recurrences for an upper bound $W^U_{i,\ell}(j) \ge W_{i,\ell}(s_j)/s$.

$$
\begin{aligned}
W^U_{0,1}(0) &= 1, \\
W^U_{0,\ell}(0) &= 0 \qquad \ell \ge 2, \\
W^U_{0,1}(j+1) &= \left(1+\frac{k-1}{k}\log\lambda'\right) W^U_{0,1}(j). \\
W^U_{i,k-1}(j+1) &= W^U_{i,k-1}(j)\left(1+\frac{1}{k}\log\lambda'\right) + W^U_{i-1,1}(j+1)\frac{1}{k}\log\lambda', \quad i \ge 1, \\
W^U_{i,\ell}(j+1) &= W^U_{i,\ell}(j)\left(1+\frac{k-\ell}{k}\log\lambda'\right) + \frac{\ell+1}{k} W^U_{i,\ell+1}(j+1)\log\lambda', \quad i \ge 1, \ell \ne k-1.
\end{aligned}
$$

Let $G^U_{i,\ell}(z)$ be the generating function for $W^U_{i,\ell}(j),\ j \ge 0$, and let $\gamma'_\ell = 1 + ((k-\ell)/k)\log\lambda'$. It follows that

$$G^U_{0,1}(z) = \frac{1}{1-\gamma'_1 z},$$

and generally, we obtain

$$G^U_{i,k-1}(z) = \gamma'_{k-1} z\, G^U_{i,k-1}(z) + \left(\frac{1}{k}\log\lambda'\right) G^U_{i-1,1}(z),$$

$$G^U_{i,\ell}(z) = \gamma'_\ell z\, G^U_{i,\ell}(z) + \left(\frac{\ell+1}{k}\log\lambda'\right) G^U_{i,\ell+1}(z), \quad i \ge 1, \ell \ne k-1.$$

Thus

$$
\begin{aligned}
G^U_{i,k-1}(z) &= \frac{1}{k}\frac{\log\lambda'}{1-\gamma'_{k-1}z}\, G^U_{i-1,1}(z), \quad i \ge 1, \qquad (24) \\
G^U_{i,\ell}(z) &= \frac{\ell+1}{k}\frac{\log\lambda'}{1-\gamma'_\ell z}\, G^U_{i,\ell+1}(z) \quad i \ge 1, \ell \ne k-1.
\end{aligned}
$$

It follows inductively that

$$G^U_{i,1}(z) = \left(\frac{k!(\log\lambda'_1)^{k-1}}{k^k(1-\gamma'_1 z)\cdots(1-\gamma'_{k-1}z)}\right)^i \frac{1}{1-\gamma'_1 z}, \qquad (25)$$

and for $\ell = 2, ..., k-2$

$$G^U_{i,\ell}(z) = \frac{1}{k} \prod_{j=\ell}^{k-1} \frac{j+1}{1-\gamma'_j z} \left(\frac{\log \lambda'_1}{k}\right)^{k-\ell} G^U_{i-1,1}(z). \tag{26}$$

The expressions (24), (25), (26) differ from (21), (22), (23) in that λ' replaces λ and multiplicative powers of z are suppressed. It will be seen in Section 3 that the W^L and W^U are sufficiently close to obtain tight bounds on the the expected occupancy of each level.

3 Random k-trees: Asymptotic expression for maximum height

We now show how to extract the coefficients of our generating functions. For reasons of symmetry of the generating function it is easier for us to focus on $W^X_{N,2}(t)$ for $X = L, U$ and suitable $N \to \infty$. Choosing $\ell = 2$ will suffice. The height of the rooted k-tree depends on the index N of the maximum level set, and not on $\ell = 1, ..., k$. As the case $k = 2$ is already known from [18], we assume $k \geq 3$.

3.1 Extraction of coefficients for a lower bound on the expected size of level sets

We first discuss the case for $G^L(z) = G^L_{N,2}(z)$, where from (22) and (23)

$$G^L(z) = \frac{k}{2z \log \lambda_1} \left(\frac{z^{k-1} k! (\log \lambda_1)^{k-1}}{k^k (1-\gamma_1 z) \cdots (1-\gamma_{k-1} z)} \right)^N.$$

Using $[z^m]G(z)$ to denote the coefficient of z^m in the formal expansion of $G(z)$, let $w^L(m) = W^L_{N,2}(m) = [z^m]G^L(z)$. To extract these coefficients, let $M = m - N(k-1) - 1$, so that

$$[z^m]G^L(z) = \frac{k}{2 \log \lambda_1} \left((\log \lambda_1)^{k-1} k!/k^k \right)^N [z^M](f(z))^N, \tag{27}$$

where

$$f(z) = \frac{1}{(1-\gamma_1 z) \cdots (1-\gamma_{k-1} z)}. \tag{28}$$

We want the smallest N such that $[z^m]G^L(z) \to 0$, i.e. $W^L_{N,2}(t) \to 0$. It will be simpler for our analysis if we can assume that such an N satisfies $N = c \log t$, where $c = \Theta(1/k)$. By inspection, as $\gamma_\ell > 1$, the coefficients of $(f(z))^N$ are at least as large as the coefficients of $1/(1-z)^{N(k-1)}$. Recall that $m = (\log t/s)/\log \lambda_0$. Thus

$$
\begin{aligned}
[z^m]G^L(z) &\geq \frac{k}{2\log\lambda_1}\left((\log\lambda_1)^{k-1}k!/k^k\right)^N \binom{N(k-1)-1+M}{M} \\
&\geq \frac{k}{2\log\lambda_1}\left((\log\lambda_1)^{k-1}k!/k^k\right)^N \frac{m^{N(k-1)-1}}{(N(k-1))!} \\
&= \Theta\left(\frac{1}{\log t}\right)\left(\frac{k}{N}\right)^{1/2}\left(\frac{\log(t/s)}{Nk}\right)^{N(k-1)}.
\end{aligned}
$$

As $N \geq 1$ and assuming $k \geq 3$, the value of $[z^m]G^L(z)$ tends to infinity with t for any $Nk \leq (\log(t/s))/2$. We can thus assume $N = c\log(t/s)$ where $c = \Theta(1/k)$. If so we have $N/m \to 0$ and $N = N(t) \to \infty$. Using $M = m - N(k-1) - 1$, as above

$$
\frac{N}{M} = (1 + O(1/\omega))c\log\lambda_0 = c'\log\lambda_1 \tag{29}
$$

where $c' = \Theta(1/k)$, and $\log\lambda_1 = \log\lambda_0(1 + o(1/\omega))$ (see (15)).

We next describe a general technique (based on [14]) to obtain an asymptotic expression for $[z^M](f(z))^N$ in terms of an implicitly defined parameter $\widehat{x}$. The method can be broken into six steps.

M1 Write

$$
[z^M](f(z))^N = \frac{f(x)^N}{x^M}[z^M]\left(\frac{f(zx)}{f(x)}\right)^N.
$$

M2 Let $Y(x)$ be a random variable with probability generating function $\mathbf{E}z^Y = f(zx)/f(x)$. By inspection of the generating function, (see (28)) the random variable Y has positive probabilities on the non-negative integers. Let $Y_1, ..., Y_N$ be i.i.d. as Y.

$$
[z^M]\left(\frac{f(zx)}{f(x)}\right)^N = [z^M]\mathbf{E}(z^{Y_1+\cdots+Y_N}) = \mathbf{Pr}(Y_1 + \cdots + Y_N = M).
$$

M3 Obtain the moments $\mu(x), \sigma^2(x)$ of $Y(x)$ from

$$
\begin{aligned}
\mu(x) = \mathbf{E}Y &= \frac{d}{dz}\mathbf{E}z^Y\bigg|_{z=1} = x\frac{f'(x)}{f(x)} \\
\sigma^2(x) - \mu(x) + \mu(x)^2 = \mathbf{E}Y(Y-1) &= \frac{d^2}{dz^2}\mathbf{E}z^Y\bigg|_{z=1} = x^2\frac{f''(x)}{f(x)}
\end{aligned}
$$

M4 Choose Y so that $\mu(x) = \mathbf{E}Y = M/N$. Solve $\mu(x) = M/N$ for x.

M5 We have chosen $\mathbf{E}(Y_1 + \cdots + Y_N) = M$. Provided $\sigma^2(x)$ is bounded, and as the random variable Y has lattice width $h = 1$, by the Local Limit Theorem (see e.g. [8] or [11])

$$\mathbf{Pr}(Y_1 + \cdots + Y_N = M) = (1 + O(1/N))\frac{1}{\sqrt{2\pi\sigma^2 N}}.$$

M6 From **M1**, **M2** and **M5**,

$$[z^M]f(z)^N = (1 + O(1/N))\frac{1}{\sqrt{2\pi\sigma^2 N}}\frac{f(x)^N}{x^M}.$$

The value of x is obtained from the condition that $\mu(x) = M/N$ in **M4**, and the value of $\sigma^2(x) < \infty$ from **M3**.

We apply this method to $f(z)$ from (28). For step **M3** we find

$$\mu(x) = \sum_{\ell=1}^{k-1} \frac{\gamma_\ell x}{1 - \gamma_\ell x} \tag{30}$$

$$\sigma^2(x) = \mu(x) + \sum_{\ell=1}^{k-1} \frac{(\gamma_\ell x)^2}{(1 - \gamma_\ell x)^2}. \tag{31}$$

Considering **M4**, from (29) we can relate $\mu(x)$ to c' by

$$\mu(x) = \frac{M}{N} = \frac{1}{c' \log \lambda_1}. \tag{32}$$

Recall that $\gamma_\ell = 1 + (k - \ell)/k \ \log \lambda_1$, and that $\lambda_1 = (1 + 1/\omega + o(1/\omega^2))$. In order to find the value of $\widehat{x}$ from (30) note that

$$\max_\ell \gamma_\ell = \max_\ell (1 + (k - \ell)/k \ \log \lambda_1) = \gamma_1.$$

The smallest singularity of (30) is at $x = 1/\gamma_1 = 1 - O(1/\omega)$. Intuitively, as $M/N \to \infty$ it must be that $\widehat{x} \to 1/\gamma_1$ from below. From (30) it follows that if $\mu(x) > 0$ then $x > 0$. The function $g(x) = \sum_{\ell=1}^{k-1} (\gamma_\ell x)/(1 - \gamma_\ell x)$ is monotone increasing in x from $g(0) = 0$. Thus the solution $x > 0$ to $g(x) = \mu(x)$ is unique.

Based on this, for some $a = a(k)$, to be determined, let

$$\widehat{x} = \frac{1 - a \log \lambda_1}{\gamma_1} = \frac{1 - a \log \lambda_1}{1 + ((k-1)/k) \log \lambda_1} = 1 - O\left(\frac{a+1}{\omega}\right) \tag{33}$$

From

$$\frac{1}{c' \log \lambda_1} = \mu(\widehat{x}) \le \frac{k\gamma_1 \widehat{x}}{1 - \gamma_1 \widehat{x}} \le \frac{k}{a \log \lambda_1},$$

it follows that $a > 0$. Also as we assumed $c' = \Theta(1/k)$, we have that $a = O(1)$. From these observations and (33) we see that

$$\widehat{x} = 1 - O(1/\omega). \tag{34}$$

The next step is to prove $a = \Theta(1)$ so that (40) (see below) is bounded, and establish the relationship between a and c given in (37). From (30) we have

$$\begin{aligned} \mu(\widehat{x}) =& \frac{1 - a\log\lambda_1}{\log\lambda_1}\left(\sum_{\ell=1}^{k-1} \frac{1 + \frac{k-\ell}{k}\log\lambda_1}{\frac{\ell-1}{k} + a + \frac{a(k-\ell)}{k}\log\lambda_1}\right) & (35)\\ =& \left(1 + O\left(\frac{k}{\omega}\right)\right)\frac{k}{\log\lambda_1}\sum_{\ell=0}^{k-2}\frac{1}{\ell + ak}. & (36) \end{aligned}$$

Using $M/N = (1 + O(k/\omega))/c\log\lambda_0$, we see from (36) that

$$\frac{1}{kc} = \left(1 + O\left(\frac{k}{\omega}\right)\right)\sum_{\ell=0}^{k-2}\frac{1}{\ell + ak}. \tag{37}$$

Note that

$$\int_x^{x+j+1}\frac{dy}{y} \le \frac{1}{x} + \cdots + \frac{1}{(x+j)} \le \frac{1}{x} + \int_x^{x+j}\frac{dy}{y}. \tag{38}$$

Putting $x = ak$ we see that

$$\log\frac{k(a+1)-1}{ka} \le \sum_{\ell=0}^{k-2}\frac{1}{\ell + ak} \le \frac{1}{ka} + \log\frac{k(a+1)-2}{ka-1}. \tag{39}$$

This implies that

$$\frac{1}{kc} \ge (1 + o(1))\log\frac{k(a+1)-1}{ka},$$

so that

$$a \ge \frac{1 - 1/k}{e^{(1+o(1))/ck} - 1}.$$

As we assumed $c = \Theta(1/k)$, it follows for $k \ge 2$ that a is bounded below by a positive constant, and thus $a = \Theta(1)$. This bound on a combined with (39) implies that

$$\sum_{\ell=0}^{k-2}\frac{1}{\ell + ak} = \log\frac{k(a+1)-1}{ka} + \frac{\zeta_1}{k} = \log\frac{a+1}{a} + \frac{\zeta_2}{k} \tag{40}$$

where $|\zeta_1|, |\zeta_2| = O(1)$.

Thus crudely, for some $B = \Theta(1)$

$$\mu(\widehat{x}) = B\frac{k}{\log\lambda_1}.$$

Armed with this, our next task is to approximate $\sigma^2(\widehat{x})$, as given in (31). Writing $\sigma^2(x) = \mu(x) + \phi(x)$ and substituting (33) we find, for some $B' = \Theta(1)$ that

$$\phi(\widehat{x}) = B' \frac{k}{\log^2 \lambda_1}.$$

Thus

$$\sigma^2(\widehat{x}) = B \frac{k}{\log \lambda_1} + B' \frac{k}{\log^2 \lambda_1}.$$

Proceeding to step **M6** and using $m/N = 1/(c \log \lambda_0)$, $M = m - N(k-1) - 1$ and (27) we have that

$$\begin{aligned} W^L_{N,2}(m) = [z^m]G^L(z) =& \frac{k\widehat{x}}{2\log \lambda_1} \frac{1+O(1/N)}{\sqrt{2\pi\sigma^2 N}} \left[\frac{k!}{k^k} (\log \lambda_1)^{k-1} \frac{f(\widehat{x})}{\widehat{x}^{1/c\log\lambda_0}} \widehat{x}^{k-1} \right]^N \\ =& \Theta\left(\frac{k^{1/2}}{N^{1/2}} \right) \quad [\Phi(k,a)]^N. \end{aligned} \tag{41}$$

The final step is to put $\Phi(k,a)$ into a more tractable form by removing the parameter $c = c(a,k)$. Our aim is to prove

$$\Phi(k,a) = \left(1 + O\left(\frac{k^2}{\omega}\right)\right) \frac{\Gamma(k)\Gamma(ka)}{\Gamma(ka+k-1)} \ \exp\left(\sum_{\ell=0}^{k-2} \frac{ka+k-1}{\ell + ak} \right). \tag{42}$$

This can be done as follows:

F1. From the definition of $\widehat{x}$ in (33)

$$\widehat{x}^{-1/c\log\lambda_0} = \exp\left(\frac{1}{kc} (ka+k-1)\left(1 + O(1/\omega)\right) \right). \tag{43}$$

F2. It follows from (37) that

$$\frac{1}{kc} = \left(1 + O\left(\frac{k}{\omega}\right)\right) \sum_{\ell=0}^{k-2} \frac{1}{\ell + ak}.$$

As $c = \Theta(1/k)$ the right hand side sums to a constant, so that (43) can be written as

$$\widehat{x}^{-1/c\log\lambda_0} = \left(1 + O\left(\frac{k^2}{\omega}\right)\right) \exp\left(\sum_{\ell=0}^{k-2} \frac{ka+k-1}{\ell+ak} \right).$$

F3. From the definition of $\widehat{x}$

$$\widehat{x}^{k-1} = 1 + O\left(\frac{k}{\omega}\right).$$

F4. From (28) and the definition of $\widehat{x}$

$$\frac{(\log\lambda_1)^{k-1}}{k^{k-1}} f(\widehat{x}) = \left(1+\frac{k-1}{k}\log\lambda_1\right)^{k-1}\prod_{\ell=1}^{k-1}\frac{1}{\ell-1+ak+a(k-\ell)\log\lambda_1}$$
$$= \frac{1+O(k/\omega)}{(ka)(ka+1)\cdots(ka+k-2)}.$$

Putting F1 to F4 together gives us (42).

3.2 Extraction of coefficients for an upper bound on the expected size of level sets

We now consider $w^U(m) = W^U_{N,2}(m) = [z^m]G^U(z)$ where $G^U(z) = G^U_{N,2}(z)$. Observe first that if we ignore the effect of the switch from λ_1 to λ' then

$$G^L(z) = G^U(z) \times z^{(k-1)N-1}.$$

Then with m as in Section 3.1 we have from (27) that with $M' = m-1$ and λ', γ'_ℓ replacing λ_1, γ_ℓ in f,

$$\begin{aligned} w^U(m) = [z^m]G^U(z) &= \frac{k}{2\log\lambda'}\left((\log\lambda')^{k-1}k!/k^k\right)^N [z^{M'}]f(z)^N \\ &\le \frac{k}{2\log\lambda'}\left((\log\lambda')^{k-1}k!/k^k\right)^N \frac{f(\widehat{x})^N}{\widehat{x}^{M'}} \\ &= \Theta(k)\left(\left(1+O\left(\frac{k}{\omega}\right)\right)\Phi(k,a)\right)^N, \end{aligned} \tag{44}$$

where $\Phi(k,a)$ is given in (41).

3.3 Asymptotic value of maximum height

For a given height $N = c\log(t/s)$, we get lower and upper bounds for $W_{N,2}(t)$, the expected number of $[N,(2,k-2)]$ relatively configured k-cliques from (41) and (44), which in turn depend on $\Phi(k,a)$. Provided $w^L(m) \sim w^U(m)$, and we can prove concentration of the level set sizes around these bounds, the maximum height $h_s(t)$ can be obtained from the value a which makes $\Phi(k,a) = 1$ in (42). By expanding $\Phi(k,a)$ around this value of a we prove that the value $w^U(m) \to 0$ for larger values of c, whereas $w^L(m) \to \infty$ for smaller values of c.

Once we find the value of a such that $\Phi(k,a) = 1$, we can obtain $c(a)$ via (37). Our analysis of behavior around $\Phi(k,a) = 1$ depends on k. Basically there are three cases. $k = 2$, $k \ge 3$ constant, and $k \to \infty$.

Setting aside the details for now, for $k \geq 3$ constant, the implicit relationships (42) and (37) are the content of Theorem 1. In the case that $k \to \infty$, the value of a solving $\Phi(k,a) = 1$ can be obtained explicitly as $a = 1 + o(1)$. This solution allows us to obtain an explicit asymptotic of $c \sim 1/k \log 2$ from (37).

3.3.1 Case $k \to \infty$

We assume $k = o(\sqrt{\omega})$ and use the asymptotic expansion of (42). As $\Gamma(y) = e^{O(1/y)}\sqrt{2\pi}\ \ y^{y-1/2}e^{-y}$ $\Phi(k,a)$ can be written as

$$\Phi(k,a) = \left(1 + O\left(\frac{1}{k}\right) + O\left(\frac{k^2}{\omega}\right)\right)\ \left(\frac{(a+1)^3 2\pi k}{a}\right)^{1/2} \times \left(\frac{a^a}{(a+1)^{a+1}} \exp\left((a+1-1/k)\sum_{\ell=0}^{k-2}\frac{1}{ak+\ell}\right)\right)^k. \tag{45}$$

It is easiest to expand directly about $a = 1$. The value of $\Phi(k,1)$ is

$$\Phi(k,1) \sim (16\pi k)^{1/2}\frac{1}{4^k}(2-\beta)^{2k-1},$$

where $0 \leq \beta \leq 1/k$. This value β is deduced as follows. Putting $a = 1$, we can bound the sum in (45) by

$$\log\frac{2k-1}{k} = \int_k^{2k-1}\frac{dy}{y} \leq \frac{1}{k} + \cdots + \frac{1}{2k-2} \leq \int_{k-1}^{2k-2}\frac{dy}{y} = \log\frac{2k-2}{k-1}.$$

We see that for some $0 \leq \beta \leq 1/k$,

$$\sum_{\ell=0}^{k-2}\frac{1}{k+\ell} = \log(2-\beta). \tag{46}$$

Denote the final bracketed term on the RHS of (45) above by $\Psi(k,a)^k$. Note that

$$\Psi(k,1) = \frac{1}{4}(2-\beta)^{2-1/k}. \tag{47}$$

The expansion of $\Phi(k,\alpha)$ in $\alpha = a(1+\epsilon)$ can thus be obtained by expanding $\Psi(k,\alpha)$ about $a = 1$. We write $\Psi(k, a(1+\epsilon)) = F_1 e^{F_2}$, and note the following simplifications.

$$F_1 = \frac{(a(1+\epsilon))^{a(1+\epsilon)}}{(a(1+\epsilon)+1)^{a(1+\epsilon)+1}} = \frac{a^a}{(a+1)^{a+1}}\left(\frac{a^a}{(a+1)^a}\right)^\epsilon \frac{(1+\epsilon)^{a(1+\epsilon)}}{(1+a\epsilon/(a+1))^{a+1+\epsilon a}} \tag{48}$$

$$= \frac{a^a}{(a+1)^{a+1}}\left(\frac{a^a}{(a+1)^a}\right)^\epsilon e^{O(\epsilon^2)}. \tag{49}$$

The second line comes from an expansion of the last term on the right hand side of (48), using $(1+x) = \exp(\log(1+x))$, in which the first order terms disappear.

$$\begin{aligned} F_2 =& (a(1+\epsilon)+1-1/k)\sum_{\ell=0}^{k-2}\frac{1}{ak(1+\epsilon)+\ell} \\ =& (a+1-1/k)\sum_{\ell=0}^{k-2}\frac{1}{ak+\ell} + \epsilon\left(\sum_{\ell=0}^{k-2}\frac{a}{ak+\ell} - a(k(a+1)-1)\sum_{\ell=0}^{k-2}\frac{1}{(ak+\ell)^2}\right) + O(\epsilon^2). \quad (50) \end{aligned}$$

Thus,

$$\Psi(k, a(1+\epsilon)) = \Psi(k,a) \quad \left(e^{O(\epsilon)}\left(\frac{a}{a+1}\right)^a \exp\left(-a\sum_{\ell=0}^{k-2}\frac{k-\ell-1}{(ak+\ell)^2}\right)\right)^\epsilon, \tag{51}$$

which, for $\epsilon > 0$ decreases faster than $(a/(a+1))^{a\epsilon}$.

Applying (46) with $b = 2$ and $a = 1$ to the second term in (50) gives

$$(2k-1)\sum_{\ell=0}^{k-3}\frac{1}{(k+\ell)^2} = 1-\theta,$$

where $\theta = O(1/k)$. As a result, from (46), (47), (49), (50) and (51)

$$\Psi(k, 1+\epsilon) = \Psi(k,1)\left(e^{O(\epsilon)}\left(1-\frac{\beta}{2}\right)e^{-1+\theta}\right)^\epsilon = \frac{1}{4}(2-\beta)^{2-1/k}\left(e^{O(\epsilon)}\left(1-\frac{\beta}{2}\right)e^{-1+\theta}\right)^\epsilon. \tag{52}$$

The coefficients $w^L(m)$, $w^U(m)$ we wish to evaluate at $a = 1+\epsilon$ are given in (41) and (44), respectively by

$$w^L(m) = W^L_{N,2}(m) = \Theta\left(\frac{k^{1/2}}{N^{1/2}}\right)[\Phi(k,a)]^N \tag{53}$$

$$w^U(m) = W^U_{N,2}(m) \le \Theta(k)\left[\left(1+O\left(\frac{k}{\omega}\right)\right)\Phi(k,a)\right]^N. \tag{54}$$

From (45) and (52) and $\theta = O(1/k)$.

$$\Phi(k, 1+\epsilon) = \left(1+O\left(\frac{1}{k}\right)+O\left(\frac{k^2}{\omega}\right)+O(\epsilon)+O(k\epsilon^2)\right) \;(4\pi k)^{1/2}\left(\left(1-\frac{\beta}{2}\right)^{2+\epsilon} e^{-\epsilon}\right)^k.$$

The the $O(1/k)$ is from $1/(1-\beta/2)$, the $O(\epsilon)$ is from $e^{\theta\epsilon k}$ and the $O(k\epsilon^2)$ from $e^{kO(\epsilon^2)}$. These come from raising the expression in (52) to the power k.

Upper bound on height. Choose $|\epsilon| = A(\log k)/k$ for some constant $A > 0$, then as $0 \le \beta \le 1/k$, for some $0 \le \xi \le 1$,

$$\Phi(k, 1+\epsilon) = (2+o(1))\sqrt{\pi}e^{-\xi}\sqrt{k}e^{-\epsilon k} = k^{1/2-A+o_k(1)} \text{ as } k \to \infty.$$

Suppose first that $A > 1/2$, say $A = 1$. Then for large enough k,

$$W_{N,2}(m) \le sw^U(m) \le s\Theta(k)k^{-N/3} \to 0.$$

Here we use $k = o(\sqrt{\omega})$ (see below(3)) to ensure convergence to zero.

We show in Lemma 4 that w.h.p. the height of $G_k(\tau)$ is bounded by $O(\log \tau)$. It follows that w.h.p. the height

$$h(t; k) \le O(\log s) + N + 1. \tag{55}$$

Indeed, we have shown that w.h.p. there are no $[N, (2, k-2)]$ relatively configured cliques and the clique generation process means that in this case there will be no $[N+1, (\ell, k-\ell)]$ relatively configured cliques.

From (32) we have $\mu(\widehat{x}) \sim \frac{1}{c \log \lambda_1}$. From (36) and (40) we have $\mu(\widehat{x}) \sim \frac{k}{\log \lambda_1} \cdot \log \frac{a+1}{a}$. It follows that $c \sim \frac{1}{k \log 2}$. From (55) we have w.h.p. that

$$h(t; k) \le O\left(\frac{\log t}{\omega}\right) + c\log(t/s) + 1 \sim \frac{\log t}{k \log 2}. \tag{56}$$

This proves the upper bound in Theorem 1 for the case where $k \to \infty$.

Lower bound on height. Now consider the lower bound. Putting $A < 0$ we get from (53) that

$$W_{N,2}(m) \ge w^L(m) \ge \Theta\left(\frac{k^{1/2}}{N^{1/2}}\right) k^{-AN} \to \infty.$$

We show in Section 4 that this is good enough to prove that $h(t; k) \ge (1 - o(1))c \log t$ w.h.p. This establishes a lower bound asymptotic to (56). Thus as asserted by Theorem 1

$$h(t; k) \sim \frac{\log t}{k \log 2}.$$

3.3.2 Case k constant

The case $k = 2$ can be resolved by our methods, but it is proved in [18] and the paper is already long enough, we omit this case. For $k \ge 3$, the statement of Theorem 1 follows from (42), and the following details.

$$\frac{\Gamma(k)\Gamma(ka)}{\Gamma(ka+k-1)} = \frac{(k-1)!}{(ka+k-2)(ka+k-3)\cdots(ka)}.$$

Let a be the unique positive solution to $\Phi(k,a) = 1$. Let $\alpha = a(1+\epsilon)$, then

$$
\begin{aligned}
(k\alpha + k - 2)\cdots(k\alpha) =& (ka + k - 2)\cdots(ka)\prod_{\ell=0}^{k-2}\left(1 + \frac{\epsilon ka}{ka+\ell}\right) \\
=& (ka + k - 2)\cdots(ka)\exp\left(\epsilon ka\sum_{\ell=0}^{k-2}\frac{1}{ka+\ell} + O(\epsilon^2 k)\right). \qquad (57)
\end{aligned}
$$

Using (57) to deal with the exponential term in the definition of $\Phi(k,a)$ in (42), we see that

$$
\Phi(k,\alpha) = \Phi(k,a)\exp\left(-a(k(a+1)-1)\sum_{\ell=0}^{k-2}\frac{1}{(ak+\ell)^2}\right)^{\epsilon k} \times e^{O(\epsilon^2 k)}.
$$

We now see from (54) that if $\epsilon > 0$ and $\epsilon N \to \infty$ then $w^U(m) \to 0$ and so from (56) we see that w.h.p. $h(t;k) \leq (1+o(1))c\log t$ where the value of c is given by (37). This verifies the upper bound in Theorem 1 for this case.

When $\epsilon < 0$ and $-\epsilon N \to \infty$ we see from (53) that $w^L(m) \to \infty$. In Section 4 we show that $w^L(m) \to \infty$ suffices to prove with high probability that $h(t;k) \geq (1-o(1))c\log t$. This verifies the lower bound in Theorem 1 for this case.

4 Concentration of occupancy of level sets around expected value $W_{i,\ell}(t)$

The coefficient $W^L_{N,2}(t)$ is the expected value of a random variable $\boldsymbol{W}$ corresponding to a subprocess of $G_k(t)$. Recall that $h_s(t)$ is our estimate of the expected height of $G_k(t)$ above $G_k(s)$. If we can prove concentration of $\boldsymbol{W}$ from below for $H = (1-\epsilon)h_s(t)$, then the height of $G_k(t)$ is at least H w.h.p. To do this we follow the method of Devroye [6], which we translate into our discrete step context. This method couples the growth of the level sets with a suitably defined Galton-Watson process. We first explain our approach. Because we observe the process at a given step t the total number of vertices added is fixed, and the proof requires an additional twist.

It is convenient to consider coupling our discrete process with a continuous time process. To do this, we replace the step parameter t of the previous sections by n and reserve variables such as t, T, τ for times in the continuous process.

Our basic view of the discrete process starting from the clique set S of $G_k(s)$ is as a set of bins $C_1, ..., C_i, ..., C_s$. At step s each bin C_i contains a single ball v_i, corresponding to a single clique. Suppose that at step $n \geq s$ bin C_i contains ν_i balls. At the next step, step $n+1$, the probability that ball v_{n+1} goes into bin C_i is ν_i/n. Given the occupancy ν_i of C_i we can subsequently construct a branching $\mathcal{T}(\nu_i)$ rooted at clique v_i as a k-tree process of length ν_i.

As mentioned above, we wish to use the method in Devroye [6] to prove concentration of the lower bound. The main problem for us, is that the occupancies of the bin system $C_S = (C_1, ..., C_i, ..., C_s)$ in the discrete process are not independent. Let ν_i be the occupancy of C_i then $\nu_1 + \nu_2 + \cdots + \nu_s = n$. Using a continuous time device we construct independent sub-processes which occur in C_S w.h.p.

To avoid confusion between the continuous time and discrete processes in the subsequent discussion we adopt the following notation. The discrete process at step n is a system of balls in bins. The continuous time process at time t, is a system of particles in cells. For the continuous time system consisting of particles $C = \{b_1, b_2, \dots, \}$, each particle $b \in C$ divides independently into b, b' with waiting time X_b a random variable with (negative) exponential distribution rate parameter $\rho = 1$. If the continuous time system is observed at time T and contains n particles (i.e. we have $C = \{b_1, ..., b_n\}$) then:

(i) The probability p_j that b_j is the next particle to divide is $p_j = 1/n$.

(ii) The waiting time from T to the division event of particle b_j is independent exponential with rate parameter $\rho = 1$.

(iii) The rate parameter for the next division in the entire system of n particles is $\rho_n = n$.

These results follow from the memoryless properties of the exponential distribution.

A pure birth process of this type is known as a Yule process, see Feller [12]. Given an initial population of θ particles in a cell C at time $t = 0$, the population $\Pi_\theta(\tau)$ of C at time τ has distribution $P_n(\tau) = \mathbf{Pr}(\Pi_\theta(\tau) = n)$ given by

$$P_n(\tau) = \binom{n-1}{n-\theta} e^{-\theta\tau} \left(1 - e^{-\tau}\right)^{n-\theta}. \tag{58}$$

This is the probability of $k = n - \theta$ failures and $r = \theta$ successes in n Bernoulii trials, where there is a success on the nth trial. The probability of success is $p = e^{-\tau}$. The expected number of failures k before the r-success is $r(1-p)/p$. Thus

$$\mathbf{E}\Pi_\theta(\tau) = \theta + \frac{\theta(1-p)}{p} = \theta e^{\tau}. \tag{59}$$

In our case the cell C can be regarded either as a single cell C_S with $\theta = s$ at $t = 0$, or as s sub-cells with $\theta = 1$ at $t = 0$; the latter corresponding to the balls in bins system of the discrete process. By choosing a time $\tau_n = \log(n/\theta)$, from (59) the expected size of the population is n. We use this relationship to switch between the discrete and the continuous time processes. If we observe a given cell C at time τ and C has occupancy N then the rooted branching $\mathcal{T}(\tau)$ is identical with $\mathcal{T}(N)$ in the discrete process. It we start at time 0 with a single cell C with occupancy $\theta = 1$, and stop at time τ with occupancy $\Pi(\tau) = \Pi_1(\tau)$, we can restart identically distributed processes $C_1, ..., C_{\Pi(\tau)}$ stopping at 2τ, and so on. We now fix our attention on a given cell C with $\theta = 1$ at $t = 0$.

In the discrete process, choose $\lambda = e^{1/\omega}$ so that $s = n^{\log \lambda} = n^{1/\omega}$. Here we will assume that (3) holds with t replaced by n. Let $s_j = s\lambda^{jL}, j = 0, 1, ...$ where $L = (1/2)\log(n/s)$. Now fix $\tau = s\lambda^L$. For a given bin C_i, after τ steps the expected occupancy is $\nu = \tau/s$, where

$$\nu = \tau/s = \lambda^L = e^{L/\omega} = \left(\frac{n}{s}\right)^{1/2\omega} = s^{(1/2)(1-1/\omega)} \sim \sqrt{s}. \tag{60}$$

In the corresponding continuous time process, let

$$T_j = jL\log\lambda + \log s = \log s_j$$

so that

$$T_{j+1} - T_j = L\log\lambda = \log(\tau/s) = T, \text{ say.}$$

Intuitively T_j is the equivalent of s_j, and T is the equivalent of τ. For a cell starting with $\theta = 1$ particles, from (59), (60)

$$\mathbf{E}\Pi(T) = e^T = \tau/s \sim \sqrt{s}.$$

Because of the memoryless property we restart the Yule processes at T_j, $j = 0, 1, ...$, assigning $i = 1$ particles per cell. Starting at T_j each cell grows independently up to T_{j+1}, etc.

A cell C is **good**, if after time T has elapsed,

(i) The occupancy $\Pi(T) \geq \nu$,

(ii) The branching constructed on the first τ/s particles in the cell has height at least $h = c(1-\epsilon)\log\nu$ where $\epsilon = o(1)$.

If C is good, let $\widetilde{W}_h$ be the occupancy of level h in this process, otherwise let $\widetilde{W}_h = 0$. In this way we define a Galton-Watson process with population sizes $X_j,\ j \geq 0$ as follows. $X_0 = 1$, $X_1 = \widetilde{W}_h$ and in general X_{j+1} is the progeny of the surviving particles at level j. Thus if $X_j = \xi$ then $X_{j+1} = X_{j,1} + \cdots + X_{j,\xi}$ where $X_{j,\ell}, \ell = 1, ..., \xi$ are independently distributed as X_1.

$$\mathbf{E}\widetilde{W}_h \geq \mathbf{Pr}(\Pi(T) \geq \nu) \times \widehat{W}$$

where $\widehat{W} = W^L_{h,2}(\nu)$ is a lower bound on the expected number of cliques (balls) at height h at time ν defined in Section 2.3. There is the caveat that s is replaced by $s' = s^{o(1)}$, chosen so that $s^{o(1)} \to \infty$ with s. We run the discrete process to generate the first ν balls in the box (particles in the cell), starting the branching from a base set of s' balls as in Section 2.3.

In (58), let $\theta = 1$, replace τ with T and n with ν. Then

$$\begin{aligned}
\mathbf{Pr}(\Pi(T) \geq \nu) &= \sum_{N \geq \nu} P_N(T) \\
&= \sum_{N \geq \nu} \frac{s}{\tau}\left(1 - \frac{s}{\tau}\right)^{N-1} \\
&= \left(1 - \frac{s}{\tau}\right)^{\tau/s-1} \geq \frac{1}{2e}.
\end{aligned}$$

If we choose $c(a)$ so that the RHS of (53) tends to infinity then we have

$$\mathbf{E}\widetilde{W}_h \geq \widehat{W}/2e > 1.$$

In the associated Galton-Watson process we have $\mu = \mathbf{E}X_1 = \mathbf{E}\widetilde{W}_h > 1$. For a Galton-Watson process with mean $\mu > 1$, the probability of ultimate survival is $1 - q$ where $q < 1$ is the smallest solution of $q = F(q)$. Here $F(x)$ is the probability generating function of X_1. Let $M = \max X$. We do not know $F(x)$, but as $M \leq \nu$ and $\mu > 1$, we use a result from Devroye [6] to upper bound q by

$$q \leq 1 - \frac{\mu}{M}.$$

Thus

$$q \leq 1 - \frac{1}{\nu}. \tag{61}$$

Let $\sigma = \log(n(1-\delta)/s)$ for $\delta = o(1)$. Observing the population $\Pi_S(\sigma)$ of the complete s-cell Yule process C_S at time σ, from (59) we have

$$\mathbf{E}\Pi_S(\sigma) = se^{\sigma} = n(1-\delta).$$

Let $N = \Pi_S(\sigma)$ be the population of the complete process at time σ, and let A be the event that $N \leq n$. We will establish in Lemma 3 below that $\mathbf{Pr}(\overline{A}) = o(1)$.

Let B be the event that the height H of $\mathcal{T}(N)$ satisfies

$$H \geq h\frac{\sigma}{T} = c(1-\epsilon)\log\nu\,\frac{\log(n(1-\delta)/s)}{\log\nu} = c(1-\epsilon')\log n/s$$

where $\epsilon' = o(1)$. Consider the complementary event $\overline{B}$ that none of the s independent Galton-Watson branching processes survives past generation $\lfloor \sigma/T \rfloor$. From (60) $\nu \sim \sqrt{s}$, and using (61) we have

$$\mathbf{Pr}(\overline{B}) \leq q^s \leq e^{-(1-o(1))\sqrt{s}} = o(1).$$

If the event A occurs, then $N \leq n$ and the corresponding tree $\mathcal{T}(N)$ is a subtree of $\mathcal{T}(n)$. Thus

$$\mathbf{Pr}(\text{height of } \mathcal{T}(n) \geq (1-\epsilon)c\log n/s) \geq 1 - \mathbf{Pr}(\overline{A}) - \mathbf{Pr}(\overline{B}) = 1 - o(1).$$

Finally observe that $\log(n/s) \sim \log n$ and this completes the proof for the lower bound on height $h_s(n)$.

Lemma 3. *Let* $\sigma = \log n(1-\delta)/s$. *Provided* $\delta \geq \sqrt{(K \log n)/s}$, *and* $s = o(\sqrt{n})$ *we have*

$$\mathbf{Pr}(\overline{A}) = \sum_{N \geq n+1} P_N(\sigma) = O(n^{-(K-3)/2}).$$

Proof From (58), with $\theta = s$, and $\tau = \sigma$ and $n = N$, we have

$$P_N(\sigma) = \binom{N-1}{N-s} e^{-s\sigma} \left(1 - e^{-\sigma}\right)^{N-s}. \tag{62}$$

Thus for $N \geq n+1$

$$\begin{aligned} \frac{P_{N+1}}{P_N} &= \frac{N}{N-s+1} \left(1 - \frac{s}{n(1-\delta)}\right) \\ &\leq 1 + s\left(\frac{1}{N-s} - \frac{1}{n(1-\delta)}\right) \\ &\leq 1 - \frac{s\delta}{2n(1-\delta)}. \end{aligned}$$

Thus,

$$\sum_{N \geq n+1} P_N = O\left(\frac{n}{s\delta}\right) P_{n+1}.$$

However, from (62)

$$\begin{aligned} P_{n+1} &= O\left(\frac{e}{1-\delta} e^{-1/(1-\delta)}\right)^s \\ &= O\left(e^{-s\delta^2/2}\right). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{N \geq n+1} P_N &= O\left(\frac{n}{s\delta} e^{-s\delta^2/2}\right) \\ &= O\left(n^{-(K-3)/2}\right). \end{aligned}$$

□

4.1 Upper bound on height

Lemma 4. *The height* $h(t)$ *of a random* k*-tree* $G_k(t)$ *is* $O(\log t)$ *w.h.p.*

Proof A crude calculation suffices to establish a w.h.p. upper bound of $O(\log t)$. Consider a shortest path $v_t, u_1, ..., u_i, v_1$ back from v_t to the root vertex v_1 in $G_k(t)$. As half of the cliques $C = K_k$ in $G_k(t)$ were added by time $t/2$,

$$\mathbf{Pr}(v_t \text{ chooses a clique } C \text{ in } G_k(t/2)) \geq \frac{1}{2}.$$

Thus the expected distance to the root must be (at least) halved by the edge $v_t u_1$. Whatever the label s of $u_1 = v_s$, this halving occurs independently at the next step. This must terminate w.h.p. after $c \log t$ steps, for some suitably large constant c, as we now prove.

If v_t is at height $h = c \log t$ then the h trials must have resulted in less than $h/3$ halving steps, for otherwise $h \leq 2h/3 + \log_2 t$, a contradiction for large c. But the probability of this is at most $e^{-h/144}$. Putting $h = 300 \log t$ we see that the probability of height h is at most $te^{-h/144} \leq 1/t \to 0$. Thus w.h.p. the height of $G_k(t)$ is $O(\log t)$. $\square$

5 Random Apollonian networks

We are interested in the height of $A_k(t)$ rooted at vertex c_1. Once again the height of $A_k(t)$ is the maximum distance of a vertex from the root. The first problem is to describe the structure of $A_k(t)$ relative to this BFS tree. The following example using $k = 3$ is instructive of our labeling method. In $A_3(0)$, let the initial clique C_0 be a triangle with vertex set $\{a, b, c\}$. Assume vertex a is at level 0 of the BFS tree and b, c at level 1. The index of the lowest level of C_0 is $i = 0$ and C_0 is oriented $(1, 2)$ giving a $[0, (1, 2)]$ configured triangle. Insertion of a vertex v in the interior of abc replaces this triangle by three new triangles abv, acv, bcv. Triangles abv, acv are configured $[0, (1, 2)]$ and bcv configured $[1, (3, 0)]$ in that all three vertices of this triangle lie in level $i = 1$ of the BFS tree. Once a clique has been subdivided, it is no longer considered as part of the Apollonian network. In the above example triangle abc is no longer available for subdivision. To distinguish this case, we call the cliques available for subdivision *embedded*.

In general, suppose clique $C = K_k$ is configured $[i, (\ell, k-\ell)]$ with vertex set $\{u_1, ..., u_\ell, v_{\ell+1}, ...v_k\}$. If $\ell = 2, ..., k$ then inserting a vertex w in the interior of C, removes C and produces ℓ cliques of type $[i, (\ell - 1, k - \ell + 1)]$ and $k - \ell$ cliques of type $[i, (\ell, k - \ell)]$. If $\ell = 1$, then insertion of a vertex inside a clique of type $[i, (1, k - 1)]$ forms one clique of type $[i + 1, (k, 0)]$ and $k - 1$ cliques of type $[i, (1, k - 1)]$.

At each step k embedded cliques are created and one is discarded, as it has been subdivided. Thus, as proved above Theorem 2 the number of embedded cliques in $A_k(t)$ is $(k-1)t+1$. This leads to the following recurrences for the expected number $W_{i,\ell}(t)$ of $[i, (\ell, k - \ell)]$ configured cliques at step t.

$$W_{0,1}(0) = 1, \qquad W_{i,\ell}(0) = 0 \text{ otherwise.}$$

$$W_{0,1}(t+1) = W_{0,1}(t) + \frac{k-2}{(k-1)t+1} W_{0,1}(t).$$

$$W_{i,k}(t+1) = W_{i,k}(t) - \frac{1}{(k-1)t+1} W_{i,k}(t) + \frac{1}{(k-1)t+1} W_{i-1,1}(t). \tag{63}$$

For $1 \leq \ell \leq k-1$,

$$W_{i,\ell}(t+1) = W_{i,\ell}(t) + \frac{k-\ell-1}{(k-1)t+1} W_{i,\ell}(t) + \frac{\ell+1}{(k-1)t+1} W_{i,\ell+1}(t).$$

5.1 Solution of recurrences

The system of recurrences for $W_{i,\ell}(t)$ and their solution is very similar to the case for k-trees. We give an outline description only, pointing out where differences arise. The main difference is that (63) contains a negative term. However, as (63) can be rewritten as

$$W_{i,k}(t+1) = W_{i,k}(t) \left(1 - \frac{1}{(k-1)t+1}\right) + \frac{1}{(k-1)t+1} W_{i-1,1}(t),$$

the lower bound substitution of $W_{i,\ell}(s_j)$ for $W_{i,\ell}(t)$ is still valid. We obtain (e.g.) the following system of lower bound recurrences, in place of (16) – (20).

$$\begin{aligned}
W^L_{0,1}(0) &= 1 \\
W^L_{0,1}(j+1) &= W^L_{0,1}(j) \left(1 + \frac{k-2}{k-1} \log \lambda\right) \\
W^L_{i,k}(j+1) &= W^L_{i,k}(j) \left(1 - \frac{1}{k-1} \log \lambda\right) + W^L_{i-1,1}(j) \frac{1}{k-1} \log \lambda \\
W^L_{i,\ell}(j+1) &= W^L_{i,\ell}(j) \left(1 + \frac{k-\ell-1}{k-1} \log \lambda\right) + W^L_{i,\ell+1}(j) \frac{\ell+1}{k-1} \log \lambda, \qquad 1 \leq \ell \leq k-1.
\end{aligned}$$

For $\ell = 1, ..., k$ let

$$\gamma_\ell = 1 + \frac{k-\ell-1}{k-1} \log \lambda.$$

The lower bound generating functions satisfy

$$\begin{aligned}
G_{0,1}(z) &= \frac{1}{1-\gamma_1 z} \\
G_{i,k}(z) &= \frac{1}{k-1} \frac{z \log \lambda}{1-\gamma_k z} G_{i-1,1}(z) \\
G_{i,\ell}(z) &= \frac{\ell+1}{k-1} \frac{z \log \lambda}{1-\gamma_\ell z} G_{i,\ell+1}(z),
\end{aligned}$$

leading to

$$G_{i,1}(z) = \left(\frac{z^k k! (\log \lambda)^k}{(k-1)^k} \frac{1}{(1-\gamma_1 z) \cdots (1-\gamma_k z)} \right)^i \frac{1}{1-\gamma_1 z}.$$

Work with $G_{N,2}(z) = G_N(z)$ as before, where

$$\begin{aligned} G_{N,2}(z) =& \frac{k-1}{2} \frac{1-\gamma_1 z}{z \log \lambda} G_{N,1}(z) \\ =& \frac{k-1}{2} \frac{1-\gamma_1 z}{z \log \lambda} \left(\frac{z^k k! (\log \lambda)^k}{(k-1)^k} \frac{1}{(1-\gamma_1 z) \cdots (1-\gamma_k z)} \right)^N \frac{1}{1-\gamma_1 z}. \end{aligned}$$

Making substitutions $M = m - kN + 1$, $\widehat{x} = (1 - a \log \lambda)/\gamma_1$ and so forth leads to the following expression for $\Phi(k, a)$ (to be compared with (42)),

$$\Phi(k, a) = \frac{k! \Gamma(a(k-1))}{\Gamma((a+1)(k-1)+1)} \exp \left(((k-1)a + k - 2) \sum_{\ell=0}^{k-1} \frac{1}{\ell + a(k-1)} \right),$$

subject to the asymptotic identity (to be compared with (37)),

$$\frac{1}{c(k-1)} = \sum_{\ell=0}^{k-1} \frac{1}{\ell + a(k-1)}.$$

In the case that $k \to \infty$ we can expand about $a = 1$ in a manner identical to k-trees to obtain the asymptotic height

$$h(t; k) \sim \frac{\log t}{k \log 2}.$$

The case $k \geq 3$ constant, is similar.

The concentration of the upper bound follows easily and the concentration of the lower bound from methods similar to Section 4. The main difference is that, in the continuous time model, on division a particle b is replaced by $k \geq 3$ progeny, as opposed to two progeny, which was the case for k-trees.

Let N be the number of (multiple) births, in a Yule process in which each particle has k children and then dies, i.e. there is an overall increase in population of $k-1$ per birth. If the original population size is s at time $t = 0$, then the population size after N-th birth is $\beta_N = (k-1)N + s$. The probability of N births having occurred by time t is given by

$$p_N(t) = \prod_{i=1}^{N} \frac{(k-1)(i-1)+s}{(k-1)i} \quad \times e^{-st}(1 - e^{-(k-1)t})^N.$$

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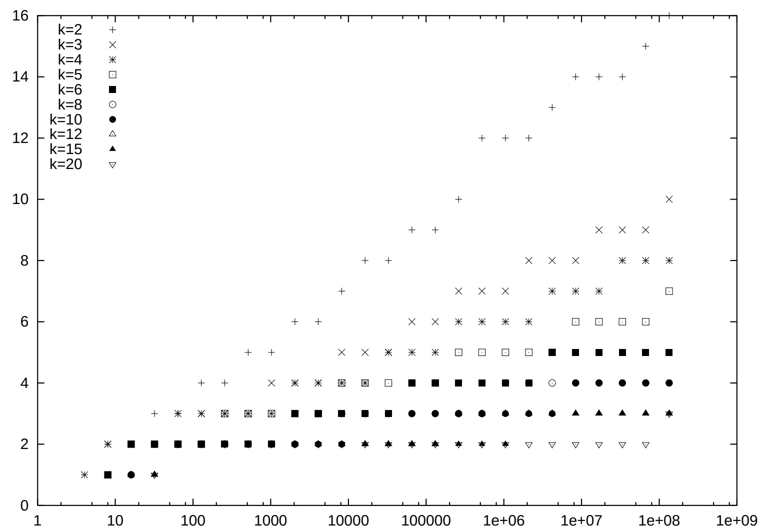

Figure 1: Experimental results for the height of random k-trees for $k = 2, 3, 5, 6, 8, 10, 12, 15, 20$

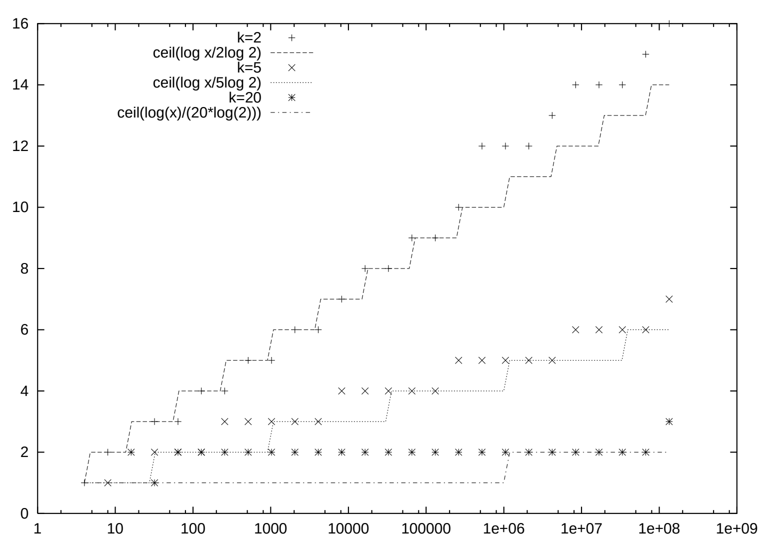

Figure 2: Experimental results for k-tree height fitted to $\lceil \log(t)/(k \log 2) \rceil$ for $k = 2, 5, 20$