On the number of perfect matchings, Hamilton cycles and spanning trees in ϵ-regular non-bipartite graphs

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Abstract

A graph $G = (V, E)$ on n vertices is super ϵ-regular if (i) all vertices have degree in the range $[(d-\epsilon)n, (d+\epsilon)n]$, dn being the average degree, and (ii) for every pair of disjoint sets $S, T \subseteq V$, $|S|, |T| \geq \epsilon n$, $e(S, T)$ is in the range $[(d-\epsilon)|S||T|, (d+\epsilon)|S||T|]$. We show that the number of perfect matchings lies in the range $[((d-2\epsilon)^\nu \frac{n!}{\nu! 2^\nu}, (d+2\epsilon)^\nu \frac{n!}{\nu! 2^\nu}]$, the number of Hamilton cycles lies in the range $[(d-2\epsilon)^n n!, (d+2\epsilon)^n n!]$ and the number of spanning trees lies in the range $[(d-2\epsilon)^{n-1} n^{n-2}, (d+2\epsilon)^{n-1} n^{n-2}]$.

1 Introduction

Let $G = (V, E)$ be a graph with $|V| = n$. Let $0 < d < 1$ and $\epsilon > 0$ be constants (independent of n) where ϵ is assumed to be small compared with d. We assume that the *density* of G is dn i.e. $|E|/\binom{n}{2} = dn$. Suppose that the following two conditions hold:

- If d_G denotes vertex degree in G then

$$(d-\epsilon)n \leq d_G(v) \leq (d+\epsilon)n \qquad \text{for all } v \in V. \tag{1}$$

- If for $S, T \subseteq V$, $S \cap T = \emptyset$ we let $e(S, T)$ denote the number of edges of G with one end in S and the other in T and $d(S, T) = \frac{e(S,T)}{|S|\,|T|}$ then

$$|d(S, T) - d| \leq \epsilon \text{ for all } S, T \subseteq V,\ S \cap T = \emptyset,\ |S|, |T| \geq \epsilon n. \tag{2}$$

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A graph satisfying (1),(2) said to be *super ϵ-regular*. We assume that $n = 2\nu$ is even. Let $m(G)$ denote the number of perfect matchings in G, let $h(G)$ denote the number of Hamilton cycles in G and let $t(G)$ denote the number of spanning trees of G. In this paper we prove

Theorem 1 *If ϵ is sufficiently small and n is sufficiently large then*

(a)

$$(d-2\epsilon)^\nu \frac{n!}{\nu!2^\nu} \leq m(G) \leq (d+2\epsilon)^\nu \frac{n!}{\nu!2^\nu}.$$

(b)

$$(d-2\epsilon)^n n! \leq h(G) \leq (d+2\epsilon)^n n!.$$

(c)

$$(d-2\epsilon)^{n-1} n^{n-2} \leq t(G) \leq (d+2\epsilon)^{n-1} n^{n-2}.$$

In all cases the bounds are "close" to the expected number of in the random graph $G_{n,d}$. The results here are strongly related to the result of Alon, Rödl and Ruciński [1]. They considered bipartite graphs H with vertex partition A, B where $|A| = |B| = n$. Assuming (1) and (2) for $S \subseteq A$ and $T \subseteq B$ they proved

Theorem 2 *[1]*

$$(d-2\epsilon)^n n! \leq m(G) \leq (d+2\epsilon)^n n!.$$

Michael Krivelevich has made some interesting observations on Theorem 1: First of all, part (b) of Theorem 1 improves Corollary 2.9 of Thomason [7] which estimates the number of Hamilton cycles in a pseudo-random graph. Secondly, if G is dn-regular and the second eigenvalue of the adjacency matrix of G is at most ηdn for small η, then G is super $\epsilon(\eta)$-regular (see for example Chung [3] Theorem 5.1) and so our result holds for such graphs. Finally, an examination of the proof of part(c) reveals that we do not use ϵ-regularity. We have in fact proved

Theorem 3 *Let G be a connected graph with minimum degree at least $(d-\epsilon)n$ and maximum degree at most $(d+\epsilon)n$ then*

$$(d-2\epsilon)^{n-1} n^{n-2} \leq t(G) \leq (d+2\epsilon)^{n-1} n^{n-2}.$$

We prove Theorem 1(a) in the next section, Theorem 1(b) in Section 3 and Theorem 3 in Section 4.

2 Perfect Matchings

Let A, B, $|A| = |B| = \nu$ be a partition of V. We re-express (2) in terms of ν i.e.

$$|d(S,T) - d| \leq \epsilon \text{ for all } S \subseteq A,\, T \subseteq B|S|, |T| \geq 2\epsilon\nu. \tag{3}$$

Furthermore, if A, B is a random partition and $H = H(A, B)$ is the bipartite sub-graph of G induced by A, B then **whp** $d_H(v) \in [(d - \epsilon - o(1))\nu, (d + \epsilon + o(1))\nu]$ for all $v \in V$. Thus the conditions of Theorem 2 are satisfied with ν replacing n and 2ϵ replacing ϵ. It follows immediately that

$$m(G) \geq (1 - o(1))\binom{n}{\nu} \times \nu!(d - 2\epsilon)^\nu \times \frac{1}{2^\nu} = (1 - o(1))\frac{n!}{\nu! 2^\nu}(d - 2\epsilon)^\nu. \tag{4}$$

The factor $\frac{1}{2^\nu}$ accounts for the fact that each perfect matching occurs in 2^ν different graphs H, assuming we consider the partition A, B distinct from B, A. There is slack in the calculation in [1] and this will absorb the $1 - o(1)$ term and so (4) proves the lower bound in Theorem 1.

For the upper bound we follow [1] and use the Minc conjecture [6] proved by Bregman [2]. For a partition A, B and $v \in A$ let $d_B(v)$ denote the number of G-neighbours of v in B. The Minc conjecture then states that

$$m(H) \leq \prod_{v \in A} (d_B(v))!^{1/d_B(v)}.$$

Thus

$$m(G) \leq \frac{1}{2^\nu} \sum_{A,B} \prod_{v \in A} (d_B(v))!^{1/d_B(v)}. \tag{5}$$

For a fixed A, we let $A_1 = \{v \in A : \; d_B(v) > (d+\epsilon)\nu\}$. Property (1) implies that $|A_1| \leq \epsilon n$. Now since $(x!)^{1/x}$ increases with x, we see, after using Stirling's approximation and (1), that

$$\begin{aligned}
\prod_{v \in A} (d_B(v))!^{1/d_B(v)} &\leq \left(\frac{d+\epsilon}{e}\nu\right)^{|A \setminus A_1|} \left(\frac{d+\epsilon}{e}n\right)^{|A_1|} \left(1 + O\left(\frac{\ln n}{n}\right)\right)^\nu \\
&\leq \left(\frac{d+\epsilon}{e}\nu\right)^\nu 2^{\epsilon n} n^{O(1)}.
\end{aligned}$$

Hence

$$m(G) \leq \frac{1}{2^\nu}\binom{n}{\nu}\left(\frac{d+\epsilon}{e}\nu\right)^\nu 2^{\epsilon n} n^{O(1)} \leq (d + 2\epsilon)^\nu \frac{n!}{\nu! 2^\nu}$$

completing the proof of part (a) of Theorem 1. □

3 Hamilton Cycles

A Hamilton cycle is the union of two perfect matchings and so $h(G) \leq \frac{1}{2}m(G)^2$ and the upper bound in part (b) of Theorem 1 follows from the upper bound in part (a).

The lower bound requires more work. For $1 \leq k \leq \lfloor n/3 \rfloor$, let Φ_k be the set of all 2-factors in G containing exactly k cycles, and let $\Phi = \cup_k \Phi_k$ be the set of all 2-factors. Let $f_k = |\mathcal{F}_k|$ so that $f_1 = h(G)$. If M is a perfect matching of G, let a_M denote the number of perfect matchings of G that are disjoint from M. Since deleting M only disturbs ϵ-regularity marginally, we see by part (a) that $a_M \geq (d-2\epsilon)^\nu \frac{n!}{\nu! 2^\nu}$. Thus

$$A_M = \sum_{M \in G} a_M \geq \left((d-2\epsilon)^\nu \frac{n!}{\nu! 2^\nu} \right)^2 \geq (d-2\epsilon)^n n! \times \frac{1}{3n^{1/2}}. \tag{6}$$

On the other hand, we have

$$A_M \leq \sum_{k=1}^{\lfloor n/3 \rfloor} 2^k f_k. \tag{7}$$

We will show by a relatively crude argument that where $k_1 = \lceil \frac{4}{(d-2\epsilon)(d-\epsilon)} \rceil$

$$\frac{f_{k+1}}{f_k} \leq n^3 \qquad 1 \leq k \leq k_1. \tag{8}$$

We then use an idea from Dyer, Frieze and Jerrum [4]. In this paper they show that if an n vertex graph G has minimum degree $\delta(G) \geq (\frac{1}{2}+\alpha)n$ for a positive constant α, then a polynomial fraction of the 2-factors of G are Hamilton cycles. We extend their argument to ϵ-regular graphs.

Let $\beta = \frac{200}{(d-2\epsilon)(d-\epsilon)^2}$. Let $k_0 = \lfloor \beta \ln n \rfloor$, and for $1 \leq k \leq n$, define $\gamma(k) = n^\beta k! (\beta \ln n)^{-k}$, and

$$\phi(k) = \begin{cases} \gamma(k), & if\ k \leq k_0; \\ \gamma(k_0), & otherwise. \end{cases}$$

Lemma 1 *Let ϕ be the function defined above. Then*

1. *ϕ is non-increasing and satisfies*

$$\min\{\phi(k-1), \phi(k-2)\} = \phi(k-1) \geq (\beta \ln n) k^{-1} \phi(k);$$

2. *$\phi(k) \geq 1$, for all k.*

Proof Observe that γ is unimodal, and that k_0 is the value of k minimizing $\gamma(k)$; it follows that ϕ is non-increasing. When $k \leq k_0$, we have $\phi(k-1) = \gamma(k-1) = (\beta \ln n) k^{-1} \gamma(k) = (\beta \ln n) k^{-1} \phi(k)$; otherwise, $\phi(k-1) = \gamma(k_0) = \phi(k) \geq (\beta \ln n) k^{-1} \phi(k)$. In either case, the inequality in part 1 of the lemma holds.

Part 2 of the lemma follows from the chain of inequalities

$$\frac{1}{\phi(k)} \leq \frac{1}{\gamma(k_0)} \leq \frac{(\beta \ln n)^{k_0}}{n^\beta k_0!} \leq n^{-\beta} \sum_{k=0}^{\infty} \frac{(\beta \ln n)^k}{k!} = n^{-\beta} \exp(\beta \ln n) = 1.$$

$\square$

Define

$$\Psi = \big\{(F, F') : F \in \Phi_k, F' \in \Phi_{k'}, k' < k, \text{ and } F \oplus F' \text{ is a 6-cycle}\big\},$$

where $\oplus$ denotes symmetric difference. Observe that $\Gamma = (\Phi, \Psi)$ is an acyclic directed graph; let us agree to call its component parts *nodes* and *arcs* to avoid confusion with the vertices and edges of G. Observe also that if $(F, F') \in \Psi$ is an arc, then F' can be obtained from F by deleting three edges and adding three others, and that this operation can decrease the number of cycles by at most two. Thus every arc $(F, F') \in \Psi$ is directed from a node F in some Φ_k to a node F' in Φ_{k-1} or Φ_{k-2}.

Our proof strategy is to define a positive weight function w on the arc set Ψ such that the total weight of arcs leaving each node (2-factor) $F \in \Phi_{\geq k_1}$ is significantly greater than the total weight of arcs entering F. We will show below that

$$\sum_{F^+:(F,F^+)\in\Psi} w(F, F^+) \geq 100\phi(k) n^2 \ln n \qquad F \in \mathcal{F}_k, k \geq k_1 \tag{9}$$

$$\sum_{F^-:(F^-,F)\in\Psi} w(F^-, F) \leq 9\phi(k) n^2 H_n \qquad F \in \mathcal{F}_k, k \geq 1 \tag{10}$$

where $H_n = \sum_{i=1}^n i^{-1} \leq \ln n + 1$ is the nth harmonic number.

Now let

$$W_{k,l} = \sum_{\substack{F\in\mathcal{F}_k, F'\in\mathcal{F}_l \\ (F,F')\in\Psi}} w(F, F').$$

Then (9) and (10) imply that for $k \geq k_1$,

$$\begin{aligned} W_{k+2,k} + W_{k+1,k} &\leq 9 f_k \phi(k) n^2 H_n &(11)\\ W_{k,k-1} + W_{k,k-2} &\geq 100 f_k \phi(k) n^2 \ln n. &(12) \end{aligned}$$

Now (12) implies that either

(i) $W_{k,k-1} \geq 50 f_k \phi(k) n^2 \ln n$ so that from (11)(k-1) we have

$$\frac{f_{k-1}}{f_k} \geq 5 \frac{\phi(k)}{\phi(k-1)}$$

or

(ii) $W_{k,k-2} \geq 50 f_k \phi(k) n^2 \ln n$ so that from (11)(k-2) we have

$$\frac{f_{k-2}}{f_k} \geq 5 \frac{\phi(k)}{\phi(k-2)}$$

It follows that if $k \geq k_0 + 2$ then

$$f_k \leq 5^{-(k-k_0)/2} \max\{f_{k_0+1}, f_{k_0}\}.$$

Then from (7) we see that

$$A_M \leq \frac{\sqrt{5}}{\sqrt{5}-2} \sum_{k=1}^{k_0+1} 2^k f_k \leq \frac{\sqrt{5}}{\sqrt{5}-2} 2^{k_0+1} \sum_{k=1}^{k_0+1} f_k. \tag{13}$$

Furthermore, since $F \in \mathcal{F}_k, k > k_1$ implies that

$$\sum_{F^+:(F,F^+)\in\Psi} w(F, F^+) - \sum_{F^-:(F^-,F)\in\Psi} w(F^-, F) \geq 1$$

the total weight of arcs entering Φ_{k_1} is an upper bound on the number of 2-factors in G with more than k_1 cycles and the maximum total weight of arcs entering a single node in Φ_{k_1} is an upper bound on the ratio $\rho = \frac{f_{k_1+1}+f_{k_1+2}+\cdots+f_{\lfloor n/3 \rfloor}}{f_{k_1}}$. Thus

$$\rho \leq 9\phi(1)n^2 H_n = O(n^{2+\beta}).$$

Combined with (13) and (8) we see that

$$A_M \leq n^{O(1)} f_1$$

and the lower bound in Theorem 1(b) follows from (6), modulo taking advantage of slack to absorb the $n^{O(1)}$ term.

3.1 Proofs of (9) and (10)

The weight function $w : \Psi \to \mathbf{R}^+$ we employ is defined as follows. For any arc (F', F) with $F \in \Phi_k$: if the 2-factor F is obtained from F' by coalescing two cycles of lengths l_1 and l_2 into a single cycle of length $l_1 + l_2$, then $w(F', F) = (l_1^{-1} + l_2^{-1})\phi(k)$; if F results from coalescing three cycles of length l_1, l_2 and l_3 into a single one of length $l_1 + l_2 + l_3$, then $w(F', F) = (l_1^{-1} + l_2^{-1} + l_3^{-1})\phi(k)$.

Let $F \in \Phi_k$ be a 2-factor with $k > 1$ cycles $C_1, C_2, \ldots, C_k$, of lengths $n_1, n_2, \ldots, n_k$. We proceed to bound from below the total weight of arcs *leaving* F. For this purpose imagine that the cycles $C_1, C_2, \ldots, C_k$ are oriented in some way, so that we can speak of each oriented edge (u, u') in some cycle C_i as being "forward" or "backward". For each vertex a we can then let $(a, \pi(a))$ be the unique forward edge containing a. Since we are interested in obtaining a *lower* bound, it is enough to consider only arcs (F, F^+) from F of a certain kind: namely, those for which the 6-cycle $C = F \oplus F^+$ is of the form $C = (x, x', y, y', z, z')$, where $(x, x') \in F$ is a forward cycle edge, $(y, y') \in F$ is a forward edge in a cycle distinct from the first, and $(z, z') \in F$ is a backward cycle edge. The edge (z, z') may be in the same cycle as either (x, x') or (y, y'), or in a third cycle. Observe that (x', y), (y', z) and (z', x)

must necessarily be edges of F^+. It is routine to check that any cycle $C = (x, x', y, y', z, z')$ satisfying the above constraints does correspond to a valid arc from F. The fact that (z, z') is oriented in the opposite sense to (x, x') and (y, y') plays a crucial role in ensuring that the number of cycles decreases in the passage to F^+ when only two cycles are involved.

First, we estimate the number of cycles C for which a fixed (x, x') is contained in a particular cycle C_i of F. We say that C is *rooted* at C_i. Let Z' be the neighbour set of x in G and $Z = \pi(Z')$. Similarly, let Y be the set of neighbours of x' which do not belong to C_i and let $Y = \pi(Y')$. If $|Y'| \geq \epsilon n$ then by ϵ-regularity there are at least $(d - 2\epsilon)n$ vertices $z \in Z$ which have at least $(d - \epsilon)|Y'| \geq (d - \epsilon)((d - \epsilon)n - n_i)$ neighbours y' in Y'. Let $\delta_i = 1_{n_i \leq (d-2\epsilon)n}$. We see that $\delta_i = 1$ implies (x, x') is contained in at least $(d - 2\epsilon)(d - \epsilon)((d - \epsilon)n - n_i)n$ cycles. Note also that $\sum_{i=1}^{k} \delta_i \geq k - \frac{1}{d-2\epsilon}$.

We can now bound the total weight of arcs leaving F. Each arc (F, F^+) defined by a cycle C rooted at C_i has weight at least $n_i^{-1} \min\{\phi(k-1), \phi(k-2)\}$, which, by Lemma 1, is bounded below by $(\beta \ln n)(kn_i)^{-1}\phi(k)$. Thus the total weight of arcs leaving F is bounded as follows:

$$
\begin{aligned}
\sum_{F^+ : (F, F^+) \in \Psi} w(F, F^+) &\geq \sum_{i=1}^{k} (d - 2\epsilon)(d - \epsilon)((d - \epsilon)n - n_i)n\delta_i n_i \frac{(\beta \ln n)\phi(k)}{kn_i} && (14) \\
&\geq \beta(d - 2\epsilon)(d - \epsilon)\phi(k) \left(d - \epsilon - \frac{1}{k(d - 2\epsilon)} - \frac{1}{k} \right) n^2 \ln n && (15) \\
&\geq \beta(d - 2\epsilon)(d - \epsilon)\phi(k) \frac{d - \epsilon}{2} n^2 \ln n, \\
&\geq 100\phi(k) n^2 \ln n && (16)
\end{aligned}
$$

where we have used the fact that $k \geq k_1$. Note that the presence of a unique backward edge, namely (z, z'), ensures that each cycle C has a distinguishable root, and hence that the arcs (F, F^+) were not overcounted in summation (14). This completes the proof of (9).

We now turn to the corresponding *upper* bound on the total weight of arcs $(F^-, F) \in \Psi$ *entering* F. It is straightforward to verify that the cycle $C = (x, x', y, y', z, z') = F^- \oplus F$ must contain three edges — (x, x'), (y, y') and (z, z') — from a single cycle C_i of F, the remaining edges coming from F^-. The labeling of vertices in C can be made canonical in the following way: assume an ordering on vertices in V, and assign label x to the smallest vertex. The condition $(x, x') \in F$ uniquely identifies vertex x', and the labeling of the other vertices in the cycle C follows.

Removing the three edges (x, x'), (y, y') and (z, z') from C_i leaves a triple of simple paths of lengths (say) $a - 1$, $b - 1$ and $c - 1$: these lengths correspond (respectively) to the segment containing x, the segment containing x', and the remaining segment. Going round the cycle C_i, starting at x' and ending at x, the vertices x, x', y, y', z, z' may appear in one of eight possible sequences:

$$
\begin{gathered}
x', y', y, z', z, x; \\
x', z, z', y, y', x; \\
x', z, z', y', y, x;
\end{gathered}
$$

$$
\begin{gathered}
x', z', z, y, y', x; \\
x', y', y, z, z', x; \\
x', y, y', z', z, x; \\
x', z', z, y', y, x; \\
x', y, y', z, z', x.
\end{gathered}
$$

For a given triple of lengths (a, b, c), each of the above sequences corresponds to at most n_i possible choices for the edges (x, x'), (y, y') and (z, z'), yielding a maximum of $8n_i$ in total. To see this, observe that the edge (x, x') may be chosen in n_i ways (minimality of x fixes the orientation of the edge), and that the choice of (x, x') combined with the information provided by the sequence completely determines the triple of edges.

The eight sequences divide into five possible cases, as the first four sequences lead to equivalent outcomes (covered by case 1 below). Taken in order, the five cases are:

1. For at most $4n_i$ of the choices for the edges (x, x'), (y, y') and (z, z'), $C_i \oplus C$ is a single cycle;
2. for at most n_i choices, $C_i \oplus C$ is a pair of cycles of lengths b and $a + c$;
3. for at most n_i choices, $C_i \oplus C$ is a pair of cycles of lengths a and $b + c$;
4. for at most n_i choices, $C_i \oplus C$ is a pair of cycles of lengths c and $a + b$;
5. for at most n_i choices, $C_i \oplus C$ is a triple of cycles of lengths a, b and c.

The first case does not yield an arc (F^-, F), since the number of cycles does not decrease when passing from $F^- = F \oplus C$ to F, but the other four cases do have to be reckoned with.

The total weight of arcs entering F can be bounded above as follows:

$$
\begin{aligned}
\sum_{F^- : (F^-, F) \in \Psi} w(F^-, F) &\le \sum_{i=1}^{k} n_i \phi(k) \sum_{\substack{a,b,c \ge 1 \\ a+b+c=n_i}} \left[\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) + \right. \\
&\qquad \left. \left(\frac{1}{a} + \frac{1}{b+c} \right) + \left(\frac{1}{b} + \frac{1}{a+c} \right) + \left(\frac{1}{c} + \frac{1}{a+b} \right) \right] \\
&= \sum_{i=1}^{k} n_i \phi(k) \sum_{\substack{a,b,c \ge 1 \\ a+b+c=n_i}} \left[\frac{6}{a} + \frac{3}{b+c} \right] \\
&\le \sum_{i=1}^{k} n_i \phi(k) n \sum_{a=1}^{n_i - 1} \left[\frac{6}{a} + \frac{3}{n_i - a} \right] \\
&\le 9 \phi(k) n^2 H_n.
\end{aligned}
$$

This completes the proof of (10).

3.2 Proof of (8)

We show that if $F \in \mathcal{F}_k$ and $2 \leq k \leq k_1$ then there is at least one arc $(F, F') \in \Psi$. Since each F' is the terminus of at most n^3 arcs, (8) follows immediately.

Let C_1 be the largest cycle of F. Then $|C_1| \geq n/k_1 \geq \frac{d^2}{5}n$.

Case 1: $|C_1| \leq n - 3\epsilon n$.

ϵ-regularity implies that there are at most ϵn vertices X which have fewer than $(d-\epsilon)|C_1|$ neighbours in C_1. As there are at least $3\epsilon n$ vertices not in C_1, there are vertices $x_1, x_2 \notin X$ which are neighbours on a cycle $C_2 \neq C_1$. Let $A_i, i = 1,2$ be the neighbour sets of x_i on C_1 and let $B_i = \pi(A_i)$ for $i = 1,2$. By assumption, $|B_i| \geq \frac{1}{5}d^2(d-\epsilon)n$ for $i = 1,2$ and so we can choose $B_i' \subseteq B_i$, $i = 1,2$ such that $B_1' \cap B_2' = \emptyset$ and $|B_1'| = |B_2'| \geq \frac{1}{6}d(d-\epsilon)n$. ϵ-regularity implies that there is at least one edge joining B_1', B_2'. Suppose this is the edge (b_1, b_2). Then $x_1, \pi^{-1}(b_1), b_1, b_2, \pi^{-1}(b_2), x_2, x_1$ defines the requisite 6-cycle.

Case 2: $|C_1| > n - 3\epsilon n$.

Just take any two vertices which are neighbours on a cycle other than C_1. Each has at least $(d - 4\epsilon)n$ neighbours in C_1 and we can argue the existence of a 6-cycle as in the previous case. □

4 Spanning Trees

For the lower bound let $\Omega = \{f : V \to V : (v, f(v)) \in E, \text{for all } v \in V\}$ be the set of functions defined by each $v \in V$ choosing a neighbour $f(v)$. Clearly

$$|\Omega| = \prod_{v \in V} d_G(v) \geq (d-\epsilon)^n n^n. \tag{17}$$

Each $f \in \Omega$ defines a digraph $D_f = (V, A_f)$, $A_f = \{(v, f(v) : v \in V\}$. A weak component of D_f consists of a cycle C with a rooted forest whose roots are in C. Suppose that D_f has k_f weak components. We obtain a spanning tree of G by (i) deleting the lexicographically first edge of each cycle of D_f and then (ignoring orientation) extending the k_f components to a spanning tree. We claim that if $\alpha = 4/\sqrt{d-\epsilon}$ and

$$\Omega_1 = \{f \in \Omega : k_f \leq \alpha\sqrt{n}\}$$

then

$$|\Omega_1| \geq |\Omega|/2. \tag{18}$$

Assume that (18) holds. Each spanning tree is obtained by deleting k_f edges of a D_f and then adding $k_f - 1$ edges. It follows that each spanning tree can be obtained in at most $\binom{N}{\alpha\sqrt{n}}^2$, $N = \binom{n}{2}$ ways from a member of Ω_1. Thus

$$t(G) \geq \tfrac{1}{2} n^{-4\alpha\sqrt{n}} (d-\epsilon)^n n^n$$

and the lower bound in (c) follows.

Proof of (18)
Let f be chosen randomly from Ω and write

$$k_f = \sum_{v \in V} \frac{1}{|K_v|}$$

where K_v is the weak component containing v.
We will argue that

$$\mathbf{Pr}(|K_v| \le k) \le \frac{k^2}{(d-\epsilon)n} \qquad k \ge 1. \tag{19}$$

Given (19) we have

$$\begin{aligned}
\mathbf{E}(|K_v|^{-1}) &\le \sum_{k=1}^{\sqrt{(d-\epsilon)n}} \frac{1}{k}(\mathbf{Pr}(|K_v| \le k) - \mathbf{Pr}(|K_v| \le k-1)) + \frac{1}{\sqrt{(d-\epsilon)n}} \\
&\le \sum_{k=1}^{\sqrt{(d-\epsilon)n}} \frac{\mathbf{Pr}(|K_v| \le k)}{k(k+1)} + \frac{1}{\sqrt{(d-\epsilon)n}} \\
&\le \sum_{k=1}^{\sqrt{(d-\epsilon)n}} \frac{1}{(d-\epsilon)n} + \frac{1}{\sqrt{(d-\epsilon)n}} \\
&= \frac{2}{\sqrt{(d-\epsilon)n}}
\end{aligned}$$

Thus

$$\mathbf{E}(k_f) \le \frac{2\sqrt{n}}{\sqrt{d-\epsilon}}$$

and (18) follows from the Markov inequality.
To verify (19), start with v and follow $v, f(v), f^2(v), \ldots$, until there is a repetition in the sequence. The probability of a repetition at the ith step is at most $\frac{i}{(d-\epsilon)n}$, since there are always at least $(d-\epsilon)n$ random choices for $f^i(v)$. If there are no repetitions by step k then $|K_v| > k$. Thus

$$\mathbf{Pr}(|K_v| \le k) \le \sum_{i=1}^{k} \frac{i}{(d-\epsilon)n}$$

and (19) follows. □
For the lower bound let $\Omega^* = \{f : V \to V : (v, f(v)) \in E \text{ or } f(v) = v \text{ for all } v \in V\}$. Then

$$t(G) \le |\Omega^*| \le ((d+\epsilon)n + 1)^n \le (d+2\epsilon)^{n-1} n^{n-2}.$$

To see this consider the following injective map from the spanning trees of G into Ω^*: orient each edge of tree T towards vertex 1 and then put $f(1) = 1$.
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