Edge-Disjoint Paths in Expander Graphs

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Abstract

Given a graph $G = (V, E)$ and a set of $\kappa$ pairs of vertices in $V$, we are interested in finding for each pair $(a_i, b_i)$, a path connecting $a_i$ to $b_i$, such that the set of $\kappa$ paths so found is edge-disjoint. (For arbitrary graphs the problem is $NP$-complete, although it is in $P$ if $\kappa$ is fixed.)

We present a polynomial time randomized algorithm for finding edge disjoint paths in an $r$-regular expander graph $G$. We show that if $G$ has sufficiently strong expansion properties and $r$ is sufficiently large then all sets of $\kappa = \Omega(n / \log n)$ pairs of vertices can be joined. This is within a constant factor of best possible.

1 Introduction

Given a graph $G = (V, E)$ with $n$ vertices, and a set of $\kappa$ pairs of vertices in $V$, we are interested in finding for each pair $(a_i, b_i)$, a path connecting $a_i$ to $b_i$, such that the set of $\kappa$ paths so found is edge-disjoint.

For arbitrary graphs the related decision problem is $NP$-complete, although it is in $P$ if $\kappa$ is fixed — Robertson and Seymour [17]. Peleg and Upfal [16] presented a polynomial time algorithm for the case where $G$ is a (sufficiently strong) bounded degree expander graph, and $\kappa \leq n^e$ for a small constant $\epsilon$ that depends on the expansion property of the graph. This result has been improved and extended by Broder, Frieze, and Upfal [2, 3], Frieze [5], Leighton and Rao [12] and Leighton, Rao and Srinivasan [13, 14]: In these papers $G$ has to be a (sufficiently strong) bounded degree expander and $\kappa$ can grow as fast as $n/(\log n)^\theta$, where $\theta$ depends only on the expansion properties of the input graph, but is at least 2.

Let $D$ be the median distance between pairs of vertices in $G$. Clearly it is not possible to connect more than $O(m/D)$ pairs of vertices by edge-disjoint paths, for all choices of pairs, since some choice would require more edges than all the edges available. In the case of an $r$-regular expander, this absolute upper bound on $\kappa$ is $O(n / \log n)$ (assuming $r$

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is independent of $n$). In this paper, we show that if $G$ has sufficiently strong expansion properties and $r$ is sufficiently large then all sets of $\kappa = \Omega(n / \log n)$ pairs of vertices can be joined. This therefore, is within a constant factor of the optimum. The precise definition of “sufficiently strong” is given after the theorem.

**Theorem 1** Let $G = (V, E)$ be an $n$-vertex, $r$-regular graph. Suppose that $G$ is a sufficiently strong expander. Then there exist $\epsilon_1, \epsilon_2 > 0$ such that $G$ has the following property: For all sets of pairs of vertices $\{(a_i, b_i) \mid i = 1, \ldots, \kappa\}$ satisfying:

(i) $\kappa = \lceil \epsilon_1 r n / \log n \rceil$.

(ii) For each vertex $v$, $|\{i : a_i = v\}| + |\{i : b_i = v\}| \leq \epsilon_2 r$.

There exist edge-disjoint paths in $G$, each of length $O(\log n)$, joining $a_i$ to $b_i$, for each $i = 1, 2, \ldots, \kappa$. Furthermore, there is a polynomial time randomized algorithm for constructing these paths.

$\epsilon_1, \epsilon_2$ depend only on certain expansion parameters $\alpha, \beta, \gamma$ defined below. They do not depend on $n$ or $r$.

The algorithm we use is based on the one used in Frieze and Zhao [7] which dealt with random $r$-regular graphs. In [7] we can take $\kappa = \lceil \epsilon_1 (r \log r) n / \log n \rceil$.

### 1.1 Preliminaries

We define expanders in terms of edge expansion (a weaker property than vertex expansion).

Let $G = (V, E)$ be a graph and let $n = |V|$. For $S \subset V$ let $\text{out}(S) = \text{out}_G(S)$ be the number of edges with one end-point in $S$ and one end-point in $V \setminus S$, that is

$$\text{out}(S) = \left| \left\{ (u, v) \mid (u, v) \in E, u \in S, v \notin S \right\} \right|.$$

Similarly,

$$\text{in}(S) = \left| \left\{ (u, v) \mid (u, v) \in E, u \notin S, v \in S \right\} \right|.$$

A graph $G = (V, E)$ is a $\theta$-expander, if for every set $S \subset V$, $|S| \leq n/2$, we have $\text{out}(S) \geq \theta |S|$.

An $r$-regular graph $G = (V, E)$ is called an $(\alpha, \beta, \gamma)$-expander if for every set $S \subset V$

$$\text{out}(S) \geq \begin{cases} (1 - \alpha) r |S| & \text{if } |S| \leq \gamma n \\ \beta r |S| & \text{if } \gamma n < |S| \leq n/2 \end{cases}$$

We naturally assume that $\beta < 1 - \alpha$.

By “sufficiently strong” in Theorem 1, we mean that $\beta, \gamma$ are arbitrary and $\alpha$ is sufficiently small. Then everything will work provided $r$ is sufficiently large.

Since $2 \text{in}(S) + \text{out}(S) = r |S|$ we see that in an $(\alpha, \beta, \gamma)$-expander

$$\text{in}(S) \leq \alpha r |S|/2 \quad \text{if } |S| \leq \gamma n.$$

In particular random regular graphs and the (explicitly constructible) Ramanujan graphs of Lubotsky, Phillips and Sarnak [15] are $(\alpha, \beta, \gamma)$-expanders. (See discussion in [2].)
The paper contains a few unspecified absolute constants. Exact values could be given but it is easier for us and the reader if we simply give the relations between them. New constants will be introduced as $C_0, \ldots$, sometimes without further comment. Furthermore, specific constants have been chosen for convenience. We made no attempt to optimize them, and, in general, we only claim that inequalities dependent on $n$ or $r$ hold for $n$ or $r$ sufficiently large.

For a graph $G = (V, E)$ and $v \in V$ we let $d_G(v)$ denote the degree of $v$ in $G$. We use $\delta(G)$ and $\Delta(G)$ to denote the smallest and largest degrees respectively. For a set $S \subseteq V$ we let $\bar{S} = V \setminus S$ and define its neighbor set, $N_G(S)$, as

$$N_G(S) = \{ v \in \bar{S} : \exists w \in S \text{ such that } \{v, w\} \in E \}.$$ 

For $v \in V$ and $S \subseteq V$ we let $d_G(v, S) = |N_G(v) \cap S|$.

Let $\Phi_S = \text{out}(S)/|S|$ and let the (edge)-expansion $\Phi = \Phi(G)$ of $G$ be defined by

$$\Phi = \min_{S \subseteq V} \frac{\Phi_S}{|S|}.$$ 

We need an algorithm for splitting a strong expander into ten expander graphs. We could use the algorithm of [2] or [6]. The latter gives a better split and we arbitrarily choose to use it. $\epsilon > 0$ is a small constant. The expansion requirements for the algorithm are

$$\frac{r}{\log r} \geq 70\epsilon^{-2} \text{ and } \Phi \geq 40\epsilon^{-2} \log r,$$  

which for us means

$$\beta \geq 40\epsilon^{-2} r^{-1} \log r.$$  

The result we need from this paper (Theorem 2) is:

**Theorem 2** Suppose that (2), (3) hold and that $G$ is an $r$-regular $(\alpha, \beta, \gamma)$-expander. Then there is a randomised polynomial time algorithm $O(n^2 \log \delta^{-1})$\footnote{The paper only claims $n^{O(\log r)}$ expected time but changing the definition of $X_0$ in [6] to deal with smaller $|S|$ easily yields this improvement} which with probability at least $1 - \delta$ constructs $E_1, E_2, \ldots, E_{10}$ such that the edge-expansion $\Phi_i$ of $G_i = (V, E_i)$ satisfies

$$\Phi_i \geq (1 - \epsilon) \frac{\Phi}{10} - (\alpha + 2\epsilon) r,$$

for $i = 1, 2, \ldots, 10.$

## 2 Overview of the algorithm

Our algorithm divides naturally into the three phases sketched below.
Phase 0: Partition $G$ into ten edge-disjoint graphs $G_i = (V, E_i)$, $1 \leq i \leq 10$. Phase 1 will use only the graphs $G_1$ and $G_2$; Phase 2 will use only the graphs $G_3$, $G_4$ and $G_5$; and Phase 3 will use only the graphs $G_6 - G_{10}$.

Phase 1: Choose two random sets $\tilde{A}, \tilde{B}$ of $\kappa$ vertices in $V$. Connect the endpoints $A = \{a_i : i = 1, 2, \ldots, \kappa\}$ to the newly chosen points $\tilde{A}$ in an arbitrary manner via edge-disjoint paths in $G_1$ using a flow algorithm. Similarly, connect the endpoints $B = \{b_i : i = 1, 2, \ldots, \kappa\}$ to the newly chosen points $\tilde{B}$, this time using $G_2$. Let $\tilde{a}_i$ (resp. $\tilde{b}_i$) be the vertex connected to $a_i$ (resp. $b_i$). The original problem is now reduced to finding edge-disjoint paths from $\tilde{a}_i$ to $\tilde{b}_i$ for each $i$.

Phase 2: We split this into parts (a),(b),(c).

(a) At this point we want $\tilde{a}_i$, $i = 1, 2, \ldots, \kappa$ to be a random ordering of a random set of vertices and so we randomly re-order $a_1, a_2, \ldots, a_\kappa$ to ensure this. We do the same with $\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_\kappa$. We then randomly generate $x_1, x_2, \ldots, x_\kappa$ from $V$ with replacement.

(b) For each $i$ in turn, we connect $\tilde{a}_i$ to $x_i$ by a path of minimum length in $G_3$. We remove the edges of this path from $G_3$.

(c) For each $i$ in turn, we connect $\tilde{b}_i$ to $x_i$ by a path of minimum length in $G_4$. We remove the edges of this path from $G_4$.

Most pairs $(\tilde{a}_i, \tilde{b}_i)$ will be successfully connected via $x_i$ in this phase. For such a pair, the final path from $a_i$ to $b_i$ is the concatenation of the paths indicated as follows

$$a_i - \tilde{a}_i - x_i - \tilde{b}_i - b_i$$

It is important in our analysis to ensure that random walks are done on subgraphs which are expander graphs. We use $G_5$ as a backup for ensuring that this is done.

Phase 3: At the end of Phase 2, there will with probability $\geq 1/2$, be at most $n/(\log n)^4$ pairs $(\tilde{a}_i, \tilde{b}_i)$ which have not been joined by paths. We use the algorithm of [5] to join them by edge disjoint paths, using only the edges of $G_6 - G_{10}$, and then construct the final paths from $a_i$ to $b_i$ as above.

To prove Theorem 1 it suffices to show that:

- Phases 0 and 1 will succeed for all choices of $a_1, \ldots, a_\kappa$ and almost every choice of $\tilde{A}, \tilde{B}$.
- Phases 2 and 3 are successful for almost every choice of $\tilde{A}, \tilde{B}$ and any bijection $\tilde{A} \rightarrow \tilde{B}$

3 Detailed description of the algorithm

The input to our algorithm is a sufficiently strong $(\alpha, \beta, \gamma)$-expander graph $G$ and a set of pairs of vertices $\{(a_i, b_i) \mid i = 1, \ldots, \kappa\}$ satisfying the premises of Theorem 1. The output is a set of $\kappa$ edge-disjoint paths, $P_1, \ldots, P_\kappa$ such that $P_i$ connects $a_i$ to $b_i$. 
3.1 Phase 0.

We start by partitioning $G$ into ten edge-disjoint graphs $G_i = (V, E_i)$, for $1 \leq i \leq 10$. We use the algorithm split of Theorem 2. We take $\epsilon = \alpha$ in the theorem and assume that $\beta \gg \alpha$. Thus each $G_i$ satisfies

$$\Phi_i = \Phi(G_i) \geq \beta_0 r,$$

(4)

$$\beta_0 r \leq \delta(G_i) \leq \Delta(G_i) < r,$$

(5)

where

$$\beta_0 = \frac{\beta}{10} - 4\alpha > 3\alpha > 0.$$

(6)

3.2 Phase 1.

Choose $\tilde{A}, \tilde{B} \subseteq V$ uniformly and randomly without replacement. We are going to replace the problem of finding paths from $a_i$ to $b_i$ by that of finding paths from $\tilde{a}_i$ to $\tilde{b}_i$.

We connect $A$ to $\tilde{A}$ via edge-disjoint paths in the graph $G_1$ using network flow techniques. We construct a network as follows

- Each undirected edge of $G_1$ gets capacity 1.
- Each $v \in V$ becomes a source of capacity $|\{i : a_i = v\}|$ and each member of $\tilde{A}$ becomes a sink of capacity 1.

Then we find a flow from $A$ to $\tilde{A}$ that satisfies all demands. Since the maximum flow has integer values, it decomposes naturally into $|A|$ edge-disjoint paths (together perhaps with some cycles). If a path joins $a_i$ to $z \in \tilde{A}$, then we let $\tilde{a}_i = z$.

We carry out a similar construction involving $B$ and $\tilde{B}$ in $G_2$.

Thus Phase 1 finds edge-disjoint paths $W_i^{(1)}$ from $a_i$ to $\tilde{a}_i$ and $W_i^{(4)}$ from $\tilde{b}_i$ to $b_i$, $1 \leq i \leq \kappa$, where the vertices $\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2, \ldots, \tilde{a}_\kappa, \tilde{b}_\kappa \in V_2$ are chosen uniformly at random without replacement. On the other hand there may be some difficult conditioning involved in the pairing of $a_i$ with $\tilde{b}_i$, $1 \leq i \leq \kappa$. We deal with this in Phase 2(a).

3.3 Phase 2.

3.3.1 Algorithm GenPaths.

We (try to) construct edge-disjoint paths connecting $\tilde{a}_i, x_i, \tilde{b}_i$ for $1 \leq i \leq \kappa$. For $1 \leq i \leq \kappa$ we try to connect $\tilde{a}_i$ to $x_i$ in graph $\Gamma_3$ by a shortest path $W_i^{(2)}$. Here $\Gamma_j = (V_j, F_j), j = 3, 4, 5$ denotes $G_j$ after the deletion of some vertices and edges. We construct these paths in the order $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_\kappa$. The edges of each such path are deleted before the next path is constructed. This keeps the paths edge-disjoint. This constitutes Phase 2(b).

In Phase 2(c) we use the same ideas and $\Gamma_4$ to join $\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_\kappa$ to $x_1, x_2, \ldots, x_\kappa$ by a shortest path $W_i^{(3)}$. 

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It is important for the above analysis to ensure that the construction of any shortest path takes place on a graph $\Gamma = (K, F)$ which is an expander. We can ensure this by keeping the degrees of vertices in the $\Gamma_j$ close to their degree in $G_j$. This may involve deleting some vertices after a walk. We use the routine REMOVE to do this.

If the proposed start vertex $v$ of a path on $\Gamma$ does not lie in $K$ then we try to connect it back to $K$ by a path in $\Gamma_5$. The terminal endpoint of this walk is denoted by $v'$. We use a subroutine CONNECTBACK for this purpose. We do not expect to succeed all the time and our failures are kept in a set $L$ for later consideration.

The walk from $a_i$ to $b_i$ is then the catenation of walks $W^{(t)}_i$, $t = 1, \ldots, 4$. These walks may each include a short walk $W_{CB}$ at the beginning provided by CONNECTBACK.

1. Algorithm GenPaths
2. begin
3. $\Gamma_i \leftarrow G_i$, $i = 3, 4, 5$.
4. for $i = 1$ to $\kappa$ do
5. Execute REMOVE($\Gamma_3$)
6. Execute CONNECTBACK($V_3, a_i, a'_i, i, W_{CB}$)
7. if $i \notin L$ then
8. Execute CONNECTBACK($V_3, x_i, x'_i, i, W_{CB}$)
9. if $i \notin L$ then
10. Construct a shortest path $W^{(2*)}_i$ from $a'_i$ to $x'_i$ in $\Gamma_3$.
11. $W^{(2*)}_i \leftarrow (W_{CB}, W^{(2*)}_i)$, $\Gamma_3 \leftarrow \Gamma_3 \setminus E(W^{(2*)}_i)$
12. fi
13. for $i = 1$ to $\kappa$ do
14. Execute REMOVE($\Gamma_4$)
15. Execute CONNECTBACK($V_4, b_i, b'_i, i, W_{CB}$)
16. if $i \notin L$ then
17. Execute CONNECTBACK($V_4, x_i, x''_i, i, W_{CB}$)
18. if $i \notin L$ then
19. Construct a shortest path $W^{(3*)}_i$ from $b'_i$ to $x''_i$ in $\Gamma_4$.
20. $W^{(3*)}_i \leftarrow (W_{CB}, W^{(3*)}_i)$, $\Gamma_4 \leftarrow \Gamma_4 \setminus E(W^{(3*)}_i)$
21. fi
22. od
23. end GenPaths

3.3.2 Subroutine REMOVE

The purpose of REMOVE is to delete vertices which might prevent a graph (or graphs) from being an expander. In GenPaths we apply REMOVE to $\Gamma_3$, or $\Gamma_4$. In CONNECTBACK we apply REMOVE to $\Gamma_5$.

In Step 4 we remove the set of vertices $R_0$ which have so far lost more than $\beta_0 r/4$ edges through the deletion of shortest paths. We then iteratively (Steps 5–12) remove vertices
which have at least $\beta_0 r / 4$ neighbours among previously removed vertices. We therefore see that for $t = 3, 4$

\[ v \in V_t \text{ implies } d_{\Gamma_t}(v) \geq d_{G_t}(v) - \beta_0 r / 2 \geq \beta_0 r / 2. \]  

(7)

1. **Algorithm** \textsc{remove}$(\Gamma_t)$
2. \textbf{begin}
3. \textbf{begin} $R_0 = \{v \in V_t : d_{\Gamma_t}(v) < d_{G_t}(v) - \beta_0 r / 4\}.$
4. \textbf{end}
5. $\ell \leftarrow 0.$
6. \textbf{begin}
7. $\hat{R}_t \leftarrow V_t \setminus R_t.$
8. $d \leftarrow \max \{d_{\Gamma_t}(v, R_t) : v \in \hat{R}_t\}.$
9. \textbf{if } $d \leq \beta_0 r / 4$ \textbf{terminate} \textsc{remove}, otherwise
10. \textbf{begin} $R_t \leftarrow R_t \cup \{w\}; V_t \rightarrow V_t \setminus \{w\}$ where $w \in \hat{R}_t$ is such that $d_{\Gamma_t}(w, R_t) = d.$
11. $\ell \leftarrow \ell + 1$
12. \textbf{end}
13. \textbf{end} \textsc{remove}

We can see from (7) that throughout the algorithm

\[ \Phi_{\Gamma_t} \geq \Phi_t - \beta_0 r / 2 \geq \beta_0 r / 2 \quad \text{for } t = 3, 4. \]

(8)

Indeed, (7) implies that for $S \subseteq V_t$ we have

\[ \text{out}_{\Gamma_t}(S) \geq \text{out}_{G_t}(S) - \beta_0 r |S|/2 \geq (\Phi_t - \beta_0 r / 2)|S|. \]

3.3.3 **Subroutine** \textsc{connectback}.

The purpose of \textsc{connectback} is to connect a vertex $x$ by a random walk to a set $K = V_3$ or $V_4$ of vertices of large degree in a particular subgraph. (If $x$ already has large degree then \textsc{connectback} does nothing except to relabel $x$ as $x'$). All walks are done on vertices $V_5$ and in Step 3 we check that the start point $x$ lies in $V_5$. If not, we put $i$ into $L$, where $x = \tilde{a}_i, \tilde{b}_i, x_i$ or $x'_i$. Edge disjoint paths for the pairs $(a_i, b_i), i \in L$ are found in Phase 3. Let

$$ \omega = [\log \log n]^2. $$

We do a random walk $W_{CB}$ from $x$ until we reach $K$ or make $\omega$ steps. In the latter case we add the corresponding $i$ to $L$. 

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1. **subroutine** `ConnectBack(K, x, x', i, W_{CB})`
2. **begin**
3.  **if** $x \in K$ **then** $x' \leftarrow x$ **exit** **fi** **else**
4.  **Execute** `Remove(\Gamma_5)`
5.  **if** $x \notin V_5$ **then** $L \leftarrow L \cup \{i\}$ **exit** **fi** **else**
6.  Do a random walk $W_{CB}$ starting at $x$ in $\Gamma_5$, until $K$ is reached or $\omega$ steps have been taken.
7.  In the latter case $L \leftarrow L \cup \{i\}$ and we **exit** **else**
8.  $\Gamma_6 \leftarrow \Gamma_5 \setminus W_{CB}$
9. **end** `ConnectBack`

### 3.4 Phase 3.

There is still the set $L$ of pairs $(\tilde{a}_i, \tilde{b}_i)$ which have not been connected by paths. We will show later that with probability at least $1 - o(1)$, $|L|$ is at most $n/(\log n)^4$. As such, these pairs can be dealt with by the algorithm of [5], using graphs $G_6 - G_{10}$.

### 4 Analysis of Phase 1

In this section we show that if (4) holds and

$$\beta_0 r \geq 1 \text{ and } \epsilon_2 \leq \beta_0$$

then after we run `Split`, we can find edge-disjoint paths from $a_i$ to $\tilde{a}_i$ in $G_1$ and edge disjoint paths from $b_i$ to $\tilde{b}_i$ in $G_2$, for $1 \leq i \leq \kappa$, for any choice of $a_1, \ldots, b_\kappa$ consistent with the premises of Theorem 1, and every choice for $\tilde{a}_1, \ldots, \tilde{a}_\kappa, \tilde{b}_1, \ldots, \tilde{b}_\kappa$.

Let $A$ and $\tilde{A}$ be as defined in Section 3.2. For $S \subseteq V$, let

$$\alpha(S) = |\{i : a_i \in S\}| \text{ and } \xi(S) = |S \cap \tilde{A}|.$$

For sets $S, T \subseteq V$, let $e_{G_1}(S, T)$ denote the number of edges of $G_1$ with an endpoint in $S$ and the other endpoint in $T$. It suffices to prove that

$$e_{G_1}(S, \tilde{S}) \geq \xi(\tilde{S}) - \alpha(S), \quad \forall S \subseteq V. \quad (10)$$

Given (10), the existence of the required flow in $G_1$ is a special case of a theorem of Gale [8] (see Bondy and Murty [1] Theorem 11.8). In which case we see that (10) implies a successful run of Phase 2.

Now

$$\alpha(\tilde{S}) = \kappa - \alpha(S) \geq \kappa - \epsilon_2 r |S|$$

and so

$$\xi(\tilde{S}) - \alpha(\tilde{S}) \leq |\tilde{A} \cap \tilde{S}| - \kappa + \epsilon_2 r |S| \leq \epsilon_2 r |S|.$$
Thus (4) verifies (10) for $|S| \leq n/2$ provided we have $\epsilon_2 \leq \beta_0$. For $|S| > n/2$ we have $\Phi_1 \geq 1$ and then
\[
e_{G_1}(S, \tilde{S}) = e_{G_1}(\tilde{S}, S) \geq \Phi_1 |\tilde{S}| \geq |\tilde{A} \cap \tilde{S}| \geq |\tilde{A} \cap S| - \kappa + \alpha(S) = \xi(\tilde{S}) - \alpha(\tilde{S})
\]
and so Phase 1 succeeds with respect to $A, \tilde{A}$. The same argument applies to $B, \tilde{B}$. To ensure these paths are of length $O(\log n)$ we can solve a minimum cost maximum flow problem as indicated in Kleinberg and Rubinfeld [11].

5 Analysis of Phase 2

**Lemma 1** Throughout the algorithm
\[
|V_j| \geq (1 - \gamma_0)n, \quad j = 3, 4,
\]
where
\[
\gamma_0 = \frac{\beta_0 \gamma}{10}.
\]

**Proof:** First consider $V_3$. We know from (8) that $\Gamma_3$ is a $(\beta_0 r/2)$-expander throughout the execution of Phase 2. We can use the strong edge-expansion of $\Gamma_3$ to prove some vertex-expansion and conclude the diameter of $\Gamma_3$ is at most $\tau_1 = \lceil 2 \log_{1+\beta_0/2} n \rceil + 1$. Indeed, in a $\theta r$-expander, every set $S, |S| \leq n/2$, has at least $\theta |S|$ neighbours. Thus the total number of edges in the paths that are removed from $G_3$ is $\leq \kappa \tau_1$. Hence the vertices $B_3$ of $G_3$ which are incident with $\beta_0 r/4$ edges of these paths satisfy
\[
|B_3| \leq \frac{4 \kappa \tau_1}{\beta_0 r} \leq \frac{\gamma_0 n}{3}
\]
provided
\[
\epsilon_1 \leq \frac{\beta_0 \gamma_0}{25} \log \left( 1 + \frac{\beta_0}{2} \right).
\]

Let $X = \{x_1, x_2, \ldots, \}$ be the remaining vertices removed by REMOVE. We claim that if $|B_3| \leq \gamma_0 n$ then $|X| \leq 2 |B_3| \leq \frac{2 \gamma_0}{3} n$ implying that $|V_3| \geq (1 - \gamma_0) n$.

Indeed, if $X_i = \{x_1, x_2, \ldots, x_i\}$ then $X_i \cup B_3$ has $i + |B_3|$ vertices and contains at least $i \beta_0 r/4$ edges. The existence of $x_i, i = 2 |B_3|$ contradicts (1) with $S = X_i \cup B_3$. So,
\[
\text{in}(S) \geq |B_3| \beta_0 r/2 \geq |S| \beta_0 r/6 > |S| \alpha r/2
\]
using (6). This proves the lemma for $V_3$ and the argument for $V_4$ is identical. \(\Box\)

Our next task is to bound the size of the set $L$ of pairs of vertices which are left to Phase 3. For this we need to establish some facts about random walks on graphs.
5.1 Random Walks

A random walk on an undirected graph \( G = (V, F) \) is a Markov chain \( \{X_t\} \) on \( V \) associated with a particle that moves from vertex to vertex according to the following rule: The probability of a transition from vertex \( v \), of degree \( d_v \) to a vertex \( w \) is \( 1/(2d_v) \) if \( \{v, w\} \in E \) and 0 otherwise. The particle stays at \( v \) with probability \( 1/2 \). This removes the possibility of periodicity and allows us to use the conductance bound of Jerrum and Sinclair. Its stationary distribution, denoted by \( \pi \), is given by \( \pi(v) = \frac{d_v}{2|E|} \) for \( v \in V \).

Let \( P \) be the transition matrix of the associated Markov chain. Let \( \lambda \) be the second largest eigenvalue of \( P \). According to Jerrum and Sinclair [18]

\[
\lambda \leq 1 - \frac{\Psi^2}{2}
\]  

(12)

where \( \Psi \) denotes the conductance of a random walk on \( G \).

Here,

\[
\Psi = \min_{\pi(S) \leq 1/2} \frac{1}{\pi(S)} \sum_{w \in S} \pi(v)P(v, w)
\]

\[
\geq \min_{\pi(S) \leq 1/2} \frac{1}{\pi(S)} \sum_{w \in S} \frac{d_v}{\Delta|V|} \cdot \frac{1}{2d_v}
\]

\[
= \min_{\pi(S) \leq 1/2} \frac{\text{out}(S)}{2\Delta|V|\pi(S)}
\]

\[
\geq \frac{\Phi \delta}{2\Delta^2}.
\]  

(13)

Another fact we will need is

\[
|P^t(v, w) - \pi(w)| \leq \sqrt{\frac{\Delta}{\delta}} \lambda^t.
\]  

(14)

A proof of this can be found for example in [18].

Now consider our random walks. Arguing as in Lemma 1 we first note that since \( \kappa \omega = o(n) \) we will have \( |V_5| = n - o(n) \) throughout the algorithm.

The minimum and maximum degrees of \( \Gamma_5 \) will satisfy

\[
\beta_0 r / 2 \leq \delta \leq \Delta \leq r.
\]

Thus \( \Gamma_5 \) has at least \( (1 - o(1))\beta_0 r n / 4 \) edges and then for sufficiently large \( n \), the steady state for a random walk on \( \Gamma_5 \) will always satisfy

\[
\frac{\beta_0}{2n} \leq \pi(v) \leq \frac{2}{\beta_0(1 - o(1))n}
\]

for all \( v \in V_j \),

where \( \pi \) denotes the steady state distribution of a random walk on \( \Gamma_5 \).
From (8) and (13) we see that the conductance $\Psi$ satisfies

$$\Psi \geq \frac{\beta_0^2}{8}. \quad (15)$$

Applying (12) we see that the second eigenvalue $\lambda$ of a random walk on $\Gamma_5$ always satisfies

$$\lambda_5 \leq 1 - \frac{\beta_0^4}{64}.$$

Using this in (14) we obtain that

$$|P_{\Gamma_j}^{(t)}(u, v) - \pi_j(v)| \leq e^{-t\beta_0^4/64}. \quad (16)$$

So we see that if

$$\tau_0 = 256\beta_0^{-4}\ln n$$

then

$$|P_{\Gamma_j}^{(\tau_0)}(u, v) - \pi_j(v)| = O(n^{-4}). \quad (17)$$

We also need a large deviation result. This can be taken from the works of Dinwoodie [4], Gillman [9] and Kahale [10]. We quote the consequences of Theorem 2.1 of [9]: Let $q$ be the distribution of the start vertex of a random walk on a graph $G$. Let $S$ be a fixed set of vertices of $G$. Let $Y$ denote the number of visits to $S$ in the first $t$ steps.

$$\Pr(Y - t\pi(S) \leq -u) \leq \left(1 + \frac{(1 - \lambda)u}{10t}\right)N_q e^{-(1-\lambda)u^2/(20t)}, \quad (18)$$

where

$$N_q = \left(\sum_{v \in V} \frac{q(v)^2}{\pi(v)}\right)^{1/2}.$$

5.2 Analysis of CONNECTBACK

Fix $j = 3$ or $4$. Consider all calls to connect back a vertex to $V_j$. Let $L = L_5 \cup L_7$ where $L_\theta$ consists of the indices added to $L$ in Step $\theta$ of CONNECTBACK.

To probabilistically bound $|L_5|$ we first bound the expected value of the number $M_j, j = 3, 4$ of vertices which are incident with 5 or more walks in executions of Step 6 of CONNECTBACK which connecting back to $V_j$. Fix a $j = 3$ or $4$ and enumerate these walks as $W_1, W_2, \ldots, W_m, m \leq 2\kappa$. Here walk $W_i$ can have one or zero vertices if the proposed start vertex $z$ satisfies $z \in V_j$ or $z \notin V_j$. Then for $c = 9/\beta_0^2$,

$$\mathbb{E}(M_j) \leq \sum_{i_1, i_2, \ldots, i_6, v} \Pr(W_{i_t}, t = 1, \ldots, 5 \text{ go through } v)$$

$$\leq \left(\frac{2\kappa}{5}\right)n \left(\frac{c\omega}{n - o(n)}\right)^5 = O\left(\frac{\omega^5n}{(\log n)^{5\epsilon}}\right). \quad (19)$$
Explanation of (19) We first show that $\omega/(n - o(n))$ bounds the probability that walk $W_t = (w_1, w_2, \ldots, w_p)$ passes through $v$, given $W_t, 1 \leq s < t$ pass through $v$.

Suppose first that $j = 3$. Then, given $X = \{a_1, a_2, \ldots, a_{j-1}\}$, (or $X = \{x_1, x_2, \ldots, x_{j-1}\}$) $w_1 = a_{j_1}$ is chosen randomly from $V_3 \setminus X$ (or $w_1 = x_{j_1}$ is chosen randomly from $V$).

$$\Pr(x_1 = v) \leq \frac{1}{n - o(n)} \leq \frac{3}{\beta_0} \pi(v).$$

(20)

If $j = 4$ the argument is identical.

By induction on $t$ we get

$$\Pr(x_t = v) = \sum_{w \in V_6} \Pr(x_{t-1} = w)P(w, v) \leq \sum_{w \in V_6} \frac{3}{\beta_0} \pi(w)P(w, v) = \frac{3}{\beta_0} \pi(v) \leq \frac{9}{\beta_0^2 n}$$

and we have the claimed bound of $\frac{9\omega}{\beta_0^2 n}$ for the (conditional) probability that $W_t$ goes through $v$. There are at most $(\binom{n}{r})$ choices for $W_t, 1 \leq s < t$ and $n$ choices for $v$ and (19) follows.

It follows from (19) and the Markov inequality that

$$\Pr\left(M_j \geq \frac{n}{8(\log n)^4}\right) = o(1).$$

In addition to these $M = M_3 + M_4$ vertices we consider those vertices which are removed by \textsc{remove}( GAP). Arguing as in Lemma 1 we see that if $M = o(n)$ then $|L_5| \leq 3M$. We deduce that

$$\Pr\left(|L_5| \geq \frac{n}{2(\log n)^4}\right) = o(1).$$

(21)

We now estimate the probability that $i \in L_7$. We apply (18) with $G = \Gamma_5, S = V_j, j = 3$ or 4, and $q(v) = \frac{1+o(1)}{|V_5|}$ for $v \in V_5$. Then we have

$$1 - \lambda \geq \frac{\beta_0^4}{64^4}, N_q \leq \left(\frac{3}{\beta_0}\right)^{1/2}$$

and

$$\pi(S) \geq 1 - \frac{3\gamma_0}{\beta_0} \geq \frac{1}{2}.$$

Putting $t = \omega$ and $u = t\pi(S)$ we get

$$\Pr(W \cap V_j = \emptyset) = O(e^{-\beta_0^4 \omega/5120})$$

and so

$$\mathbb{E}\left(|L_7|\right) = O(e^{-\beta_0^4 \omega/5120 \kappa})$$

and

$$\Pr\left(|L_7| \geq \frac{n}{2(\log n)^4}\right) = o(1).$$

Combining this with (21) we see that $\Pr(|L| \geq n/(\log n)^4) = o(1)$. 

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6 Analysis of Phase 4

We join the pairs in $L$ using the algorithm of [5]. The algorithm is capable of joining $\Omega(n/(\log n)^{2+\alpha(1)})$ distinct pairs, provided the graph has sufficient edge-expansion. Notice that $\tilde{a}_i, \tilde{b}_i$ are chosen as distinct vertices. We briefly describe how we can make this algorithm route $m \leq \frac{n}{(\log n)^r}$ pairs using the graphs $G_6 - G_{10}$, assuming only that $\Phi_{e_6}, \ldots, \Phi_{e_{10}} \geq 1$. Let $\lambda = \lceil \log n \rceil$.

(a) The aim here is to choose $w_j, W_j, 1 \leq j \leq 2m$ such that (i) $w_j \in W_j$, (ii) $|W_j| = \lambda + 1$, (iii) the sets $W_j, 1 \leq j \leq 2m$ are pairwise disjoint and (iv) $W_j$ induces a connected subgraph of $G_8$.

As in [12] we can partition an arbitrary spanning tree $T$ of $G_8$. Since $T$ has maximum degree at most $r$ we can find $2m$ vertex disjoint subtrees $T_j, 1 \leq j \leq 2m$ of $T$, each containing between $\lambda + 1$ and $(r-1)\lambda + 2$ vertices. We can find $T_1$ as follows: choose an arbitrary root $\rho$ and let $Q_1, Q_2, \ldots, Q_r$ be the subtrees of $\rho$. If there exists $l$ such that $Q_l$ has between $\lambda + 1$ and $(r-1)\lambda + 2$ vertices then we take $T_1 = Q_l$. Otherwise we can search for $T_1$ in each $Q_l$ with more than $(r-1)\lambda + 2$ vertices. Since $T \setminus T_1$ is connected, we can choose all of the $T_j$’s in this way. Finally, $W_j$ is the vertex set of an arbitrary $\lambda + 1$ vertex subtree of $T_j$ and $w_j$ is an arbitrary member of $W_j$ for $j = 1, 2, \ldots, 2m$.

(b) Let $S_A, S_B$ denote the set of sources and sinks that need to be joined. Using a network flow algorithm in $G_6$ connect in an arbitrary manner the vertices of $S_A \cup S_B$ to $W = \{w_1, \ldots, w_{2m}\}$ by $2m$ edge disjoint paths. The expansion properties of $G_6$ ensure that such paths always exist.

Let $\tilde{a}_k$ (resp. $\tilde{b}_k$) denote the vertex in $W_i$ that was connected to the original end-point $a_k$ (resp. $b_k$). Our problem is now to find edge disjoint paths joining $\tilde{a}_k$ to $\tilde{b}_k$ for $1 \leq k \leq m$.

(c) If $w_i$ has been renamed as $\tilde{a}_k$ (resp. $\tilde{b}_k$) then rename the elements of $W_i$ as $\tilde{a}_{k,\ell}$ (resp. $\tilde{b}_{k,\ell}$), $1 \leq \ell \leq \lambda$. Choose $\xi_j, 1 \leq j \leq \lambda m$ and $\eta_j, 1 \leq j \leq \lambda m$ independently at random from the steady state distribution $\pi$ of a random walk on $G_{10}$. Using a network flow algorithm as in (b), connect $\{\tilde{a}_{k,\ell} : 1 \leq k \leq m, 1 \leq \ell \leq \lambda\}$ to $\{\xi_j : 1 \leq j \leq \lambda m\}$ by edge disjoint paths in $G_8$. Similarly, connect $\{\tilde{b}_{k,\ell} : 1 \leq k \leq m, 1 \leq \ell \leq \lambda\}$ to $\{\eta_j : 1 \leq j \leq \lambda m\}$ by edge disjoint paths in $G_9$. Rename the other endpoint of the path starting at $\tilde{a}_{k,\ell}$ (resp. $\tilde{b}_{k,\ell}$) as $\tilde{a}_{k,\ell}$ (resp. $\tilde{b}_{k,\ell}$). Once again the expansion properties of $G_8, G_9$ ensure that such paths always exist.

(d) Choose $\hat{x}_{k,\ell}, 1 \leq k \leq m, 1 \leq \ell \leq \lambda$ independently at random from the steady state distribution $\pi$ of a random walk on $G_{10}$. Let $\hat{W}'_{k,\ell}$ (resp. $\hat{W}''_{k,\ell}$) be a random walk of length $\theta \log n$ from $\hat{a}_{k,\ell}$ (resp. $\hat{b}_{k,\ell}$) to $\hat{x}_{k,\ell}$. Here $\theta$ is sufficiently large that a random walk of this length on $G_{10}$ is “well mixed”. The use of this intermediate vertex $\hat{x}_{k,\ell}$ helps to break some conditioning caused by the pairing up of the flow algorithm.

Let $B'_k$ (resp. $B''_k$) denote the bundle of walks $W'_{k,\ell}, 1 \leq \ell \leq \lambda$ (resp. $W''_{k,\ell}, 1 \leq \ell \leq \lambda$). Following [14] we say that $W'_{k,\ell}$ is bad if there exists $k' \neq k$ such that $W'_{k,\ell}$ shares an edge with a walk in a bundle $B'_{k'}$ or $B''_{k'}$. Each walk starts at an independently chosen vertex and moves to an independently chosen destination. The steady state of a random walk is uniform on edges and so at each stage of a walk, each edge is equally likely to be crossed.
Thus
\[ \Pr(W'_{k,\ell} \text{ is bad}) \leq \frac{2\lambda m \beta^2 (\log n)^2}{\beta_0 n} = O\left(\frac{1}{\log n}\right). \]

We say that index \( k \) is bad if either \( B_k \) or \( B''_k \) contain more than \( \lambda/3 \) bad walks. If index \( k \) is not bad then we can find a walk from \( \hat{a}_{k,\ell} \) to \( \hat{b}_{k,\ell} \) through \( \hat{x}_{k,\ell} \) for some \( \ell \) which is edge disjoint from all other walks. This gives a walk
\[ a_k - \hat{a}_k - \hat{a}_{k,\ell} - \hat{x}_{k,\ell} - \hat{b}_{k,\ell} - \hat{b}_k - b_k, \]
which is edge-disjoint from all other such walks.

The probability that index \( k \) is bad is at most
\[ 2 \Pr(B(\lambda, O(1/\log n)) \geq \lambda/3) = O(n^{-2}). \]

So with probability 1-o(1) there are no bad indices. \( \square \)

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**References**


