ON THE INDEPENDENCE NUMBER OF RANDOM CUBIC GRAPHS

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Abstract

We show that as $n \to \infty$, the independence number $\alpha(G)$, for almost all 3-regular graphs G on n vertices, is at least $(6\log(3/2) - 2 - \epsilon)n$, for any constant $\epsilon > 0$. We prove this by analyzing a greedy algorithm for finding independent sets.

1 INTRODUCTION

This paper is concerned with the independence number of random cubic graphs. For a graph G, the independence number $\alpha(G)$ is the size of the largest set of vertices not containing any edge.

The independence numbers of random graphs have been studied by a number of authors. For r-regular graphs, Frieze and Luczak [3] showed that if G_{r-reg} is randomly chosen from the set of all r-regular graphs with vertex set $[n] =$

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$\{1,2,\ldots,n\}$ then for any fixed $\epsilon > 0$ there exists a fixed r_ϵ such that if $r_\epsilon \le r \le n^{1/3}$ then

$$\mathbf{Pr}\left(\left|\frac{\alpha(G_{r-reg})}{n} - \frac{2}{r}(\log r - \log\log r + 1 - \log 2)\right| \ge \frac{\epsilon}{r}\right) = o(1).$$

This tells us nothing about small values of r, e.g. r=3, i.e. cubic graphs. Bollobás [2] gives the following bounds in his book:

$$\mathbf{Pr}\left(\frac{7}{18} - o(1) \le \frac{\alpha(G_{3-reg})}{n} \le .4591\cdots\right) = 1 - o(1).$$

The main result of this paper is

Theorem 1 *For any constant $\epsilon > 0$,*

$$\lim_{n\to\infty} \mathbf{Pr}(\alpha(G_{3-reg}) \ge (6\log(3/2) - 2 - \epsilon)n) = 1.$$

Observe that $6\log(3/2) - 2 = .432\cdots$.

We prove this theorem by analysing a simple algorithm MINGREEDY. This algorithm repeatedly chooses a vertex v of minimum degree, adds it to its current independent set and then deletes it along with all of its neighbours. In detail, given a graph G we have

MINGREEDY
Input G;
$S := \emptyset$;
while $(V(G) \ne \emptyset)$ **do**
begin
    $V_{\min}$:= set of vertices of minimum degree in G;
    choose v from $V_{\min}$ with uniform probability;
    $G := G - \{v\} - \Gamma_G(v)$;
    $S := S \cup \{v\}$;
    remove from G all isolated vertices if any;
end;
output S;

Let $\mu(G)$ denote the (expected) size of the independent set produced by MINGREEDY. We prove

Theorem 2 *For any constant $\epsilon > 0$ and sufficiently large n,*

$$\mathbf{E}(\mu(G_{3-reg})) \geq (6\log(3/2) - 2 - \epsilon)n.$$

A simple martingale argument shows that $\alpha(G_{3-reg})$ is concentrated around its mean and consequently Theorem 1 follows immediately from Theorem 2.

2 CONFIGURATIONS

Our first task is to describe our model of a random cubic graph. We will use the *configuration model* of Bollobás [2] which is a simple and useful description of that used by Bender and Canfield [1].

Suppose we are given a degree sequence $1 \leq d_1, d_2, \ldots d_\nu \leq \Delta$. Let $W_i = \{i\} \times [d_i]$ for $i \in [\nu]$ and $W = \bigcup_{i=1}^{\nu} W_i$. A configuration is a partition of the *points* W into $\mu = 3\nu/2$ pairs. Let $\Omega_\nu = \{\text{configurations}\}$ and let F_ν be chosen uniformly from Ω_ν. Then let $\gamma(F_\nu)$ be the multigraph $([\nu], \{\{i,j\} : \{(i,x),(j,y)\} \in F_\nu \text{ for some } x \in [d_i], y \in [d_j]\})$.

The properties that we need of this model are:

Property 1 *conditional on $\gamma(F_\nu)$ being simple, it is equally likely to be any simple graph with the given degree sequence,*

Property 2 *assuming Δ is an absolute constant (here three will suffice),*

$$\Pr(\gamma(F_\nu) \textit{ is simple }) = \exp\left\{-\frac{\lambda}{2} - \frac{\lambda^2}{4}\right\}\left(1 + O\left(\frac{1}{\mu}\right)\right), \qquad (1)$$

where $\lambda = \frac{1}{\mu}\sum_{i=1}^{\nu}\binom{d_i}{2}$.

We make a simple observation which is the basis of our analysis of MINGREEDY when applied to $G_n = \gamma(F_n)$. F_n being a random matching of $3n/2$ labelled points, it can be constructed by repeatedly choosing an arbitrary point u from the set P of the current unmatched points and matching u

with a randomly chosen point from $P - \{u\}$. Now the step in MINGREEDY where vertices are removed can be regarded as a sequence of edge removals from F_n. When applying MINGREEDY to G_n, we may think of F_n as being constructed in parallel to MINGREEDY. Each edge in F_n constructed is precisely the current edge being removed by MINGREEDY. In particular, we have the following observation stated in the next lemma. We shall write from now on $N_i(t)$ $(i = 1, 2, 3)$ as the number of vertices of degree i in the graph $G(t)$ at the end of the t-th iteration of MINGREEDY. We shall also write $N(t) = (N_1(t), N_2(t), N_3(t))$ and $M(t)$ as the number of edges in $G(t)$.

Lemma 1 *Given $N(t)$, $G(t)$ is a multigraph with vertex set $V_1 \cup V_2 \cup V_3$, where V_1, V_2, V_3 are random disjoint subsets of $[n]$ with sizes N_1, N_2, N_3 respectively, obtained from a random configuration F on $W = \cup_{i=1,2,3} \cup_{v \in V_i} \{v\} \times [i]$. Consequently, $\{N(t)\}_{t \geq 0}$ is a Markov chain with initial state $N(0) = (0, 0, n)$.*

We will prove

Theorem 3 *For any constant $\epsilon > 0$ and sufficiently large n,*

$$\mathbf{E}(\mu(G_n)) \geq (6 \log(3/2) - 2 - \epsilon)n.$$

This is not sufficient to prove either of Theorems 1 or 2. On the other hand we know from martingale arguments that the independence number of G_n is concentrated around its mean and so Theorem 1 then follows from Properties 1 and 2 of the model. We will continue in "multigraph mode" until the end of the paper where we will show how the proof can equally well be applied to simple graphs and obtain both of Theorems 1 and 2.

3 SOME PRELIMINARIES

We first consider the transition probabilities of $N_1(t)$. Suppose that $N(t) = (N_1(t), N_2(t), N_3(t))$ is given. We write $\Delta N_i(t) = N_i(t+1) - N_i(t)$ and given $N(t)$, $p_i = p_i(t) = iN_i(t)/(2M(t))$. Suppose that in the t-th iteration of MINGREEDY, a vertex u of minimal degree $\delta(t)$ is picked. Let $\gamma(t)$

be the number of edges joining u or a neighbour of u to a vertex not in $\{u\} \cup \{\text{neighbours of } u\}$. Consider the case where $\delta(t) = 1$ first. Then

$$\mathbf{Pr}_1(\gamma(t) = i) = \begin{cases} p_{i+1} + O(1/M), & \text{if } i = 0, 1, 2, \\ 0, & \text{otherwise,} \end{cases}$$

where we write $\mathbf{Pr}_i$ as the probability conditional on $N(t)$ and $\delta(t) = i$. Next, each of the $\gamma(t)$ edge inspections increases $N_1(t)$ by 1 with probability $p_2 + O(1/M)$ and decreases $N_1(t)$ by 1 with probability $p_1 + O(1/M)$. Since $\delta(t) = 1$, $N_1(t)$ is decreased by 1 automatically. Therefore, the transition probabilities in the case where $\delta(t) = 1$ are given by

$$\begin{aligned} & \mathbf{Pr}_1(\Delta N_1(t) = k - 1) \\ = \ & \text{coefficient of } x^k \text{ in } \sum_j (p_1 x^{-1} + p_3 + p_2 x)^j \mathbf{Pr}_1(\gamma(t) = j) + O(1/M). \end{aligned}$$

Hence,

$$\mathbf{Pr}_1(\Delta N_1(t) = i) = \begin{cases} p_3 p_2^2 + O(1/M), & \text{if } i = 1, \\ 2p_3^2 p_2 + p_2^2 + O(1/M), & \text{if } i = 0, \\ p_3^3 + p_3 p_2 + O(p_1), & \text{if } i = -1, \\ O(p_1), & \text{if } i = -2, \\ O(p_1), & \text{if } i = -3, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, we have

$$\mathbf{Pr}_2(\gamma(t) = 2 + i) = \begin{cases} O(1/M), & \text{if } i = -1, -2, \\ \binom{2}{i} p_2^{2-i} p_3^i + O(1/M), & \text{if } i = 0, 1, 2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} & \mathbf{Pr}_2(\Delta N_1(t) = k) \\ = \ & \text{coefficient of } x^k \text{ in } \sum_j (p_3 + p_2 x)^j \mathbf{Pr}_2(\gamma(t) = j) + O(1/M). \end{aligned}$$

This gives that

$$\mathbf{Pr}_2(\Delta N_1(t) = i) = \begin{cases} p_3^2p_2^4 + O(1/M), & \text{if } i = 4, \\ 4p_3^3p_2^3 + 2p_3p_2^4 + O(1/M), & \text{if } i = 3, \\ 6p_3^4p_2^2 + 6p_3^2p_2^3 + p_2^4 + O(1/M), & \text{if } i = 2, \\ 4p_3^5p_2 + 6p_3^3p_2^2 + 2p_3p_2^3 + O(1/M), & \text{if } i = 1, \\ p_3^6 + 2p_3^4p_2 + p_3^2p_2^2 + O(1/M), & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

For $\delta(t) = 3$, it is enough to check that

$$\mathbf{Pr}_3(\Delta N_1(t) = i) = \begin{cases} 1 - O(1/M), & \text{if } i = 0, \\ O(1/M), & \text{if } i = 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check the following transition probabilities of M.

$$\mathbf{Pr}_1(\Delta M(t) = i) = \begin{cases} p_3 + O(1/M), & \text{if } i = -3, \\ p_2 + O(1/M), & \text{if } i = -2, \\ p_1 + O(1/M), & \text{if } i = -1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{Pr}_2(\Delta M(t) = i) = \begin{cases} p_3^2 + O(1/M), & \text{if } i = -6, \\ 2p_3p_2 + O(1/M), & \text{if } i = -5, \\ p_2^2 + O(1/M), & \text{if } i = -4, \\ O(1/M), & \text{if } i = -2, -3 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathbf{Pr}_3(\Delta M(t) \leq 9) = 1.$$

We next state a result proved in Frieze, Radcliffe and Suen [4] concerning the behaviour of $N_3(t)$ with respect to $M(t)$.

Lemma 2 *For any fixed $\epsilon > 0$,*

$$\mathbf{Pr}\left(\exists t \text{ such that } M(t) \geq n^{1/2}\ln^3 n \text{ and } \left|N_3(t)\sqrt{n}\left(\tfrac{3}{2M(t)}\right)^{3/2} - 1\right| \geq \epsilon\right)$$
$$= O(n^{-2}).$$

Proof (sketch) We shall only sketch briefly why it is true. Note that for each edge uv removed by MINGREEDY, one of the end-points, say u, is picked from the vertices of minimal (but non-zero) degrees, or u is predetermined from a previous edge removal. The other end-point v is chosen randomly from the neighbours of u. Now since almost all cubic graphs are connected, each decrease in N_3 (except for the first edge removal) is accounted for exactly once as the end-point v (whenever v is of degree 3). Now consider the edge removal when the current graph has N_3 vertices of degree 3 and M edge. The probability that the end-point v in the edge removed is of degree 3 is precisely $3N_3/(2M)$. Thus the rate of change in N_3 with respect to M should be approximately

$$\frac{dN_3}{dM} = \frac{3N_3}{2M},$$

which gives us an approximation as stated in Lemma 2. □

We shall use Lemma 2 as follows. Define $\hat{t}_m$ as the minimum t such that $M(t) \leq m$. Then $N_i(\hat{t}_m)$ is a function of m. Lemma 2 gives a fairly accurate estimate of $N_3(\hat{t}_m)$ and hence $p_3(\hat{t}_m)$: with probability $1 - O(1/n^2)$, we have for $m \geq n^{1/2}\log^3 n$ and for any constant $\epsilon_1 > 0$ that

$$|p_3(\hat{t}_m) - (2m/3n)^{1/2}| \leq \epsilon_1(2m/3n)^{1/2}. \tag{2}$$

We shall consider the behaviour of N_1 conditional on N_3 satisfying (2). This will be done in the Section 5 by dividing the interval $[m]$ into h subintervals where h is a large integer constant. As $p(\hat{t}_m)$ does not change much for m within a subinterval, we are able to approximate N_1 by a Markov chain.

4 Approximate Chains for N_1

We next describe the Markov chain that will be used to approximate N_1. Let Z be a random variable with support on $\mathcal{N}$. We write $\lambda_i = \mathbf{Pr}(Z = i)$

and $G_Z(s) = \sum_{i\geq 0} s^i\lambda_i$ for the probability generating function of Z. We also write $\mu = \mathbf{E}[Z]$. Assume also that the radius of convergence of $G_Z(s)$ is least a constant strictly greater than 1. We next define the transition probabilities of X_t as follows.

$$\begin{aligned} \mathbf{Pr}(X_{t+1} - X_t = i \mid X_t = 0) &= \lambda_i, \\ \mathbf{Pr}(X_{t+1} - X_t = i \mid X_t \neq 0) &= \begin{cases} p, & \text{if } i = 1, \\ 1-p-q, & \text{if } i = 0, \\ q, & \text{if } i = -1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We shall assume throughout that p and q are constants such that $q - p > 0$. Let τ_i be the minimum value of t such that $X_t = 0$ given that $X_0 = i$. Thus $\tau_0 = 0$ for example. Note that τ_1 equals in distribution to
(a) 1 with probability q,
(b) $1 + L_1$ with probability $1 - p - q$,
(c) $1 + L_2 + L_3$ with probability p,
where L_1, L_2, L_3 are independent copies of τ_1. Thus, if $\varphi(s) = \mathbf{E}[\exp(s\tau_1)]$, then

$$\varphi(s) = \mathrm{e}^s(q + (1-p-q)\varphi(s) + p\varphi(s)^2),$$

giving that for $s > -\log(1 - (\sqrt{q} - \sqrt{p})^2)$,

$$\varphi(s) = \frac{1}{2p}\mathrm{e}^{-s}\left\{1 - \mathrm{e}^s(1-p-q) - \sqrt{(\mathrm{e}^s(1-p-q)-1)^2 - 4pq\mathrm{e}^{2s}}\right\}.$$

Note that

$$\mathbf{E}[\tau_1] = \frac{1}{q-p}.$$

Lemma 3 *For any $A > 0$, we have as $k \to \infty$ that*

$$\mathbf{Pr}(|\tau_k - k/(q-p)| \geq Ak^{1/2}) = O(\mathrm{e}^{-A}).$$

Proof Note that τ_k equals in distribution to the sum of k independent copies of τ_1. Thus for $s > 0$,

$$\mathbf{Pr}(\tau_k - k/(q-p) \geq Ak^{1/2})$$

$$
\begin{aligned}
&\le \mathbf{E}[\exp(s\tau_k)]\exp(-sk/(q-p) - Ask^{1/2}) \\
&= \varphi(s)^k \exp(-sk/(q-p) - Ask^{1/2}) \\
&= (1 + s/(q-p) + O(s^2))^k \exp(-sk/(q-p) - Ask^{1/2}) \\
&\le \exp(O(ks^2) - Ask^{1/2}).
\end{aligned}
$$

Take $s = k^{-1/2}$ and obtain

$$\mathbf{Pr}(\tau_k - k/(q-p) \ge Ak^{1/2}) \le O(\mathrm{e}^{-A}).$$

Similarly, one can obtain that

$$\mathbf{Pr}(\tau_k - k/(q-p) \le -Ak^{1/2}) \le O(\mathrm{e}^{-A}),$$

and the lemma follows. □

Let R_k be the time elapsed when X_t first returns to 0 for the k-th time given that $X_0 = 0$. Note that R_1 equals in distribution to one plus the sum of Z independent copies of τ_1. Hence if $\psi(s) = \mathbf{E}[\exp(sR_1)]$, then

$$\psi(s) = \mathrm{e}^s G_Z(\varphi(s)).$$

Note that $\mathbf{E}[R_1] = 1 + \mu/(q-p)$. Note also that as $s \to 0$,

$$\psi(s) = 1 + s(q-p+\mu)/(q-p) + O(s^2).$$

Since R_k equals in distribution to the sum of k independent copies of R_1, we have the following lemma by following similar arguments used in showing Lemma 3.

Lemma 4 *For any $A > 0$, we have that as $k \to \infty$,*

$$\mathbf{Pr}(|R_k - k(q-p+\mu)/(q-p)| \ge Ak^{1/2}) = O(\mathrm{e}^{-A}).$$

□

We shall also require the following lemma.

Lemma 5 *Suppose that $X_0 = 0$. Then for any $A > 0$, there are constants $C > 0$ and $\rho \in (0,1)$ such that*

$$\mathbf{Pr}(\exists j \in [k] \text{ s.t. } R_j \geq A) \leq kC\rho^A \qquad \text{for all } k > 0.$$

Since X_t can be decreased by at most 1 in each transition, it follows that for any $A > 0$, there are constants $C > 0$ and $\rho \in (0,1)$ such that

$$\mathbf{Pr}(\exists t \in [R_k] \text{ s.t. } X_t \geq A) \leq kC\rho^A \qquad \text{for all } k > 0.$$

Proof We need only show that there are constants $C > 0$ and $\rho \in (0,1)$ such that

$$\mathbf{Pr}(R_1 \geq A) \leq C\rho^A.$$

This follows from

$$\mathbf{Pr}(R_1 \geq A) \leq \psi(s)\mathrm{e}^{-As},$$

by setting s to a positive constant. □

5 PROOF OF THEOREM 3

Choose a large integer h and define

$$m_i = \lfloor \tfrac{h-i}{h} \tfrac{3n}{2} \rfloor.$$

Thus, $m_0 = 3n/2$ equals the number of edges in the initial cubic graph. Define

$$t_i = \hat{t}_{m_i} = \min\{t : M(t) \leq m_i\}.$$

Since $M(t)$ is monotone decreasing in t, we have $t_i \leq t_{i+1}$. We shall first use the following lemma to prove the theorem. Write $r = r_i = \sqrt{(h-i)/h}$. Note that from (2), $r_i = \sqrt{2m_i/(3n)} + O(1/n)$ is an approximation of $p_3(t_i)$.

Lemma 6 *For any constant $\hat{\epsilon} > 0$ and for sufficiently large h, we have with probability $1 - O(1/n^2)$ that for $i = 1, 2, \ldots, h-2$,*

$$t_{i+1} - t_i \geq (1 - \hat{\epsilon}) \frac{3n}{4h} \frac{2 - r_i^2}{(2 + r_i)}.$$

Proof of Theorem 3 Note that

$$\mathbf{E}[\mu(G_n)] \geq \mathbf{E}[t_{h-1}].$$

¿From Lemma 6, we have with probability $1 - O(1/n^2)$ that

$$\begin{aligned}
t_{h-1} &\geq \sum_{i=1}^{h-2}(t_{i+1} - t_i) \\
&\geq (1-\hat{\epsilon})\frac{3n}{4h}\sum_{i=1}^{h-2}\frac{1+i/h}{2+\sqrt{1-i/h}} + O(n/h) \\
&= (1-\hat{\epsilon})\frac{3n}{4}\left(\int_0^1 \frac{1+x}{2+\sqrt{1-x}}dx - O(1/h)\right) + O(n/h) \\
&= (1-\hat{\epsilon})\frac{3n}{4}\int_0^1 \frac{2+x}{2+\sqrt{x}}dx + O(n/h) \\
&= (1-\hat{\epsilon})\frac{3n}{4}\left(8\log(3/2) - 8/3\right) + O(n/h) \\
&= (1-\hat{\epsilon})n\left(6\log(3/2) - 2\right) + O(n/h).
\end{aligned}$$

Thus

$$\mathbf{E}[t_{h-1}] \geq (1-\hat{\epsilon})n\left(6\log(3/2) - 2 + O(1/h)\right)\left(1 - O(1/n^2)\right).$$

The theorem follows by choosing sufficiently small $\hat{\epsilon}$ and sufficiently large h.
□

The rest of the section is devoted to proving Lemma 6. We first require an upper bound of N_1. We shall need the functions

$$\begin{aligned}
\alpha(x) &= x(1-x)^2 = x^3 - 2x^2 + x, \\
\beta(x) &= x^3 + (1-x)x = x^3 - x^2 + x.
\end{aligned}$$

Lemma 7 *With probability* $1 - O(1/n^2)$*, we have that for all* $t < t_{h-1}$*,*

$$N_1(t) = O(\log^2 n).$$

Proof We introduce two chains $\hat{W}_1, W_1$ where $N_1 \leq \hat{W}_1 \leq W_1$ in distribution and consider the time intervals $t_i \leq t \leq t_{i+1}$ separately. $\hat{W}_1$ is obtained from N_1 by ignoring the influence of p_1 and W_1 is obtained by replacing $\alpha(p_3), \beta(p_3)$ by suitable constants α_i^*, β_i^*.

Consider the transition probabilities of ΔN_1 when $\delta = 1$. Note that $p_3 p_2^2 \leq \alpha(p_3)$ and that $p_3^2 + p_3 p_2 \geq \beta(p_3)$. Since

$$\mathbf{Pr}(\Delta N_1(t) \leq 4 \mid N(t), \delta(t) \neq 1) = 1,$$

it is not difficult to check by using a coupling argument that

$$N_1(t) \leq \hat{W}_1(t) + 4,$$

where $\hat{W}_1(t)$ is a process that runs alongside $N(t)$ with $\hat{W}_1(0) = 0$ and transition probabilities given by

$$\begin{aligned} &\mathbf{Pr}(\Delta \hat{W}_1(t) = i \mid \hat{W}_1(t) > 0, p_3(t)) \\ = &\begin{cases} \alpha(p_3) + O(1/M), & \text{if } i = 1, \\ 1 - \alpha(p_3) - \beta(p_3) + O(1/M), & \text{if } i = 0, \\ \beta(p_3) + O(1/M), & \text{if } i = -1, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$\mathbf{Pr}(\Delta \hat{W}_1(t) = 1 \mid \hat{W}_1(t) = 0, p_3(t)) = 1.$$

We next find an upper bound $W_1(t)$ of $\hat{W}_1(t)$, for $t_i \leq t \leq t_{i+1}$, by using the fact that with high probability p_3 does not change by very much in the interval $t_i \leq t \leq t_{i+1}$. Since for all $x \in (0,1)$,

$$\alpha(x) < \beta(x) \quad \text{and} \quad \alpha(x) + \beta(x) \in (0,1),$$

it is possible to choose sufficiently large h and sufficiently small $\epsilon_2 = \epsilon_2(h) > 0$ so that for $i \leq h - 1$ and for $i/h \leq x \leq (i+1)/h$ (i.e. $m_{i+1} \leq m \leq m_i$),

$$\alpha(\sqrt{1-x}) < \alpha(r_i) + \epsilon_2/2 < \alpha(r_i) + \epsilon_2 < \beta(r_i) - \epsilon_2 < \beta(r_i) - \epsilon_2/2 < \beta(\sqrt{1-x}).$$

Write $\alpha_i^* = \alpha(r_i) + \epsilon_2$ and $\beta_i^* = \beta(r_i) - \epsilon_2$. Now we have from (2) that with error probability $O(1/n^2)$, we may assume that for $i < h$ and for $t_i \leq t \leq t_{i+1}$,

$$\alpha(p_3(t)) \leq \alpha_i^*, \quad \beta(p_3(t)) \geq \beta_i^*, \quad \text{and} \quad \alpha_i^* < \beta_i^*. \tag{3}$$

For each $i < h$, define a process $W_1(t) = W_1^{(i)}(t)$ with transition probabilities

$$\mathbf{Pr}(\Delta W_1(t) = j \mid W_1(t) > 0) = \begin{cases} \alpha_i^*, & \text{if } j = 1, \\ 1 - \alpha_i^* - \beta_i^*, & \text{if } j = 0, \\ \beta_i^*, & \text{if } j = -1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathbf{Pr}(\Delta W_1(t) = 1 \mid W_1(t) = 0) = 1.$$

Note that if N_3 satisfies (2), then from (3), it is possible to couple for each $i < h$, the processes $\hat{W}_1(t)$ and $W_1(t)$ so that $\hat{W}_1(t_i) = W_1(t_i)$ and

$$\hat{W}_1(t) \leq W_1(t), \qquad \text{for } t_i \leq t \leq t_{i+1}.$$

Therefore, for $t_i \leq t \leq t_{i+1}$,

$$\begin{aligned} & \mathbf{Pr}(N_1(t) \geq \log^2 n) \\ \leq\ & \mathbf{Pr}(N_1(t) \geq \log^2 n \mid N_3 \text{ satisfies (2)}) + O(1/n^2) \\ \leq\ & \mathbf{Pr}(W_1(t) \geq \log^2 n) + O(1/n^2). \end{aligned} \tag{4}$$

Note that since $N_1(0) = 0$, we may assume that $W_1(0) = O(\log^2 n)$. Thus assume inductively that $W_1(t_i) = O(\log^2 n)$ and show $W_1(t) = O(\log^2 n)$ for $t_i < t \leq t_{i+1}$. Now $W_1(t)$ is a special case of the process X_t defined earlier. Let T be the minimum value of $t \geq t_i$ such that $W_1(t) = 0$. Then since $W_1(t_i) \leq \log^2 n$, we have from Lemma 3 that $T - t_i = O(\log^2 n)$ with probability $1 - O(1/n^2)$. This shows that for $t_i \leq t \leq T$, $W_1(t) = O(\log^2 n)$ with probability $1 - O(1/n^2)$. Since $t_{i+1} - t_i \leq n$, we have from Lemma 5 that with probability $1 - O(1/n^2)$,

$$W_1(t) = O(\log^2 n),$$

for all t satisfying $T \leq t \leq t_{i+1}$. The lemma now follows from (4).

Proof of Lemma 6 Assume throughout that N_3 and p_3 satisfy (2) (which incurs an error probability of $O(1/n^2)$). Thus there is a positive constant $\epsilon_3 = \epsilon_3(h)$, where $\epsilon_3(h) \to 0$ as $h \to \infty$, so that for $i \leq h-2$ and $t_i \leq t \leq t_{i+1}$,

$$p_3(t) \leq r_i + \epsilon_3,$$

where r_i is as defined before. Write $\hat{r}_i = r_i + \epsilon_3$. Define two variables Y' and Y'' by

$$\mathbf{Pr}(Y' = j) = \begin{cases} \hat{r}_i, & \text{if } j = -3, \\ 1 - \hat{r}_i, & \text{if } j = -2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathbf{Pr}(Y'' = j) = \begin{cases} \hat{r}_i^2, & \text{if } j = -6, \\ 2\hat{r}_i(1 - \hat{r}_i), & \text{if } j = -5, \\ (1 - \hat{r}_i)^2, & \text{if } j = -4, \\ 0, & \text{otherwise.} \end{cases}$$

Y' approximates the number of edges deleted in an iteration when $N_1 > 0$ ($\hat{r}_i \approx p_3$ here) and Y'' approximates the number of edges deleted in an iteration when $N_1 = 0$. Next, define a process $W_2(t) = W_2^{(i)}(t)$ which runs alongside $N(t)$ with the following transition probabilities: (δ = minimum degree),

$$\begin{aligned} \mathbf{Pr}(\Delta W_2(t) = j \mid \delta(t) = 1) &= \mathbf{Pr}(Y' = j), \quad \forall j, \\ \mathbf{Pr}(\Delta W_2(t) = j \mid \delta(t) = 2) &= \mathbf{Pr}(Y'' = j), \quad \forall j, \\ \mathbf{Pr}(\Delta W_2(t) = -9 \mid \delta(t) = 3) &= 1. \end{aligned}$$

By comparing the transition probabilities of W_2 and the distribution of $\Delta M(t)$, we see that for $t \in [t_i, t_{i+1}]$, we have

$$\Delta W_2(t) \leq \Delta M(t).$$

in distribution. Thus, if we take $W_2(t_i) = m_i - 9$ ($\leq M(t_i)$ deterministically), then we have for any $\tau > 0$,

$$\mathbf{Pr}(t_{i+1} - t_i \leq \tau) \leq \mathbf{Pr}(W_2(t_i + \tau) \leq m_{i+1}).$$

(Note: $\{t_{i+1} \leq t_i + \tau\} \Rightarrow \{M(t_i + \tau) \leq m_{i+1}\} \Rightarrow \{W_2(t_i + \tau) \leq m_{i+1}\}$.)

Let $Z_k(\tau)$ be the number of times $t \in [t_i, t_i + \tau)$ such that $\delta(t) = k$. Note that since the probability that a random cubic graph has at least three components equals $O(1/n^2)$, we can assume throughout (which incurs an error probability of $O(1/n^2)$) that $Z_3 \leq 3$. Note also that $Y'' \leq Y'$ in distribution, which implies that for any $y > 0$, the distribution of $W_2(t_i + \tau)$ conditional on

$Z_2(\tau) \le y$ is bounded below stochastically by the distribution of $W_2(t_i+\tau)$ conditional on $Z_2(\tau) = y$. That is,

$$\mathbf{Pr}(W_2(t_i+\tau) \le m_{i+1} \mid Z_2(\tau) \le y) \le \mathbf{Pr}(W_2(t_i+\tau) \le m_{i+1} \mid Z_2(\tau) = y).$$

Since $\mathbf{E}[Y''] = 2\mathbf{E}[Y'] = -(4+2\hat{r}_i)$, we have

$$\mathbf{E}[W_2(t_i+\tau) - W_2(t_i) \mid Z_2(\tau) = y] = -(\tau+y)(2+\hat{r}_i) + O(1).$$

It is therefore not difficult to check that for any constants $\eta' \in [0,1]$ and $\epsilon_4 > 0$, there is a constant $\rho \in (0,1)$ such that as $\tau \to \infty$,

$$\mathbf{Pr}(W_2(t_i+\tau) \le m_i - (1+\epsilon_4)\tau(1+\eta')(2+\hat{r}_i) \mid Z_2(\tau) \le \eta'\tau) \le \rho^\tau.$$

Take

$$\begin{aligned} \tau &= \lfloor (1-\hat{\epsilon})\tfrac{3n}{4h}(2-r_i^2)/(2+r_i) \rfloor, \\ \eta &= r_i^2/(2-r_i^2), \\ \eta' &= (1+\epsilon_5)\eta. \end{aligned}$$

Since we can choose sufficiently large h so that $\hat{r}_i$ is as close to r_i as we like, we have that

$$m_{i+1} \le m_i - (1+\epsilon_4)\tau(1+\eta')(2+\hat{r}_i),$$

by choosing sufficiently small $\epsilon_4 > 0$, $\epsilon_5 > 0$ and sufficiently large h. We claim that for any constant $\epsilon' > 0$ and sufficiently large h, we have that for $i = 1, 2, \ldots, h-2$,

$$\mathbf{Pr}(Z_2(\tau) \ge (1+\epsilon')\eta\tau) = O(1/n^2), \tag{5}$$

which will be proved later. Now using (5), we have

$$\begin{aligned} & \mathbf{Pr}(t_{i+1} - t_i \le \tau) \\ \le\ & \mathbf{Pr}(W_2(t_i+\tau) \le m_{i+1}) \\ \le\ & \mathbf{Pr}(W_2(t_i+\tau) \le m_{i+1} \mid Z_2(\tau) \le \eta'\tau) + \mathbf{Pr}(Z_2(\tau) \ge \eta'\tau) \\ \le\ & \mathbf{Pr}(W_2(t_i+\tau) \le m_i - (1+\epsilon_4)\tau(1+\eta')(2+\hat{r}_i) \mid Z_2(\tau) \le \eta'\tau)) \\ & \qquad +\mathbf{Pr}(Z_2(\tau) \ge \eta'\tau) \\ =\ & O(1/n^2). \end{aligned}$$

It therefore remains to show (5). Remember that with high probability $Z_2(\tau) = \mathrm{O}(1)$ plus the number of times $N_1 = 0$. We consider an approximate lower bound of N_1. For similar reasons as given in proof of Lemma 7, it is possible to choose sufficiently large h and sufficiently small $\epsilon_6 = \epsilon_6(h) > 0$, where $\epsilon_6(h) \to 0$ as $h \to \infty$, such that for all $i = 1, 2, \dots, h-2$ and for x satisfying $i/h \le x \le (i+1)/h$,

$$\begin{aligned} \alpha(\sqrt{1-x}) &> \alpha(r_i) - \epsilon_6, \\ \beta(\sqrt{1-x}) &< \beta(r_i) + \epsilon_6. \end{aligned}$$

Write

$$\begin{aligned} \hat{a}_i &= \alpha(r_i) - \epsilon_6, \\ \hat{b}_i &= \beta(r_i) + \epsilon_6. \end{aligned}$$

¿From the assumption that p_3 satisfies (2), we have for $i \le h-2$ and $t_i \le t \le t_{i+1}$ that,

$$\alpha(p_3(t)) \ge \hat{a}_i, \quad \text{and} \quad \beta(p_3(t)) \le \hat{b}_i. \tag{6}$$

For each $i \le h-2$, define a process $W_3(t) = W_3^{(i)}(t)$, ($\le N_1$ in distribution),

$$\mathbf{Pr}(\Delta W_3(t) = j \mid W_3(t) > 0) = \begin{cases} \hat{a}_i, & \text{if } j = 1, \\ 1 - \hat{a}_i - \hat{b}_i, & \text{if } j = 0, \\ \hat{b}_i, & \text{if } j = -1, \\ 0, & \text{otherwise,} \end{cases}$$

and for small $\epsilon_7 > 0$,

$$\begin{aligned} &\mathbf{Pr}(\Delta W_3(t) = j \mid W_3(t) = 0) \\ &= \begin{cases} r_i^2(1-r_i)^4 - \epsilon_7, & \text{if } j = 4, \\ 4r_i^3(1-r_i)^3 + 2r_i(1-r_i)^4 - \epsilon_7, & \text{if } j = 3, \\ 6r_i^4(1-r_i)^2 + 6r_i^2(1-r_i)^3 - (1-r_i)^4 - \epsilon_7, & \text{if } j = 2, \\ 4r_i^5(1-r_i) + 6r_i^3(1-r_i)^2 + 2r_i(1-r_i)^3 - \epsilon_7, & \text{if } j = 1, \\ r_i^6 + 2r_i^4(1-r_i) + r_i^2(1-r_i)^2 + 4\epsilon_7, & \text{if } j = 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Define $W_3(t_i) = N_1(t_i)$. Note that we can choose sufficiently small $\epsilon_7 > 0$ and sufficiently large h so that (under the assumption that p_3 satisfies (2)) for all j,

$$\mathbf{Pr}(\Delta N_1(t) \le j \mid N_1(t) = 0, N(t)) \ge \mathbf{Pr}(\Delta W_3(t) \le j \mid W_3(t) = 0).$$

We next bound the transition probabilities of N_1 when $N_1 \neq 0$. We first deal with the transitions of N_1 with probabilities $O(p_1)$ as follows. Assume in conjunction with N_1 there is a process of coin tosses where
(a) if $N_1 > 0$, the probability of a head appearing equals the sum of the transition probabilities of N_1 involving p_1,
(b) if $N_1 = 0$, the probability of a head appearing equals 0.
If a head appears at stage t, then N_1 makes an appropriate transition according to those probabilities involving p_1; if a tail appears then N_1 makes an appropriate transition according to those probabilities not involving p_1. In particular, we have

$$\begin{aligned} & \mathbf{Pr}(\Delta N_1(t) = j \mid N_1(t) > 0, \text{ a tail appears at stage } t) \\ = \; & \frac{1}{1 - O(p_1)} \times \begin{cases} p_3 p_2^2 + O(1/M), & \text{if } j = 1, \\ 2p_3^2 p_2 + p_2^2 + O(1/M), & \text{if } j = 0, \\ p_3^3 + p_3 p_2 + O(1/M), & \text{if } j = -1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let $\hat{W}_3(t)$ be a process with $\hat{W}_3(t_i) = N_1(t_i)$ and transition probabilities such that for all j,

$$\begin{aligned} & \mathbf{Pr}(\Delta \hat{W}_3(t) = j \mid \hat{W}_3(t) > 0) \\ = \; & \mathbf{Pr}(\Delta N_1(t) = j \mid N_1(t) > 0, \text{ a tail appears at stage } t), \\ & \mathbf{Pr}(\Delta \hat{W}_3(t) = j \mid \hat{W}_3(t) = 0) \\ = \; & \mathbf{Pr}(\Delta N_1(t) = j \mid N_1(t) = 0). \end{aligned}$$

$\hat{W}_3$ has the same distribution as N_1 conditional on no heads appearing. It is coupled with N_1 as follows: let $\sigma_1 < \sigma_2$ be times such that $N_1(\sigma_1) = N_1(\sigma_2) = 0$ and $N_1(t) \neq 0$ for $\sigma_1 < t < \sigma_2$. If there are no heads in the interval $[\sigma_1, \sigma_2]$ then $\hat{W}_3 = N_1$ throughout, otherwise the interval is ignored in defining $\hat{W}_3$.

Let $\hat{R}$ (resp. R) be the number of times $t \in [t_i, t_i + \tau)$ such that $\hat{W}_3(t) = 0$ (resp. $W_3(t) = 0$) and let H be the number of times that a head appears in stages $t \in [t_i, t_i + \tau)$. Write $Z_2 = Z_2(\tau)$ and we now bound Z_2. For a given time t, let t' be the last time before t such that $N_1(t') = 0$. Let H' be the number of times $t \in [t_i, t_i + \tau)$ such that $N_1(t) = 0$ and that no head appears in the coin tosses between t' and t. Then

$$Z_2 \leq H' + H.$$

Since

$$H' \leq \hat{R}, \quad \text{in distribition,}$$

we have

$$Z_2 \leq \hat{R} + H, \quad \text{in distribition.}$$

¿From Lemma 7, we may assume (with error probability $O(1/n^2)$) that $p_1 = O(\log^2 n/n) = o(1)$. Under this assumption (and the assumption that p_3 satisfies (2)), we have for all j,

$$\mathbf{Pr}(\Delta\hat{W}_3(t) \geq j \mid \hat{W}_3(t) > 0) \geq \mathbf{Pr}(\Delta W_3(t) \geq j \mid W_3(t) > 0).$$

Hence, we have

$$\hat{R} \leq R \quad \text{in distribution.}$$

Note that as $N_1 = O(\log^2 n)$ with probability $1 - O(1/n^2)$, we have with probability $1 - O(1/n^2)$ that for all $i = 1, 2, \ldots, h-2$,

$$H = O(\log^2 n).$$

Thus, for any j,

$$\mathbf{Pr}(Z_2 \geq j) \leq \mathbf{Pr}(R \geq j - O(\log^2 n)) + O(1/n^2). \tag{7}$$

Since $W_3(t_i) = O(\log^2 n)$ and $W_3(t)$ is an example of the process X_t considered earlier, it follows from Lemma 3 and Lemma 4 that for any small constant $\epsilon_8 > 0$,

$$\mathbf{Pr}(R \geq (1+\epsilon_8)\frac{(\hat{b}_i - \hat{a}_i)\tau}{\hat{b}_i - \hat{a}_i + \mu}) = O(1/n^2),$$

where μ in this case equals

$$\mathbf{E}[\Delta W_3(t) \mid W_3(t) = 0] = 2 - 2r_i^2 - 10\epsilon_7.$$

Since $\hat{b}_i - \hat{a}_i = r_i^2 + 2\epsilon_6$, we have for any small constant $\epsilon_8 > 0$ that with probability $O(1/n^2)$,

$$R \geq (1+\epsilon_8)\tau\frac{r_i^2 + 2\epsilon_6}{2 - r_i^2 - 10\epsilon_7 - 2\epsilon_6}.$$

Since ϵ_6 and ϵ_7 can be arbitrary small (by choosing sufficiently large h), we have for any constant $\epsilon' > 0$ that

$$R \geq (1+\epsilon')\tau \frac{r_i^2}{2-r_i^2},$$

with probability $O(1/n^2)$. We now have (5) from (7). □

6 SIMPLE GRAPHS

Let $N(t)$ denote the number of vertices of degrees 1,2,3 in the current graph $G(t)$ at the end of the t-th iteration of MINGREEDY when applied to G_{3-reg}. To prove Theorems 1 and 2 we need only verify that

(i) given $N_i(t) = n_i, i = 1, 2, 3$, $G(t)$ is equally likely to be any member of $\mathcal{G}(n_1, n_2, n_3) = \{$graphs with vertex set $V \subseteq [n]$ and n_i vertices of degree $i = 1, 2, 3\}$,

and

(ii) the transition probabilities given for the N_i stay the same as in Section 3 up to an error of $O(1/M)$.

To prove (i) we fix t and assume inductively that (i) holds at t. We can therefore assume that $G(t)$ is a random member of $\mathcal{G}(x_1, x_2, x_3)$ for some x. Now fix y such that there is a positive probability of a transition to a state y. For $G \in \mathcal{G}(y_1, y_2, y_3)$ let $I_G = \{(H, v) : H \in \mathcal{G}(x_1, x_2, x_3), v$ is of minimum degree in H, and G is obtained from H by deleting v and all of its neighbours$\}$. We must show that $|I_G|$ depends only on x, y. Let d denote the minimum i such that $x_i \neq 0$. To construct I_G for G we first

(a) choose $v, v_1, \ldots, v_d \in [n] \setminus V(G)$;

(b) add edges incident with $v, v_1, v_2, \ldots v_d$ to make a graph in $\mathcal{G}(x_1, x_2, x_3)$.

The number of choices in (a) (trivially) depends only on x, y and the same is true for the number of choices in (b), which is fixed once the degree sequence of G is fixed. Just observe that if one chooses G and edges A as in (b) and changes G without changing the degree sequence then A remains a valid choice.

To prove (ii) we must consider the configuration model as described in Section 2. Here let $d_i = 1, 1 \leq i \leq x_1, d_i = 2, x_1 < i \leq x_1 + x_2, d_i = 3, x_1 + x_2 < i \leq x_1 + x_2 + x_3$. Now choose F_ν randomly from Ω_ν and apply one step of Algorithm MINGREEDY to $\gamma(F_\nu)$. Let $\mathcal{E}_y$ denote the event that the graph remaining is in $\mathcal{G}(y_1, y_2, y_3)$. All we need to show is that

$$\mathbf{Pr}(\mathcal{E}_y \mid F_\nu \text{ is simple}) = \mathbf{Pr}(\mathcal{E}_y) + O(1/\nu). \tag{8}$$

But

$$\mathbf{Pr}(\mathcal{E}_y \mid F_\nu \text{ is simple}) = \frac{\mathbf{Pr}(F_\nu \text{ is simple} \mid \mathcal{E}_y)\mathbf{Pr}(\mathcal{E}_y)}{\mathbf{Pr}(F_\nu \text{ is simple})}$$

and so we need only show

$$\mathbf{Pr}(F_\nu \text{ is simple} \mid \mathcal{E}_y) = \mathbf{Pr}(F_\nu \text{ is simple}) + O(1/\nu)$$

or

$$\mathbf{Pr}(F_\nu \text{ is simple} \mid D) = \mathbf{Pr}(F_\nu \text{ is simple}) + O(1/\nu) \tag{9}$$

where D is the set of pairs of points deleted from F_ν in one step. But $|D| = O(1)$ and contains no loops or multiple edges. (9) follows easily from (1).

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