HAMILTON CYCLES IN RANDOM REGULAR DIGRAPHS

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March 14, 2005

Abstract

We prove that almost every r-regular digraph is Hamiltonian for all fixed $r \geq 3$.

1 Introduction

In two recent papers Robinson and Wormald [8],[9] solved one of the major open problems in the theory of random graphs. They proved

*Supported by NSF grant CCR-9024935

Theorem 1 *For every fixed $r \geq 3$ almost all r-regular graphs are hamiltonian.*

For earlier attempts at this question see Bollobás [2], Fenner and Frieze [5] and Frieze [6] who established the result for $r \geq r_0$.

In [8] (r=3) they used a clever variation on the second moment method and in [9] (for $r \geq 4$) they used this idea plus a sort of monotonicity argument.

In this paper we will study the directed version of the problem. Thus let $\Omega_{n,r} = \Omega$ denote the set of digraphs with vertex set $[n] = \{1, 2, \ldots, n\}$ such that each vertex has indegree and outdegree r. Let $D_{n,r} = D$ be chosen uniformly at random from $\Omega_{n,r}$. Then

Theorem 2

$$\lim_{n \to \infty} Pr(D \text{ is Hamiltonian}) = \begin{cases} 0 & r = 2 \\ 1 & r \geq 3 \end{cases}$$

The case $r = 2$ follows directly from the fact that the expected number of Hamilton cycles in $D_{n,2}$ tends to zero.

Our method of proof for $r \geq 3$ is quite different from [8], [9] although we will use the idea that for $r \geq 3$, a random r-regular bipartite graph is close in some probabilistic sense to a random $(r-1)$-regular bipartite graph plus a random matching.

Our strategy is close to that of Cooper and Frieze [4] who prove that almost every 3-in,3-out digraph is Hamiltonian.

2 Random digraphs and random bipartite graphs

Given $D_{n,r} = ([n], A)$ we can associate it with a bipartite graph $B = B_{n,r} = \phi(D_{n,r}) = ([n], [n], E)$ in a standard way. Here B contains an edge $\{x, y\}$ iff D contains the directed edge (x, y). The mapping ϕ is a bijection between r-regular digraphs and r-regular bipartite graphs and so B is uniform on the latter space, which we denote by $\Omega^B_{n,r}$.

For $r \geq 3$ we wish to replace $B_{n,r}$ by $B_{n,r-1}$ plus an independently chosen random perfect matching M of $[n]$ to $[n]$. This is equivalent to replacing D by $\Pi_0 \cup \hat{D}$ where Π_0 and $\hat{D}$ are independent and

(i) Π_0 is the digraph of a random permutation.

(ii) $\hat{D} = D_{n,r-1}$.

Of course Π is the union of vertex disjoint cycles. We call such a digraph a *permutation digraph*. Its cycle count is the number of cycles.

The arguments of [9] allow us to make the above replacement. A brief sketch of why this is so would certainly be in order.

Let X_M denote the number of perfect matchings in $B_{n,r}$. Arguments in [9] demonstrate the existence of $\epsilon(b) > 0$ such that for $b > 0$ fixed

$$\lim_{n\to\infty} Pr(X_M \geq \mathbf{E}(X_M)/b) \geq 1 - \epsilon(b)$$

where $\epsilon(b) \to 0$ as $b \to \infty$.

Now consider a bipartite graph $\mathcal{B} = (\Omega^B_{n,r-1}, \Omega^B_{n,r}, \mathcal{E})$. There is an edge from $G \in \Omega^B_{n,r-1}$ to $G' \in \Omega^B_{n,r}$ iff $G' = G \cup M$ where M is a perfect matching. Now choose (G, G') randomly from $\mathcal{E}$. Let A denote some event defined on $\Omega^B_{n,r}$ and $\hat{A} = \{(G, G') \in \mathcal{E} : G' \in A\}$. Then since the maximum and minimum degrees of the $\Omega^B_{n,r-1}$ vertices of $\mathcal{B}$ are asymptotically equal to $n!e^{-(r-1)}$ (Bender and Canfield [1])

$$Pr_0(\hat{A}) = (1 + o(1))Pr_1(A)$$

where $o(1)$ refers to $n \to \infty$, Pr_0 refers to the space $\mathcal{E}$ with the uniform measure and Pr_1 refers to (randomly chosen) $G = B_{n,r-1}$ plus a randomly chosen M, disjoint from $G' = B_{n,r-1}$.

On the other hand if Pr refers to $B_{n,r}$ then

$$\begin{aligned} Pr_0(\hat{A}) &= \sum_{G' \in A} \frac{X_M}{|\mathcal{E}|} \\ &= \sum_{G' \in A} \frac{X_M}{\mathbf{E}(X_M)|\Omega^B_{n,r}|} \\ &\geq (Pr(A) - \epsilon(b))/b. \end{aligned}$$

Thus

$$Pr(A) \leq \epsilon(b) + (b + o(1))Pr_1(A).$$

Thus if A is $\{\phi^{-1}(B_{n,r-1} \cup M)$ is non-Hamiltonian$\}$ (M disjoint from $B_{n,r-1}$ here) we can show that $Pr(A) \to 0$ (as $n \to \infty$) by proving that $Pr_1(A) \to 0$ (as $n \to \infty$), since b can be arbitrarily large.

Finally, if Pr_2 refers to $B_{n,r-1}$ plus a randomly chosen M (not necessarily disjoint from $B_{n,r-1}$) then $Pr_2(A) \to 0$ (as $n \to \infty$) implies $Pr_1(A) \to 0$ (as $n \to \infty$) since the probability that M is disjoint from $B_{n,r-1}$ in this case tends to the constant $e^{-(r-1)} > 0$.

We have thus reduced the proof of Theorem 2 to that of showing

$$\lim_{n\to\infty} Pr(\Pi_0 \cup \hat{D} \text{ is Hamiltonian}) = 1.$$

In fact we have only to prove the result for $r = 3$ and apply induction. Thus assume $r = 3$ from now on.

We will use a *two phase* method as outlined below.

Phase 0. Π_0 being a random permutation digraph it is almost always of cycle count at most $2\log n$, see for example Bollobás [3].

Phase 1. Using $\hat{D}$ we increase the minimum cycle size in the permutation digraph to at least $n_0 = \lceil \frac{100n}{\log n} \rceil$.

Phase 2. Using $\hat{D}$ we convert the *Phase 1* permutation digraph to a Hamilton cycle.

In what follows inequalities are only claimed to hold for n sufficiently large. The term **whp** is short for *with high probability* i.e. probability 1-o(1) as $n \to \infty$.

3 Phase 1. Removing small cycles

We partition the cycles of the permutation digraph Π_0 into sets SMALL and LARGE, containing cycles C of size $|C| < n_0$ and $|C| \geq n_0$ respectively. We define a Near Permutation Digraph (NPD) to be a digraph obtained from a permutation digraph by removing one edge. Thus an NPD Γ consists of a path $P(\Gamma)$ plus a permutation digraph $PD(\Gamma)$ which covers $[n] \setminus V(P(\Gamma))$.

We now give an informal description of a process which removes a small cycle C from a *current* permutation digraph Π. We start by choosing an (arbitrary) edge (v_0, u_0) of C and delete it to obtain an NPD Γ_0 with $P_0 = P(\Gamma_0) \in \mathcal{P}(u_0, v_0)$, where $\mathcal{P}(x, y)$ denotes the set of paths from x to y in D. The aim of the process is to produce a *large* set S of NPD's such that for each $\Gamma \in S$, (i) $P(\Gamma)$ has a least n_0 edges and (ii) the small cycles of $PD(\Gamma)$ are a subset of the small cycles of Π. We will show that **whp** the endpoints of one of the $P(\Gamma)$'s can be joined by an edge to create a permutation digraph with (at least) one less small cycle.

The basic step in an *Out-Phase* of this process is to take an NPD Γ with $P(\Gamma) \in \mathcal{P}(u_0, v)$ and to examine the edges of $\hat{D}$ leaving v. Let w be the terminal vertex of such an edge and assume that Γ contains an edge (x, w). Then $\Gamma' = \Gamma \cup \{(v, w)\} \setminus \{(x, w)\}$ is also an NPD. Γ' is acceptable if (i) $P(\Gamma')$ contains at least n_0 edges and (ii) any new cycle created (i.e. in Γ' and not Γ) also has at least n_0 edges.

If Γ contains no edge (x, w) then $w = u_0$. We accept the edge if $P(\Gamma)$ has at least n_0 edges. This would (prematurely) end an iteration, although it is unlikely to occur.

We do not want to look at very many edges of $\hat{D}$ in this construction and we build a tree T_0 of NPD's in a natural breadth-first fashion where each non-leaf vertex Γ gives rise to NPD children Γ' as described above. The construction of T_0 ends when we first have $\nu = \left\lceil \sqrt{n \log n} \right\rceil$ leaves. The construction of T_0 constitutes an Out-Phase of our procedure to eliminate small cycles. Having constructed T_0 we need to do a further *In-Phase*, which is similar to a set of Out-Phases.

Then **whp** we close at least one of the paths $P(\Gamma)$ to a cycle of length at least n_0. If $|C| \geq 2$ and this process fails then we try again with a different edge of C in place of (u_0, v_0).

We now increase the the formality of our description. We start Phase 2 with a permutation digraph Π_0 and a general iteration of Phase 2 starts with a permutation digraph Π whose small cycles are a subset of those in

Π_0. Iterations continue until there are no more small cycles. At the start of an iteration we choose some small cycle C of Π. There then follows an Out-Phase in which we construct a tree $T_0 = T_0(\Pi, C)$ of NPD's as follows: the root of T_0 is Γ_0 which is obtained by deleting an edge (v_0, u_0) of C.

We grow T_0 to a depth at most $\lceil 1.5 \log n \rceil$. The set of nodes at depth t is denoted by S_t.
Let $\Gamma \in S_t$ and $P = P(\Gamma) \in \mathcal{P}(u_0, v)$. The *potential* children Γ' of Γ, at depth $t+1$ are defined as follows.

Let w be the terminal vertex of an edge directed from v in $\hat{D}$.
Case 1. w is a vertex of a cycle $C' \in PD(\Gamma)$ with edge $(x, w) \in C'$. Let $\Gamma' = \Gamma \cup \{(v, w)\} \setminus \{(x, w)\}$.
Case 2. w is a vertex of $P(\Gamma)$. Either $w = u_0$, or (x, w) is an edge of P. In the former case $\Gamma \cup \{(v, w)\}$ is a permutation digraph Π' and in the latter case we let $\Gamma' = \Gamma \cup \{(v, w)\} \setminus \{(x, w)\}$.

In fact we only admit to S_{t+1} those Γ' which satisfy the following conditions.
C(i) The new cycle formed (Case 2 only) must have at least n_0 vertices, and the path formed must either be empty or have at least n_0 vertices. When the path formed is empty we close the iteration and if necessary start the next with Π'.

Now define W_+, W_- as follows: initially $W_+ = W_- = \emptyset$. A vertex x is added to W_+ whenever we learn any of its out-neighbours in $\hat{D}$ and to W_- whenever we learn any of its in-neighbours. $W = W_+ \cup W_-$. We never allow $|W|$ to exceed $n^{9/10}$.

The only information we learn about $\hat{D}$ is that certain specific arcs are present.

The property we need of the random graph $\hat{D}$ is that if $x \notin W_+$ and S is any set of vertices, disjoint from W, then

$$Pr(N_+(x) \cap S \neq \emptyset) = \left(1 - \left(1 - \frac{|S|}{n}\right)^2\right) \left(1 + O\left(\frac{1}{n^{1/10}}\right)\right).$$

These approximations are intended to hold conditional on any past history of the algorithm such that $|W| \leq n^{9/10}$. Furthermore, if $x \in W_+$ but only one neighbour y is known then, where $y \notin S$,

$$Pr(N_-(x) \cap S \neq \emptyset | y) = \frac{|S|}{n}\left(1 + O\left(\frac{1}{n^{1/10}}\right)\right).$$

Similar remarks are true for $N_-(x)$. Thus, since W remains small, $N_\pm(v)$ are usually (near) random pairs in $\bar{W}$.

C(ii) $x \notin W$.

An edge (v, w) which satisfies the above conditions is described as *acceptable.*

In order to remove any ambiguity, the vertices of S_t are examined in their order of construction.

Lemma 3 *Let $C \in SMALL$. Then*

$$Pr(\exists t < \lceil \log_{3/2} \nu \rceil \textit{ such that } |S_t| \geq \nu) = 1 - O((\log\log n/\log n)^2).$$

Proof. We assume we stop construction of T_0, in mid-phase if necessary, when $|S_t| = \nu$, and show inductively that **whp** $(\frac{3}{2})^t \leq |S_t| \leq 2^t$, for $t \geq 3$. Let t^* denote the value of t when we stop. Thus the overall contribution to $|W|$ from this part of the algorithm is at most $|SMALL| \times 2^{t^*+1} \leq n^{0.86}$.

In general, let X_t be the number of unacceptable edges found when constructing S_{t+1}, $(t = 1, 2, ..., t^*)$. The event of a particular edge (v, w) being unacceptable is stochastically dominated by a Bernouilli trial with probability of success $p < \log\log n/n$. (in general inequalities are only claimed for sufficiently large n). To see this observe that there is a probability of at most $201/\log n$ that in Case 2 we create a small cycle or a short path. There is an $O(n^{-1/10})$ probability that $x \in W$. Finally there is the probability that w lies in a small cycle. Now in a random permutation the expected number of vertices in cycles of size at most k is precisely k/n. Thus **whp** Π_0 contains at most $n \log\log n/(2\log n)$ vertices on small cycles and so given this, the probability that w lies on a small cycle is at most $\log\log n/(2\log n)$.
For $t \leq c$, constant, the probability of 2 or more unacceptable edges in layers $t \leq c$ is $O\left(\frac{2^{2c}(\log\log n)^2}{(\log n)^2}\right)$ and thus $|S_{t+1}| > 2|S_t| - 1 > (\frac{3}{2})^t$ for $3 \leq t \leq c$ with probability $1 - O((\log\log n/\log n)^2)$.
In order to see this, note that in the case where there is only one acceptable

edge at the first iteration, subsequent layers expand by a power of 2, and $|S_1| = 2$ otherwise.
For $t > c, c$ large, the expected number of unacceptable edges at iteration t is at most $\mu = 2p|S_t|$ and thus by standard bounds on tails of the Binomial distribution,

$$Pr\left(X_t > \lfloor |S_t|/2 \rfloor \,\big|\, |S_t| = s\right) \le \left(\frac{2e\log\log n}{\log n}\right)^{\lfloor s/2 \rfloor}.$$

This upper bound is easily good enough to complete the proof of the lemma.

□

Now T_0 has leaves Γ_i, for $i = 1, \ldots, \nu$, each with a path of length at least n_0, (unless we have already successfully made a cycle). We now execute an In-Phase. This involves the construction of trees $T_i, i = 1, 2, \ldots \nu$. Assume that $P(\Gamma_i) \in \mathcal{P}(u_0, v_i)$. We start with Γ_i and $\mathcal{D}_i$ and build T_i in a similar way to T_0 except that here all paths generated end with v_i. This is done as follows: if a current NPD Γ has $P(\Gamma) \in \mathcal{P}(u, v_i)$ then we consider adding an edge $(w, u) \in \hat{D}$ and deleting an edge $(w, x) \in \Gamma$ (as opposed to (x, w) in an Out-Phase). Thus our trees are grown by considering edges directed into the start vertex of each $P(\Gamma)$ rather than directed out of the end vertex. Some technical changes are necessary however.

We consider the construction of our ν trees in two iterations. First of all we grow the trees only enforcing condition C(ii) of success and thus allow the formation of small cycles. We try to grow them to depth $k = \lceil \log_{3/2} \nu \rceil$. We also consider the growth of the ν trees simultaneously. Let $T_{i,\ell}$ denote the set of start vertices of the paths associated with the nodes at depth ℓ of the i'th tree, $i = 1, 2 \ldots, \nu, \ell = 0, 1, \ldots, k$. Thus $T_{i,0} = \{u_0\}$ for all i. We prove inductively that $T_{i,\ell} = T_{1,\ell}$ for all i, ℓ. In fact if $T_{i,\ell} = T_{1,\ell}$ then the acceptable $\hat{D}$ edges have the same set of initial vertices and since all of the deleted edges are Π_0-edges (enforced by C(ii)) we have $T_{i,\ell+1} = T_{1,\ell+1}$.

The probability that we succeed in constructing ν trees $T_1, T_2, \ldots T_\nu$, say, is, by the analysis of Lemma 3, $1 - O((\log\log n/\log n)^2)$. Note that the number of nodes in each tree is at most $2^{k+1} \le n^{.87}$ and so the overall contribution to $|W|$ from this part of the algorithm is $O(n^{.87}\log n)$.

We now consider the fact that in some of the trees some of the leaves may have been constructed in violation of C(i). We imagine that we prune the trees $T_1, T_2, \dots T_\nu$ by disallowing any node that was constructed in violation of C(i). Let a tree be BAD if after pruning it has less than ν leaves. Now an individual pruned tree has essentially been constructed in the same manner as the tree T_0 obtained in the Out-Phase. (We have chosen k large enough so that we can obtain ν leaves at the slowest growth rate of 3/2 per node.) Thus

$$Pr(T_1 \text{ is BAD}) = O\left(\left(\frac{\log\log n}{\log n}\right)^2\right)$$

and

$$E(\text{number of BAD trees}) = O\left(\nu\left(\frac{\log\log n}{\log n}\right)^2\right)$$

and

$$Pr(\exists \geq \nu/2 \text{ BAD trees}) = O\left(\left(\frac{\log\log n}{\log n}\right)^2\right).$$

Thus

$$\begin{aligned} & Pr(\exists < \nu/2 \text{ GOOD trees after pruning}) \\ \leq\; & Pr(\text{failure to construct } T_1, T_2, \dots T_\nu) + Pr(\exists \geq \nu/2 \text{ BAD trees}) \\ =\; & O\left(\left(\tfrac{\log\log n}{\log n}\right)^2\right) \end{aligned}$$

Thus with probability 1-$O((\log\log n/\log n)^2)$ we end up with $\nu/2$ sets of ν paths, each of length at least $100n/\log n$ where the i'th set of paths have V_i say, as their set of start vertices and v_i as a final vertex. At this stage each $v_i \notin W_+$ and each $V_i \cap W_- = \emptyset$. Hence

$$\begin{aligned} Pr(\text{no } \Pi \text{ edge closes one of these paths}) &\leq \left(1 - \frac{2\nu}{n}\left(1 + O\left(\frac{1}{n^{1/10}}\right)\right)\right)^{\nu/2} \\ &= O(n^{-1}). \end{aligned}$$

Consequently the probability that we fail to eliminate a particular small cycle is
$O((\log\log n/\log n)^2)$ and we have

Lemma 4 *The probability that Phase 2 fails to produce a permutation digraph with minimal cycle length at least n_0 is $o(1)$.*

□

At this stage we have shown that $\Pi_0 \cup \hat{D}$ almost always contains a permutation digraph Π^* in which the minimum cycle size is at least n_0.

We shall refer to Π^* as the *Phase 1* permutation digraph.

4 Phase 2. Patching the Phase 1 permutation digraph to a Hamilton cycle

Let $C_1, C_2, \dots, C_k$ be the cycles of Π^*, and let $c_i = |C_i \setminus W|$, $c_1 \le c_2 \le \cdots \le c_k$, and $c_1 \ge n_0 - n^{3/4} \ge \frac{99 \log n}{n}$. If $k = 1$ we can skip this phase, otherwise let $a = \frac{n}{\log n}$. For each C_i we consider selecting a set of $m_i = 2\lfloor \frac{c_i}{a} \rfloor + 1$ vertices $v \in C_i \setminus W$, and deleting the edge (v, u) in Π^*. Let $m = \sum_{i=1}^{k} m_i$ and relabel (temporarily) the broken edges as $(v_i, u_i), i \in [m]$ as follows: in cycle C_i identify the lowest numbered vertex x_i which loses a cycle edge directed out of it. Put $v_1 = x_1$ and then go round C_1 defining $v_2, v_3, \dots v_{m_1}$ in order. Then let $v_{m_1+1} = x_2$ and so on. We thus have m path sections $P_j \in \mathcal{P}(u_{\phi(j)}, v_j)$ in Π^* for some permutation ϕ. We see that ϕ is an even permutation as all the cycles of ϕ are of odd length.

There will be a chance that we can rejoin these path sections of Π^* to make a Hamilton cycle using $\hat{D}$. Suppose we can. This defines a permutation ρ where $\rho(i) = j$ if P_i is joined to P_j by $(v_i, u_{\phi(j)})$, where $\rho \in H_m$ the set of cyclic permutations on $[m]$. We will use the second moment method to show that a suitable ρ exists **whp**. Unfortunately a technical problem forces a restriction on our choices for ρ.

Given ρ define $\lambda = \phi\rho$. In our analysis we will restrict our attention to $\rho \in R_\phi = \{\rho \in H_m : \phi\rho = \lambda, \lambda \in H_m\}$. If $\rho \in R_\phi$ then we have not only constructed a Hamilton cycle in $\Pi^* \cup \hat{D}$, but also in the *auxillary digraph* Λ, whose edges are $(i, \lambda(i))$.

Lemma 5 $(m-2)! \le |R_\phi| \le (m-1)!$

Proof. We grow a path $1, \lambda(1), \gamma^2(1), \dots, \gamma^k(1)$ in Λ, maintaining feasibility in the way we join the path sections of Π^* at the same time.

We note that the edge $(i, \lambda(i))$ of Λ corresponds in $\hat{D}$ to the edge $(v_i, u_{\phi\rho(i)})$. In choosing $\lambda(1)$ we must avoid not only 1 but also $\phi(1)$ since $\lambda(1) = 1$ implies $\rho(1) = 1$. Thus there are $m - 2$ choices for $\lambda(1)$ since $\phi(1) \neq 1$.

In general, having chosen $\lambda(1), \gamma^2(1), \ldots, \gamma^k(1), 1 \leq k \leq m-3$ our choice for $\gamma^{k+1}(1)$ is restricted to be different from these choices and also 1 and ℓ where u_ℓ is the initial vertex of the path terminating at $v_{\lambda^k(1)}$ made by joining path sections of Π^*. Thus there are either $m - (k+1)$ or $m - (k+2)$ choices for $\gamma^{k+1}(1)$ depending on whether or not $\ell = 1$.

Hence, when $k = m-3$, there *may* be only one choice for $\gamma^{m-2}(1)$, the vertex h say. After adding this edge, let the remaining isolated vertex of Λ be w. We now need to show that we can complete λ, ρ so that $\lambda, \rho \in H_m$.

Which vertices are missing edges in Λ at this stage ? Vertices $1, w$ are missing in-edges, and h, w out-edges. Hence the path sections of Π^* are joined so that either

$$u_1 \to v_h, \;\; u_w \to v_w \quad \text{or} \quad u_1 \to v_w, \;\; u_w \to v_h.$$

The first case can be (uniquely) feasibly completed in both Λ and D by setting $\lambda(h) = w, \lambda(w) = 1$. Completing the second case to a cycle in Π^* means that

$$\lambda = (1, \lambda(1), \ldots, \gamma^{m-2}(1))(w) \tag{1}$$

and thus $\lambda \notin H_m$. We show this case cannot arise.

$\lambda = \phi\rho$ and ϕ is even implies that λ and ρ have the same parity. On the other hand $\rho \in H_m$ has a different parity to λ in (1) which is a contradiction.

Thus there is a (unique) completion of the path in Λ. □

Let H stand for the union of the permutation digraph Π^* and $\hat{D}$. We finish our proof by proving

Lemma 6 *Pr(H does not contain a Hamilton cycle)* $= o(1)$.

Proof. Let X be the number of Hamilton cycles in H resulting from rearranging the path sections generated by ϕ according to those $\rho \in R_\phi$. We will use the inequality

$$Pr(X > 0) \geq \frac{E(X)^2}{E(X^2)}. \tag{2}$$

Here probabilities are now with respect to the $\hat{D}$ choices for edges incident with vertices not in W and on the choices of the m *cut* vertices.

Now the definition of the m_i yields that

$$\frac{2n}{a} - k \leq m \leq \frac{2n}{a} + k$$

and so

$$(1.99)\log n \leq m \leq (2.01)\log n.$$

Also

$$k \leq m/199, m_i \geq 199 \text{ and } \frac{c_i}{m_i} \geq \frac{a}{2.01}, \qquad 1 \leq i \leq k.$$

Let Ω denote the set of possible cycle re-arrangements. $\omega \in \Omega$ is a *success* if $\hat{D}$ contains the edges needed for the asssociated Hamilton cycle. Thus, where $\epsilon = O(1/n^{1/10})$,

$$\begin{aligned}
E(X) &= \sum_{\omega\in\Omega} Pr(\omega \text{ is a success}) \\
&= \sum_{\omega\in\Omega} \left(\frac{2}{n}(1+\epsilon)\right)^m \\
&\geq \left(\frac{2}{n}(1+\epsilon)\right)^m (m-2)! \prod_{i=1}^{k} \binom{c_i}{m_i} \\
&\geq \frac{1-o(1)}{m\sqrt{m}} \left(\frac{2m}{en}\right)^m \prod_{i=1}^{k} \left(\left(\frac{c_i e}{m_i^{1+(1/2m_i)}}\right)^{m_i} \left(\frac{\exp\{-m_i^2/2c_i\}}{\sqrt{2\pi}}\right)\right) \\
&\geq \frac{(1-o(1))(2\pi)^{-m/398}}{m\sqrt{m}} \left(\frac{2m}{en}\right)^m \prod_{i=1}^{k} \left(\frac{c_i e}{(1.02)m_i}\right)^{m_i} \\
&\geq \frac{(1-o(1))(2\pi)^{-m/398}}{m\sqrt{m}} \left(\frac{2m}{en}\right)^m \left(\frac{ea}{2.01 \times 1.02}\right)^m \\
&\geq \frac{(1-o(1))(2\pi)^{-m/398}}{m\sqrt{m}} \left(\frac{3.98}{2.0502}\right)^m \\
&\geq n^{1.3}.
\end{aligned} \tag{3}$$

Let M, M' be two sets of selected edges which have been deleted in J and whose path sections have been rearranged into Hamilton cycles according to ρ, ρ' respectively. Let N, N' be the corresponding sets of edges which have been added to make the Hamilton cycles. What is the interaction between these two Hamilton cycles?

Let $s = |M \cap M'|$ and $t = |N \cap N'|$. Now $t \leq s$ since if $(v, u) \in N \cap N'$ then there must be a unique $(\tilde{v}, u) \in M \cap M'$ which is the unique J-edge into u. We claim that $t = s$ implies $t = s = m$ and $(M, \rho) = (M', \rho')$. (This is why we have restricted our attention to $\rho \in R_\phi$.) Suppose then that $t = s$ and $(v_i, u_i) \in M \cap M'$. Now the edge $(v_i, u_{\gamma(i)}) \in N$ and since $t = s$ this edge must also be in N'. But this implies that $(v_{\gamma(i)}, u_{\gamma(i)}) \in M'$ and hence in $M \cap M'$. Repeating the argument we see that $(v_{\gamma^k(i)}, u_{\gamma^k(i)}) \in M \cap M'$ for all $k \geq 0$. But γ is cyclic and so our claim follows.

We adopt the following notation. Let $t = 0$ denote the event that no common edges occur, and (s, t) denote $|M \cap M'| = s$ and $|N \cap N'| = t$. So

$$\begin{aligned} E(X^2) &\leq E(X) + (1+\epsilon)^{2m} \sum_{\Omega} \left(\frac{2}{n}\right)^m \sum_{\substack{\Omega \\ t=0}} \left(\frac{2}{n}\right)^m \\ &\quad + (1+\epsilon)^{2m} \sum_{\Omega} \left(\frac{2}{n}\right)^m \sum_{s=2}^{m} \sum_{t=1}^{s-1} \sum_{\substack{\Omega \\ (s,t)}} \left(\frac{2}{n}\right)^{m-t} \\ &= E(X) + E_1 + E_2 \text{ say.} \end{aligned} \tag{4}$$

Clearly

$$E_1 \leq (1+\epsilon)^{2m} E(X)^2. \tag{5}$$

For given ρ, how many ρ' satisfy the condition (s, t)? Previously $|R_\phi| \geq (m-2)!$ and now $|R_\phi(s,t)| \leq (m-t-1)!$, (consider fixing t edges of Γ'). Thus

$$E_2 \leq (1+\epsilon)^{2m} E(X)^2 \sum_{s=2}^{m} \sum_{t=1}^{s-1} \binom{s}{t} \left[\sum_{\sigma_1+\cdots+\sigma_k=s} \prod_{i=1}^{k} \frac{\binom{m_i}{\sigma_i}\binom{c_i-m_i}{m_i-\sigma_i}}{\binom{c_i}{m_i}} \right] \frac{(m-t-1)!}{(m-2)!} \left(\frac{n}{2}\right)^t.$$

Now

$$\frac{\binom{c_i-m_i}{m_i-\sigma_i}}{\binom{c_i}{m_i}} \leq \frac{\binom{c_i}{m_i-\sigma_i}}{\binom{c_i}{m_i}}$$

$$\begin{aligned}
&\leq (1+o(1))\left(\frac{m_i}{c_i}\right)^{\sigma_i}\exp\left\{-\frac{\sigma_i(\sigma_i-1)}{2m_i}\right\}\\
&\leq (1+o(1))\left(\frac{2.01}{a}\right)^{\sigma_i}\exp\left\{-\frac{\sigma_i(\sigma_i-1)}{2m_i}\right\}
\end{aligned}$$

where the $o(1)$ term is $O((\log n)^3/n)$. Also

$$\sum_{i=1}^{k}\frac{\sigma_i^2}{2m_i}\geq\frac{s^2}{2m}\qquad\text{for } \sigma_1+\cdots\sigma_k=s,$$

$$\sum_{i=1}^{k}\frac{\sigma_i}{2m_i}\leq\frac{k}{2},$$

and

$$\sum_{\sigma_1+\cdots+\sigma_k=s}\prod_{i=1}^{k}\binom{m_i}{\sigma_i}=\binom{m}{s}.$$

Hence

$$\begin{aligned}
\frac{E_2}{E(X)^2} &\leq (1+o(1))e^{k/2}\sum_{s=2}^{m}\sum_{t=1}^{s-1}\binom{s}{t}\exp\left\{-\frac{s^2}{2m}\right\}\left(\frac{2.01}{a}\right)^{s}\binom{m}{s}\frac{(m-t-1)!}{(m-2)!}\left(\frac{n}{2}\right)^{t}\\
&\leq (1+o(1))n^{.01}\sum_{s=2}^{m}\sum_{t=1}^{s-1}\binom{s}{t}\exp\left\{-\frac{s^2}{2m}\right\}\left(\frac{2.01}{a}\right)^{s}\frac{m^{s-(t-1)}}{(s-1)!}\left(\frac{n}{2}\right)^{t}\\
&= (1+o(1))n^{.01}\sum_{s=2}^{m}\left(\frac{2.01}{a}\right)^{s}\frac{m^s}{s!}\exp\left\{-\frac{s^2}{2m}\right\}m\sum_{t=1}^{s-1}\binom{s}{t}\left(\frac{n}{2m}\right)^{t}\\
&\leq (1+o(1))\left(\frac{2m^3}{n^{.99}}\right)\sum_{s=2}^{m}\left(\frac{(2.01)n\exp\{-s/2m\}}{2a}\right)^{s}\frac{1}{s!}\\
&= o(1) \qquad (6)
\end{aligned}$$

To verify that the RHS of (6) is $o(1)$ we can split the summation into

$$S_1=\sum_{s=2}^{\lfloor m/4\rfloor}\left(\frac{(2.01)n\exp\{-s/2m\}}{2a}\right)^{s}\frac{1}{s!}$$

and

$$S_2=\sum_{s=\lfloor m/4\rfloor+1}^{m}\left(\frac{(2.01)n\exp\{-s/2m\}}{2a}\right)^{s}\frac{1}{s!}.$$

Ignoring the term $\exp\{-s/2m\}$ we see that

$$\begin{aligned} S_1 &\leq \sum_{s=2}^{\lfloor (.5025)\log n \rfloor} \frac{((1.005)\log n)^s}{s!} \\ &= o(n^{9/10}) \end{aligned}$$

since this latter sum is dominated by its last term.

Finally, using $\exp\{-s/2m\} < e^{-1/8}$ for $s > m/4$ we see that

$$S_2 \leq n^{(1.005)e^{-1/8}} < n^{9/10}.$$

The result follows from (2) to (6). □

Acknowledgement: we thank the referee for his/her comments.

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