An Algorithm for Finding Hamilton Cycles in Random Directed Graphs

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We describe a polynomial \((O(n^{1.5}))\) time algorithm DHAM for finding hamilton cycles in digraphs. For digraphs chosen uniformly at random from the set of digraphs with vertex set \{1, 2, ..., n\} and \(m = m(n)\) edges the limiting probability (as \(n \to \infty\)) that DHAM finds a hamilton cycle equals the limiting probability that the digraph is hamiltonian. Some applications to random “travelling salesman problems” are discussed. © 1988 Academic Press, Inc.

1. INTRODUCTION

Some of the main problems in the study of hamilton cycles in random (undirected) graphs have been solved in recent years. For example Komlós and Szemerédi [13] showed that if \(m = \frac{1}{2} n \log n + \frac{1}{2} n \log \log n + c_n n\) and \(G_n, m\) denotes the random graph sampled uniformly from the set of graphs with vertex set \(V_n = \{1, 2, ..., n\}\) and \(m\) edges, then

\[
\lim_{n \to \infty} \Pr(G_n, m \text{ is hamiltonian}) = \lim_{n \to \infty} \Pr(\delta(G_n, m) \geq 2) = \begin{cases} 
0 & \text{if } c_n \to -\infty \\
e^{-e^{-2c}} & \text{if } c_n \to c \\
1 & \text{if } c_n \to \infty,
\end{cases} (1.1)
\]

where \(\delta(G)\) is the minimum degree of graph \(G\).

Bollobás [3] strengthened this result in the following way: let \(e_1, e_2, \ldots, e_N\), where \(N = \binom{n}{2}\) be a random permutation of the edges of the complete graph \(K_n\). Let \(G_m = (V_n, (e_1, e_2, \ldots, e_m))\) and \(m^* = \min(m: \delta(G_m) \geq 2)\).
Bollobas showed that
\[
\lim_{n \to \infty} \Pr(G_{n^*} \text{ is hamiltonian}) = 1. \quad (1.2)
\]
Subsequently Bollobas, Fenner, and Frieze [5] described an algorithm HAM which can be implemented in \(O(n^{3+\varepsilon})\) time and satisfies
\[
\lim_{n \to \infty} \Pr(\text{HAM finds a hamilton cycle in } G_{n,m}) = \lim_{n \to \infty} \Pr(G_{n,m} \text{ is hamiltonian}). \quad (1.3)
\]
There are now a large number of related results, many of which can be found in Bollobas [4, Chap. VIII].

When we come to digraphs we find that previous research has left considerable gaps in our knowledge. The main result of this paper provides the analogs of (1.1) and (1.3) for digraphs. Let \(D_{n,m}\) denote the digraph sampled uniformly from the set of digraphs with vertex set \(V_n\) and \(m\) edges. Suppose now \(m = n \log n + c_n n\).

Let \(\delta^+(D), \delta^-(D)\) denote the minimum outdegree and indegree of a digraph \(D\). We prove

**Theorem 1.** There is a (randomised) polynomial \(O(n^{1.5})\) time algorithm \(DHAM\) which satisfies
\[
\lim_{n \to \infty} \Pr(DHAM \text{ finds a hamilton cycle in } D_{n,m}) = \lim_{n \to \infty} \Pr(\min(\delta^+(D_{n,m}), \delta^-(D_{n,m})) \geq 1)
\]
\[
= \begin{cases}
0 & \text{if } c_n \to -\infty \\
e^{-2e^{-c}} & \text{if } c_n \to c \\
1 & \text{if } c_n \to \infty.
\end{cases}
\]

The previous best existence result is due to McDiarmid [14] who gave a nonconstructive proof that, in the notation of Theorem 1, if \(c_n \to \log \log n\) then
\[
\lim_{n \to \infty} \Pr(D_{n,m} \text{ is hamiltonian}) = 1.
\]
He proved this by showing that for any \(0 \leq p \leq 1,\)
\[
\Pr(D_{n,p} \text{ is hamiltonian}) \geq \Pr(G_{n,p} \text{ is hamiltonian}),
\]
where \(D_{n,p}\) (resp. \(G_{n,p}\)) is the random digraph (resp. graph) with vertex set \(V_n\) in which each of the \(2N\) (resp. \(N\)) possible edges occur independently with probability \(p\).
The best previous result concerning polynomial time algorithms is due to Angluin and Valiant [2] who described an \( O(n \log n) \) time algorithm \( A \) and showed

\[
\lim_{n \to \infty} \Pr(A \text{ finds a hamilton cycle in } D_{n,p}) = 1,
\]

assuming \( p = c \log n/n \) and \( c \) is a sufficiently large constant.

Instead of proving Theorem 1 directly we shall instead prove a stronger result, the directed constructive equivalent of (1.2), from which the theorem follows.

Let \( e_1, e_2, \ldots, e_{n(n-1)} \) be a random permutation of the edges of the complete digraph \( DK_n \) with vertex set \( V_n \). Let \( E_m = \{e_1, e_2, \ldots, e_m\} \) for \( 1 \leq m \leq n(n-1) \) and \( D_m = (V_n, E_m) \).

THEOREM 2. Let \( m^* = \min\{m: \delta^+(D_m) \geq 1, \delta^-(D_m) \geq 1\} \). Then

\[
\lim_{n \to \infty} \Pr(DHAM \text{ finds a hamilton cycle in } D_{m^*}) = 1.
\]

(Note that \( \lim_{n \to \infty} \Pr(HAM \text{ finds a hamilton cycle in } G_{m^*}) = 1 \) was proved in [5].) The proof generalises easily to the case where we want \( k \) edge disjoint cycles.

THEOREM 3. Let \( k \geq 1 \) be an integer constant and let \( m^*_k = \min\{m: \delta^+(D_m) \geq k, \delta^-(D_m) \geq k\} \). Then there exists an \( O(n^{1.5}) \) time algorithm \( DHAM_k \) satisfying

\[
\lim_{n \to \infty} \Pr(DHAM_k \text{ finds } k \text{ edge disjoint hamilton cycles in } D_{m^*_k}) = 1.
\]

We will also consider the use of \( DHAM \) in the exact solution of random travelling salesman problems. The first application is to the Bottleneck Travelling Salesman Problem (BTSP). An instance of BTSP is specified by the assignment of a numerical weight to the edges of \( DK_n \). The objective is to find a hamilton circuit for which the maximum edge-weight is minimised.

Let us assume that edge-weights are drawn independently from the uniform \([0, 1]\) distribution. As a simple corollary of Theorem 2 we prove

THEOREM 4. There is a polynomial \( (O(n^2)) \) time algorithm \( DBOT \) satisfying

\[
\lim_{n \to \infty} \Pr(DBOT \text{ solves BTSP exactly}) = 1.
\]

We can use ideas from this paper and Frieze [9] to tackle the more usual travelling salesman problem (DTSP) in which we seek the hamilton circuit of minimum total weight. We change the distribution so that the edges of
$DK_n$ are independently given integer weights from \{0, 1, \ldots, B(n) - 1\}. We can then prove

**Theorem 5.** If $B(n) = O(n/\log n)$ then there is a polynomial ($O(n^2)$) time algorithm DTSPSOLVE satisfying

$$\lim_{n \to \infty} \Pr(\text{DTSPSOLVE solves DTSP exactly}) = 1.$$ 

(In order not to make the paper too long we will only give an outline proof of this result, but we hope it will suffice to convince the reader.)

2. SOME NOTATION AND PRELIMINARIES

For convenience we give here some definitions and basic results that are used throughout the paper.

$E(X)$ denotes the edge set of $X$, where $X$ can be a graph, digraph, or cycle and $V(X)$ denotes its vertex set.

$B(a, p)$ denotes the binomial random variable with parameters $a$ and $p$ and

$$BS(b, c; a, p) = \Pr(b \leq B(a, p) \leq c).$$

The following inequalities for the tails of the binomial distribution are invaluable and can be derived, for example, from Theorem 1 of Hoeffding [10]:

$$BS(0, (1 - \epsilon)ap; a, p) \leq e^{-\epsilon^2 ap/2}, \quad 0 < \epsilon < 1.$$  \hspace{1cm} (2.1a)

$$BS((1 + \epsilon)ap, \infty; a, p) \leq e^{-\epsilon^2 ap/3}, \quad 0 < \epsilon < 1.$$  \hspace{1cm} (2.1b)

Let $Q_n$ be a sequence of events. We say that $Q_n$ occurs almost always (a.a.) if $\lim_{n \to \infty} \Pr(Q_n) = 1$. It is also useful to allow a.a. to stand for "almost all", with the obvious meaning.

If $m = an \log n$ for some constant $\alpha > 0$ and $p = m/n(n - 1)$ then it is convenient to derive properties of $D_{n, m}$ from $D_{n, p}$. For a property $Q$

$$\Pr(D_{n, p} \text{ has } Q) = \sum_{m=0}^{n(n-1)} \Pr(D_{n, m} \text{ has } Q) \Pr(|E(D_{n, p})| = m') \hspace{1cm} (2.2)$$

as $D_{n, p}$ has $m'$ edges is distributed as $D_{n, m'}$. We deduce from (2.2) that

$$\Pr(D_{n, m} \text{ has } Q) \leq (1 + o(1))\sqrt{2\pi m} \log n \Pr(D_{n, p} \text{ has } Q) \hspace{1cm} (2.3)$$
and that if \( \{Q_n\} \) are monotone properties, i.e., are preserved by adding edges or are preserved by deleting edges then

\[
D_{n,p} \text{ has } Q_n \text{ a.a. implies } D_{n,m} \text{ has } Q_n \text{ a.a.}
\]  

(2.4)

The following quantities are defined later as well but are given here for quick reference:

\[
m_0 = \lceil n \log n - n \log \log \log n \rceil, \quad m_1 = \lceil n \log n + n \log \log \log n \rceil, \quad m_2 = \lceil 5n \log n / 6 \rceil, \quad \text{and} \quad m_3 = \lceil 2n \log n / 3 \rceil.
\]

\[
p_i = m_i / n(n - 1) \quad \text{for } i = 0, 1, 2, 3.
\]

\[
l_o = \left\lfloor \frac{3 \log n}{\log \log n} \right\rfloor, \quad \mu_1 = \left\lfloor \frac{(\log n)^2}{\log \log \log n} \right\rfloor, \quad \text{and} \quad \mu_2 = \left\lfloor \frac{\log n}{\log \log \log \log n} \right\rfloor.
\]

3. Overview of the Algorithm

We assume that the input to DHAM consists of the edges \( ED_n \) in random order \( e_1, e_2, \ldots, e_{n(n-1)} \). However, only the edges \( e_1, e_2, \ldots, e_{m^*} \) are available to DHAM.

In the undirected case one tries to extend a current path and use rotations (see Fig. 1) to create extra paths. This approach fails in digraphs, for obvious reasons.

The ideas behind DHAM derive from the patching algorithms of Karp [11], Karp and Steele [12], and, more recently, Dyer and Frieze [6], for finding approximate solutions to travelling salesman problems.

The algorithm is split into 3 phases. In Phase 1 we construct a small set of edges \( E^1 \subseteq E_{n^*} \) such that if \( D^1 = (V_n, E^1) \) then \( \delta^+(D^1) \geq 1 \) and \( \delta^-(D^1) \geq 1 \) and \( D^1 \) a.a. contains a set of about \( \log n \) vertex disjoint cycles covering \( V_n \).

In Phase 2 we try to "patch" the cycles together in pairs by 2-edge exchanges (see Fig. 2). We will see that at the end of this phase there is a.a. a cycle \( C \) of size \( n - o(n) \) plus a few \( O(\log n) \) others.

![Figure 1](image-url)
In the final phase we try to patch the small cycles one by one into $\tilde{C}$ by a more complex process of "double-rotations" (see Fig. 3).

4. DETAILS OF PHASE 1

PHASE 1 OF DHAM.

begin

$\sigma :=$ a random permutation of $V_n$; \{used to avoid bias in the set of cycles produced (by Phase 1 \}

if $e = (v, w)$ let $\hat{e}$ denote $(v, \sigma(w))$;

$\hat{E}^1_- := \hat{E}^1_+ := \emptyset$;

{construct a set $\hat{E}^1_+$ of about $10n$ edges for which $\delta^+(\hat{E}^1_+) \geq 1$ and the average out-degree is $\approx 10$}

$A$ for $i = 1$ to $m^*$ do

begin

if $d^+(v_i, \hat{E}^1_+) \leq 9$ then $\hat{E}^1_+ := \hat{E}^1_+ \cup \{\hat{e}_i\}$;

{where $e_i = (v_i, w_i)$ and $d^+(v, E)$ (resp. $d^-(v, E)$) is the number of edges in $E$ with tail (resp. head) $v$.}

end;

{construct a set $\hat{E}^1_-$ of about $10n$ edges, disjoint from $\hat{E}^1_+$, for which $\delta^-(\hat{E}^1_-) \geq 1$ and the average indegree is $\approx 10$}
B: for \( i = 1 \) to \( m^* \) do 
begin 
if \( e_i \not\in \hat{E}^{1+} \) and \( d^{-}(\sigma(w_i), \hat{E}^{1-}) \leq 9 \) then \( \hat{E}^{1-} := \hat{E}^{1-} \cup \{ \hat{e}_i \} \)
end
\( \hat{E} := \hat{E}^{1+} \cup \hat{E}^{1-} ; \)
BIP := bipartite graph \((V_n, W_n, A)\) where \( W_n \) is a disjoint copy of \( V_n \) and \( \{ v, w \} \in A \) iff \((v, w)\) or \((w, v)\) \(\in \hat{E}\);

\{BIP is close to the model of Walkup [15] where each vertex randomly chooses 10 neighbours. 10 is sufficiently large so that it is easy to show it a.a. contains a perfect matching, 2 in fact should suffice.\}

apply the algorithm of Even and Tarjan [8] to find a maximum matching \( M \) of BIP;
{This algorithm is not "aware" that the edges of \( E_{m^*} \) have been relabelled.}
if \( M \) is not a perfect matching then DHAM has failed else
begin
define the permutation \( \psi \) of \( V_n \) by \( M = \{(v, \psi(v)): v \in V_n\} \);
\( \sigma := \sigma^{-1} \phi ; \)
output \( F = \{(v, \phi(v)): v \in V_n\} \)
{\( F \) defines a set of vertex disjoint cycles covering \( V_n \)}
end
end

Before proving that Phase 1 is a.a. successful we discuss some preliminary results. Let \( m_0 = [n \log n - n \log \log \log n] \) and \( m_1 = [n \log n + n \log \log \log n] \). It is straightforward to show
\[
\delta^+(D_{m_0}) = \delta^-(D_{m_0}) = 0 \quad \text{and} \quad \delta^+(D_{m_1}) = \delta^-(D_{m_1}) = 1
\]
a.a. (Erdos and Renyi [7])
and hence
\[
m_0 \leq m^* \leq m_1 \quad \text{a.a.}
\]
Next let
\[
\text{TINY} = \left\{ v \in V_n: d^+(v, E_{m_0}) \leq l_0 = \left\lceil \frac{3 \log n}{\log \log n} \right\rceil \text{ or } \quad d^-(v, E_{m_0}) \leq l_0 \right\}.
\]

**Lemma 4.1.**

\[
v, w \in \text{TINY} \implies \text{dist}(v, w; GD_{m^*}) \geq 5 \quad \text{a.a.} \quad (4.1)
\]
where if \( D \) is a digraph then \( GD \) is the graph obtained by ignoring edge orientation and parallel edges in \( D \) and \( \text{dist}(v, w; G) \) is the distance from vertex \( v \) to vertex \( w \) in graph \( G \).
Proof.

\[ \Delta(GD_{n,m_0}) \leq 6 \log n \quad \text{a.a. (actually with probability } 1 - o(n^{-1})) \]

(4.2)

where \( \Delta \) denotes maximum degree. (The expected number of vertices of degree exceeding \( 6 \log n \) in \( GD_{n,p_0} \) is at most \( n \) BS \( (6 \log n, n-1; n-1, 2p_0) = o(n^{-2}) \); now apply (2.3).)

Here and throughout the paper we do not aim for best possible interim results, only for ease of calculation:

\[ |\text{TINY}| = n^{o(1)} \quad \text{a.a.} \quad (4.3) \]

(The expected size of TINY in \( GD_{n,p_0} \) is \( n \) BS \( (0, 1; n-1, p_0) = n^{o(1)} \). The markov inequality \( \Pr(X > a) \leq E(X)/a \) for a nonnegative random variable \( X \), \( E \equiv \text{Expectation here, naturally} \) implies that \( |\text{TINY}| = n^{o(1)} \) a.a.; apply (2.4.).)

We note next that

\[ \Pr(\exists \text{ a path of length } \leq 4 \text{ in } CD_{n,p_0} \text{ with distinct endpoints } u, w \in \text{TINY}) \]

\[ \leq \binom{n}{2} \sum_{k=0}^{3} \binom{n-2}{k} k! p_0^{k+1} \text{BS}(0, l_0; n-5, 2p_0)^2 = O(n^{-1-o(1)}) \]

and so, by (2.3), (4.1) holds in \( GD_{n,m_0} \). Now \( GD_{m_1} \) is obtained from \( GD_{m_0} \) by adding \( 2n \log \log \log n \) (random) edges. Assuming (4.1) holds in \( GD_{n,m_0} \) and \( \Delta(GD_{n,m_0}) \leq 6 \log n \) and \( |\text{TINY}| = n^{o(1)} \) we find the probability that any of these extra edges is incident with two vertices, both at distance 3 or less from TINY is \( O(n^{-1-o(1)})(\log n)^6 n \log \log n / n^2) = o(1) \) and the result follows. \( \square \)

**Lemma 4.2.** The following hold a.a.:

(a) Phase 1 succeeds.

(b) \( \phi \) has at most \( 2 \log n \) cycles.

**Proof.** (a) Let us first change the problem slightly so that loops are allowed and return later to the loopless case. We now assume that \( e_1, e_2, \ldots, e_n \) is a random permutation of \( V_n \times V_n \). We can apply Phase 1 to this sequence as we did above.

**Remark 1.** The distribution of the sequence \( (v_i, \sigma(w_i)) \) is independent of \( \sigma \). (This is not the case if no loops are allowed in the original sequence.)

We must show that BIP a.a. contains a perfect matching. Let us first consider the distribution of \( \hat{E}^1 \): In the construction of \( \hat{E}^1 \) each \( v \in V_n \) independently chooses \( a(v) = \min\{10, d^+(v)\} \) out-neighbours at random
from $V_n$, where $d^+(v)$ (resp. $d^-(v)$) is the out-degree (resp. in-degree) of $v$ in $D_n$. For $w \in V_n$ let $\Gamma^-(w) = \{ v \in V_n; (v, w) \in \hat{E}^{1+}\}$. Given $\hat{E}^{1+}, \hat{E}^{1-}$ has the following distribution: each $w \in V_n$ independently chooses $b(v) = \min(10, d^-(w) - |\Gamma^-(w)|)$ in-neighbours from $V_n - \Gamma^-(w) - \{ v; d^+(v) < 10\}$.

Suppose now that BIP does not contain a perfect matching. Then by Hall’s theorem there exists $K \subseteq V_n$, $|K| = k$, and $L \subseteq W_n$, $|L| = k - 1$ such that $N(K, BIP) \subseteq L$ where $N(K, BIP) = \{ y; \exists x \in K \text{ such that } \{ x, y \} \in E(BIP) \}$. Now $k \neq 1$ or $n$ since $\delta(BIP) \geq 1$ by construction. Let $K' = K \cap TINY$ and $k' = |K'|$. Then (4.1) implies (i) $|L'| \geq k'$, where $L' = N(K', BIP)$ and (ii) each $v \in K - K'$ chooses at least 9 out-neighbours in $L - L'$ in the construction of $\hat{E}^{1+}$. But the probability of the occurrence of (ii) with $k \leq n/2$ is at most

\[ \sum_{t=9}^{\lfloor n/2 \rfloor} \binom{n}{t} \binom{n}{t-1} \left( \frac{10}{n-1} \right)^{t} \quad (t \equiv k - k') \]

\[ \leq \sum_{t=9}^{\lfloor n/2 \rfloor} \left( \frac{n^2 e^2 10r^9}{t^2 n^9} \right)^t \]

\[ = O(n^{-63}). \quad (4.4) \]

The case $k > n/2$ is equivalent to the existence of a $K \subseteq W_n$, $|K| \leq n/2$ in which $|N(K, B)| < |K|$. We will show that

\[ |\Gamma^-(w)| \leq \frac{2 \log n}{\log \log n} \quad \text{for all } w \in W_n \text{ a.a.} \quad (4.5) \]

We can now argue as before, given (4.5), that this implies each $w \in K - TINY$ chooses at least 9 in-neighbours from $L - L'$ in the construction of $\hat{E}^{1-}$, where $L, L'$ are defined analogously to the previous case. The probability of this is $O(n^{-63})$ as before.

To prove (4.5) we note that $|\Gamma^-(w)|$ is dominated by $B(10n, 1/n)$ and then a simple calculation yields (4.5).

(b) Let $\phi^*$ be some fixed permutation of $V_n$. Then

\[ \Pr(\text{Phase 1 outputs } \phi^*) = \sum_{\sigma} \Pr(\psi = \sigma \phi^* | \sigma = \sigma^*) \Pr(\sigma = \sigma^*) \]

\[ = \frac{1}{n!} \sum_{\sigma} \Pr(\psi = \sigma \phi^* | \sigma = \sigma^*) \]

But Remark 1 implies $\Pr(\psi = \psi^* | \sigma = \sigma_1) = \Pr(\psi = \psi^* | \sigma = \sigma_2)$ for any permutations $\psi^*, \sigma_1, \sigma_2$. Thus each $\phi^*$ is output with the same probability.
Now it is well known (for example, [3, p. 363]) that a random permutation of $V_n$ a.a. has approximately $\log n$ cycles. We conclude that (when loops are present) Phase 1 ends a.a. with a set of $r_1 \leq 2 \log n$ (say) vertex disjoint cycles covering $V_n$. Suppose now that we only consider loopless sequences of edges. This is the same as proceeding as above except that the edges $e_i$, $i \leq m^*$, cannot be used. We show that

$$\Pr(\text{no loop in } e_1, e_2, \ldots, e_{m^*} \text{ is used in the construction of BIP}) \geq (1 - o(1)) e^{-20}$$

(4.6)

and then we can infer the result that we actually want.

We have to estimate the probability that no edge $(i, \sigma(i))$ is chosen. But referring to the discussion of the distribution of $\mathbf{E}^1$ we see that, using (4.5), this probability is at least

$$\left(1 - \frac{10}{n}\right)^n \left(1 - \frac{10}{n - \frac{2 \log n}{\log \log n}}\right)^n + o(1).$$

and (4.6) follows. $\square$

5. DETAILS OF PHASE 2

We inherit from Phase 1 the permutation $\phi$ plus the associated set of edges $\mathcal{F}$, which we now denote by $\phi_1, F_1$, respectively. Let $m_2 = \lfloor 5n \log n/6 \rfloor$.

**PHASE 2 OF DHAM.**

*begin*

$\phi_2 := \phi_1$; $F_2 := F_1$;

A: for $i = 1$ to $m_2$ do

begin

if $e_i = (x, y)$ and $x, y$ are in different cycles of $F_2$ and $e = (\phi_2^{-1}(y), \phi_2(x)) = (z, w) \in E_{m_2}$ then

begin

$F_2 := (F_2 \cup \{e_i, e\}) \setminus \{(x, w), (z, y)\}$;

$\phi_2(x); = y; \phi_2(z); = w$;

go to A

end

end

(if no patches are made we arrive here)

end.
Let $C_1, C_2, \ldots, C_r$ denote the cycles defined by $F_2, \phi_2$, at the end of Phase 2, where $|C_1| \geq |C_2| \geq \cdots \geq |C_r|$. We know that $r_2 \leq 2 \log n$ a.a. and the aim now is to show that $|C_1| = n - o(n)$ a.a.

Let now $m_3 = \lceil 2n \log n / 3 \rceil$ and $\text{LARGE} = \{v \in V_n; d^+(v, E_{m_3}) \geq l_0$ and $d^-(v, E_{m_3}) \geq l_0\}$.

**Lemma 5.1.** $|\text{LARGE}| \geq n - n^{\alpha}$ a.a.

**Proof.** Consider $D_{n, p_3}$,

$$E(\left| V_n - \text{LARGE} \right| \cap D_{n, p_3}) \leq 2n BS(0, l_0, n - 1, p_3) - O(n^{3/4 + o(1)}).$$

Hence $|V_n - \text{LARGE}| \leq n^{1/2}$ a.a. in $D_{n, p_3}$. Now apply (2.4). □

The edges in the set $(E_{m_0} - E_{m_3}) \cap \text{LARGE}$ (where $\text{LARGE} = \text{LARGE} \times \text{LARGE} - \{(v, v); v \in \text{LARGE}\}$) are only slightly conditioned by the construction of $E^1$, assuming that (4.5) holds. In this case we have, given $E^1$,

- if $m_2 < i \leq m_0$ and $e_i = (v, w)$ where $v, w \in \text{LARGE}$ then $e_i$ is equally likely to be any edge in $\text{LARGE}$ which has not (5.1) occurred previously.

Now comes a "preparatory" lemma.

**Lemma 5.2.** The following holds a.a.:

$$S \subseteq V_n, s_0 = \left\lceil n/\sqrt{\log n} \right\rceil \leq |S| \leq n - s_0 \text{ implies } (5.2)$$

$$\left|\{(v, w) \in E_{m_3}; v \in S, w \not\in S\}\right| \geq |S|(n - |S|) \log n / 2n.$$

**Proof.** We consider the corresponding result for $D_{n, p_3}$. If $S$ is fixed and $s = |S|$ then the number of edges from $S$ to $V_n - S$ in $D_{n, p_3}$ is distributed as $B(s(n - s), p_3)$. Now $\Pr(B(s(n - s), p_3) \leq 3s(n - s)p_3/4) \leq e^{-s(n - s)p_3/32}$ on using (2.1a). Applying (2.3) we find $\Pr((5.2) \text{ fails}) \leq 3\sqrt{n \log n} 2n e^{-s_0(n - s_0)p_3/32} = o(1)$. □

We can now prove

**Lemma 5.3.** $|C_1| \geq n_0 = \lceil n - \sqrt{n} / \log n \rceil$ a.a.

**Proof.** Throughout the lemma we condition on the value of $E^1 \cup E_{m_3}$. We assume this is such that $r_1 \leq 2 \log n$, $|\text{LARGE}| \geq n - n^{1/2}$, (4.5) and (5.2) hold, and where each $v \in V_n$ is incident with at most $6 \log n$ edges in $E^1 \cup E_{m_3}$. These conditions have been shown to hold a.a. We shall also assume that $|E^2| \geq n \log n / 12$, where $E^2 = \{e_i \in \text{LARGE}; m_3 < i \leq m_2\}$ (the probability that $e_i \in \text{LARGE}$ for $m_3 < i \leq m_2$ is $1 - o(1)$
Regardless of $e_{m+1}, \ldots, e_{i-1}$ and given our assumptions). None of our assumption affects (5.1), which we invoke to do our calculations. Partition $E^2$ into sets $E^2(k), k = 1, 2, \ldots, r_2$ of roughly equal size, at least $n/24$ by assumption, so that if $e_i \in E^2(k)$ and $e_j \in E^2(k+1)$ then $i < j$. Let $a_k = \min \{i: e_i \in E^2(k)\}$.

Let $A_k$ denote the event \{closeness\} when $i$ first reaches $a_k$ in loop A of Phase 2 and no patch is made for $i = a_k, a_k + 1, \ldots, a_k + r-1$. $C_1$ refers to the “current” largest cycle. We show

$$\Pr(A_k) \leq e^{-\log n/20}, \quad k = 1, 2, \ldots, r_2.$$  (5.3)

Hence $\Pr(\bigcup_{k=1}^{r_2} A_k) = o(1)$ which implies the lemma as at most $r_2 - 1$ patches can be made in Phase 2.

**Proof of (5.3).** Fix $k$ and suppose that when $i$ first reaches $a_k$, $|C_i| < n_0$. Then, where \{closeness\} denote the current cycles, there exists a smallest $t$ such that

$$n - n_0 \leq \sum_{i=0}^{t} |C_{r_2-i}| \leq n_0.$$  (5.4a)

Let $S = \bigcup_{i=0}^{t} C_{r_2-i}, T_0 = \{(v, w) \in E_{m_2}; v \in S, w \notin S\}, T_1 = \{\phi_2^{-1}(w), \phi_2(v)\}; (v, w) \in T_0$ and $T_2 = \{e = (v, w) \in T_1: v, w \in \text{LARGE}\}$. Then (5.2) implies

$$|T_1| - |T_0| \geq n_0(n - n_0)\log n/2n.$$  (5.4b)

and then

$$|T_2| \geq |T_1| - 12n^{1/2}\log n,$$  (5.4b)

as for a given $v$ there are at most $12 \log n$ neighbours of $\phi_2(v), \phi_2^{-1}(v)$. But if $A_k$ occurs then $E^2(k) \cap T_2 = \emptyset$. But then (5.1) implies that

$$\Pr(\bigcap_{k} E^2(k) \cap T_2 = \emptyset) \leq (1 - |T_2|/n(n-1))^{|E^2(k)|}. $$

Using (5.4) and $|E^2(k)| \geq n/24$ gives (5.3). $\square$

6. Details of Phase 3

We a.a. start Phase 3 with cycles $C_1, C_2, \ldots, C_{r_2}$ where $|C_1| = n - o(n)$ and $r_2 = 2 \log n$. 
PHASE 3 OF DHAM.

begin
for $i = 2$ to $r_2$ do {merge $C_i$ into (the current) $C_1$}
begin
suppose $C_i = (x_1, x_2, \ldots, x_k, x_{k+1} = x_1)$ as a sequence of vertices
for $j = 1$ to $k$ do
begin
FINDCYCLE ($C_i$, $x_j$, $x_{j+1}$, $C_1$, outcome);
if outcome = success then goto A
end {j - loop}
end {end {i - loop}
terminate - success
A: end (i - loop)
terminate - failure
end;

procedure FINDCYCLE ($C_i$, $x_j$, $x_{j+1}$, $C_1$, outcome);
begin
outcome: = failure;
suppose $C_1 = (y_1, y_2, \ldots, y_p)$ and the out-neighbours of $x_j$ are $y_{i_1}, y_{i_2}, \ldots, y_{i_l}$;
if $l = 0$ then return else
begin
$\varphi_0$: = $\{(x_{j+1}, x_{j+2}, \ldots, x_1), \ldots, x_j, y_{i_t}, y_{i_t+1}, \ldots, y_{i_{t-1}}\}$;
$\mathit{END}_0$: = $\{y_{i_t-1}; t = 1, 2, \ldots, l\}$;
for $t = 1$ to $T = \left\lceil \frac{2 \log n}{3 \log \log n} \right\rceil$ do
begin
suppose $\varphi_{t-1} = \{P_1, P_2, \ldots, P_s\}$;
$\varphi_t$: = $\emptyset$; $\mathit{END}_t$: = $\emptyset$;
for $r = 1$ to $s$ do
begin
FIGURE 4
suppose \( P_r = (u_1, u_2, \ldots, u_q) \);
if \((u_q, u_1) \in E_m^r\) then
begin
terminate FINDCYCLE successfully with
\[ C_1 = (u_1, u_2, \ldots, u_q, u_1) \]
end else
begin
construct all possible paths obtainable from \( P_r \) in the following way (see Fig. 3);
if \((u_q, u_a) \in E_m^r\) and \((u_{a-1}, u_b) \in E_m^r\) where \( b > a \) then
\[ \varphi_t := \varphi_t \cup \{ \text{ROTATE}(P_r, a, b) \}; \;
\text{END}_t := \text{END}_t \cup \{ u_{b-1} \} \]
where \( \text{ROTATE}(P_r, a, b) = (u_1, u_2, \ldots, u_{a-1}, u_b, u_{b+1}, \ldots, u_q, u_a, \ldots, u_{b-1}) \)
end \{ t-loop \}
end \{ r-loop \}
end.

We must show that Phase 3 a.a. succeeds. Let \( m_2, \text{LARGE} \) be as in Section 5. Consider the subgraph \( \tilde{H} \) of \( D_{m_0} \) induced by \( \text{LARGE} \). Now the edges of \( \tilde{H} \) can be partitioned into \( E^3 \cup E^4 \), where \( E^3 \) consists of those edges that are in \( \text{LARGE} \) but do not belong to \( E_{m_2} \). Now, assuming (4.5), whatever \( E^4 \) is, \( E^3 \) is a random \( |E^3| \) - subset of the remaining possible edges—see (5.1). But, given \( E^4 \), we obtain the same distribution for \( \tilde{H} \) by choosing a random \( |E^3| \) - subset \( E^5 \) of \( \text{LARGE} \) and then adding in \( E^4 - E^3 \) plus further randomly chosen edges.

Furthermore we can obtain \( E^3 \) by independently including each edge of \( \text{LARGE} \) with probability \( p = \log n/n \) and excluding it with probability \( 1 - p \), then randomly adding or deleting edges to get a set of the required size. Since \( |E^3| = n \log n/6 \) a.a. we will a.a. be adding edges.

To summarise, we can a.a. consider \( \tilde{H} \) to be a supergraph of a graph \( H \) distributed as \( D_{h, p} \) where \( h = |\text{LARGE}| = n - o(n) \) and the edges of \( H \) occur independently of the outcome of Phase 1 or 2.

We need to be able to show that \( |\text{END}_t| \) grows sufficiently rapidly with \( t \) inside FINDCYCLE. Unfortunately this is quite technical and requires several preliminary lemmas and constructions.

Let \( C_i(i) \) denote the cycle \( C_i \) after \( C_i \) has been merged in Phase 3 for \( i \geq 2 \). Let \( C_{\text{Initial}} = C_1 = (u_1, u_2, \ldots, u_{n_1}, u_1) \) at the start of Phase 3 and let \( n_2 = |\text{LARGE} \cap V(C_{\text{Initial}})| = n - o(n) \).

Let \( \mu_1 = [(\log n)^2/\log \log \log n] \) and partition the path \( P_{\text{Initial}} = (u_1, u_2, \ldots, u_{n_2}) \) into consecutive blocks \( A_1, A_2, \ldots, A_{\mu_1} \) each containing \( [n_2/\mu_1] \) or \( [n_2/\mu_1] \) large vertices. For \( I \subseteq M = \{ 1, 2, \ldots, \mu_1 \} \) let \( A_I = \bigcup_{i \in I} A_i \). For each \( I \subseteq M, \; |I| = |\mu_1/10| \) we let \( B_I = \{ v \in \text{LARGE}: \)}
| \{ w \in \text{LARGE} \cap A_i^j : (v, w) \in E(H) \} | \leq \log n/75 \} (= \text{those } v \text{ with substantially fewer than the expected number of out-neighbours in } A_i^j). 

\textbf{Lemma 6.1.} The following holds with probability \(1 - o(n^{-1})\):

\[ I \subseteq M, |I| = [\mu_1/10], \text{ implies } |B_I| \leq n^{0.99999}. \]

\textit{Proof.} For a fixed \(I\) and positive integer \(k, \)

\[ \Pr(|B_I| \geq k) \leq \left(\frac{h}{k}\right)^k \text{BS}(0, [\log n/75]; [n_2/\mu_1][1/10], p)^k \]

\[ \leq \left(\frac{he}{k}\right)^k \exp\left\{- \frac{1}{2} \cdot \frac{1}{25} \cdot \frac{1}{11} \cdot k \log n \right\} \]

\[ \leq \left(n^{0.999985}e/k\right)^k. \]

Hence

\[ \Pr(\exists I : |B_I| \geq n^{0.99999}) \leq 2^{\mu_1}(n^{0.000005}e)n^{0.99999} = o(n^{-1}). \]

Next let \(\mu_2 = [\log n/\log \log \log \log n]\).

\textbf{Lemma 6.2.} The following holds with probability \(1 - o(n^{-1})\): \(\exists I \subseteq M, |I| = [\mu_1/10], v \in \text{LARGE}\) and disjoint sets

\[ A = \{a_1, a_2, \ldots, a_{\mu_2}\} \subseteq \text{LARGE}, \quad B = \{b_1, b_2, \ldots, b_{\mu_2}\} \subseteq B_I \]

such that

\[ (v, a_i) \in E(H) \quad \text{and} \quad ((b_i, a_i) \in E(H) \quad \text{or} \quad a_i = \phi_2(b_i)), \]

\[ i = 1, 2, \ldots, \mu_2. \]

\textit{Proof.} For a fixed \(I \subseteq M, v \in \text{LARGE}, A, B \subseteq \text{LARGE}, \) where \(a_i = \phi_2(b_i)\) for \(i = 1, 2, \ldots, k,\) the probability that the remaining conditions are satisfied is at most

\[ p^{2^{\mu_2-k}}\text{BS}(0, [\log n/75]; [n_2/\mu_1][1/10], p)^{\mu_2} \leq p^{2^{\mu_2-k}n^{-0.00001\mu_2}}. \]

Thus

\[ \Pr(\exists I, v, A, B \text{ as in the lemma}) \leq 2^{\mu_1}n^{\mu_2} \sum_{k=0}^{\mu_2} \left(\frac{n}{\mu_2 - k}\right)^{2^{\mu_2-k}n^{-0.00001\mu_2}} \]

\[ \leq 2^{\mu_1} \sum_{k=0}^{\mu_2} (\log n)^{2^{\mu_2-k}n^{-0.00001\mu_2}} = o(n^{-1}). \]
Lemma 6.3. Let $0 < \alpha < 1$ be fixed. Then $D_{n,1}$ satisfies the following with probability $1 - o(n^{-1})$: \( \forall A, B \subseteq V_n \) such that (i) \( |A| \leq u_0 = \alpha e^{-3} n / \log n \), (ii) \( |B| \leq \alpha |A| \log n / 2 \), and (iii) \( |\{(v, w) \in E_{m_1} : v \in A, w \in B\}| \geq \alpha |A| \log n \).

Proof. We prove the result for $D_{n,1}$ and apply (2.3):

$$
\Pr(\exists A, B \text{ in } D_{n,1}) \\
\leq \sum_{k=1}^{u_0} \binom{n}{k} \left( \left\lfloor \frac{n}{k \log n/2} \right\rfloor \cdot \binom{k}{\left\lfloor \frac{n}{k \log n} \right\rfloor} \right) p^{[u \alpha \log n]} \\
\leq \sum_{k=1}^{u_0} \left( \frac{ne}{k} \left( 1 + o(1) \right) \frac{2ne}{ak \log n} \cdot \frac{e^2k^2}{4} \cdot \frac{(\log n)^2}{n^2} \right)^{\alpha \log n/2} k \\
= \sum_{k=1}^{u_0} \left( \frac{ne}{k} \left( 1 + o(1) \right) \frac{e^3k \log n}{2an} \right)^{\alpha \log n/2} k \\
= o(n^{-1}).
$$

Finally, a calculation similar to that for Lemma 4.1 yields

Lemma 6.4. The following holds with probability $1 - o(n^{-3})$:

\[ v, w \in \text{SMALL} = V_n - \text{LARGE} \implies \text{dist}(v, w; GD_{m_1}) \geq 10. \quad (6.1) \]

We can now get down to proving that Phase 3 a.a. succeeds. We implicitly assume that (4.2) and the conditions of Lemmas 6.1–6.4 hold, but see Remark 2 below.

Consider any $C_i, i \geq 2$ and the first $j$ such that both $x_i, x_{i+1} \in \text{LARGE}$. By Lemma 6.4 such a $j$ a.a. exists. We show that for such an $i, j$

$$
\Pr(\text{FINDCYCLE fails}) = O(n^{-\epsilon_1}) \quad \text{for some constant } \epsilon_1 > 0. \quad (6.2)
$$

It follows then that

$$
\Pr(\text{Phase 3 fails}) = O(n^{-6} \log n) + o(1) = o(1)
$$

and Theorem 2 will follow.

We now consider such an execution of FINDCYCLE.

Proof of (6.2). The following observation is crucial.

At any stage the paths constructed are of the following form: we take $P_{\text{initial}}$ and delete $O((\log n)^2 / \log \log n)$ edges, permute the subpaths around and rejoin them with new edges (ROTATIONs) or paths (from the cycles
Thus if \( n \) is large,

If we partition the vertices of a constructed path \( P \) into consecutive sets \( S_1, S_2, \ldots, S_9 \) each containing at least \( \lfloor n/9 \rfloor \) vertices of \( \text{LARGE} \cup V(C_{\text{initial}}) \) then for each \( i = 1, 2, \ldots, 9 \), \( S_i \) contains a complete block \( A_{t_i} \), where \( |I_i| = \lfloor \mu_1/10 \rfloor \) for some \( I_i \), each \( A_{t_i}, t \in I_i \) appears on \( P \) as it did on \( P_{\text{initial}} \).

Let a constructed path \( P = (z_1, z_2, \ldots, z_p) \) be good if \( z_p \in \text{LARGE} \) and \( \exists I \subseteq M, |I| = \lfloor \mu_1/10 \rfloor \), and \( r < s \leq p/2 \) with \( s - r \leq p/9 \) such that \( A_t \subseteq \{z_r, z_{r+1}, \ldots, z_s\} \) and \( z_p \notin B_t \).

Let now \( \hat{\varphi}_i \) denote those paths of \( \varphi_t \) which are good. Let \( \text{END}_t = \{v \in \text{END}_i : \hat{\varphi}_t \text{ contains a path from } x_{j+1} \text{ to } v \} \). By deleting paths, if necessary, we assume that for each \( v \in \text{END}_t \) there is a unique such \( P_v \in \hat{\varphi}_t \).

We first show

\[
\Pr(\hat{\varphi}_0 \leq \log n/8) = O(n^{-\alpha}). \tag{6.4}
\]

Consider \( C_i(i-1) = (w_1, w_2, \ldots, w_p) \). If we partition it into subpaths \( S_1, S_2, \ldots, S_9 \) then as in (6.3) we find a complete \( A_{t_i}, |I_i| = \lfloor \mu_1/10 \rfloor \) contained in each \( S_k \). For each \( w_i \), at least 4 of these \( A_{t_i} \)'s will be in the first half of the path \( (x_{j+1}, \ldots, x_j, w_{i+1}, \ldots, w_i, \ldots, w_q) \) if \( (x_j, w_{i+1}) \in E_{m^*} \).

Now we can assume that \( |B_{t_i} \cup B_{t_{i+1}} \cup \cdots \cup B_{t_k}| \leq 9n^{0.99999} \) by Lemma 6.1. Thus \( C_i(i-1) \) contains a set \( L \) of \( n - o(n) \) large vertices \( w_i \) such that if \( (x_j, w_{i+1}) \) exists then the path produced is good. Since the \( H \)-edges \( x_i \) to \( V(C_i(i-1)) \cap \text{LARGE} \) exist independently with probability \( \log n/7 \), (6.4) follows easily from (2.1a).

Remark 2. The definition of \( \text{LARGE} \) depends only on the first \( m \) edges and so is independent of \( H \). On the other hand, assuming the conditions of Lemmas 6.1–6.4 does (mildly) condition \( H \). However,

\[
\Pr(\hat{\varphi}_0 \leq \log n/8) \\
\leq \Pr( x_j \text{ has } \leq \log n/8 \text{ "good" neighbours } | L | \\
| L | \geq n_1 = n_0 - 9n^{0.99999} - \sqrt{n} ) + \Pr(|L| < n_1) \\
\leq (1 - p)^{n_1} + o( n^{-3} ),
\]

where we can use \( (1 - p)^{n} \) as the \( H \)-edges incident with \( x_j \) are independent of \( L \), and \( L \) will be large assuming the conditions of Lemmas 6.1–6.4.
The (almost) final piece in the jigsaw is, assuming the conditions of the previous lemmas,
\[ 1 \leq |E \tilde{\mathcal{D}}| \leq n/(456e^3(\log n)^2) \] implies
\[ |E_{\mathcal{N}}D_{t+1}| \geq (\log n/152)^3 |E \tilde{\mathcal{D}}|. \] (6.5)

To prove this let \( |E \tilde{\mathcal{D}}| \leq n/(456e^3(\log n)^2) \). Let \( v \in E \tilde{\mathcal{D}}, P_v = (z_1, z_2, \ldots, z_p) \) and \( r, s, I \) be as in the definition of a good path.

**Case 1.** \( r > p/9 \). Partition \( P_v \) into subpaths \( S_1, S_2, \ldots, S_9 \) containing complete blocks \( I_1, I_2, \ldots, I_9 \) as in (6.3). Let
\[ F_1(v) = \{ z_k: (v, z_k) \in E(H), z_k \in A_I, z_{k-1} \in \text{LARGE} - B_{I_k}, (z_{k-1}, z_k) \in C_1(i-1) \} \]
and
\[ F_2(v) = \{ z_{k-1}: z_k \in F_1(v) \}. \]

Note that Lemmas 6.1, 6.2, and 6.4 imply
\[ |F_1(v)| \geq \log n/75 - T - \mu_2 - 1 \geq \log n/76 \quad \text{for } n \text{ large.} \] (6.6)

For \( z_{k-1} \in F_2(v) \) let
\[ F_3(z_{k-1}) = \{ z_l: l > k, (z_{k-1}, z_l) \in E(H), z_l \in A_{I_l}, z_{l-1} \in \text{LARGE} - B_{I_l}, (z_{l-1}, z_l) \in C_1(i-1) \} \]
and
\[ F_4(z_{k-1}) = \{ z_{l-1}: z_l \in F_3(z_{k-1}) \}. \]

Similarly to (6.6) we have
\[ |F_3(z_{k-1})| \geq \log n/76 \quad \text{for } n \text{ large.} \] (6.7)

The important point to note is that
\[ \text{if } z_k \in F_1(v), z_l \in F_3(z_{k-1}) \text{ then ROTATE } (P_v, k, l) \]
is a good path with \( I = I_1 \). (6.8)

(Though this would not necessarily be true if we had used 8, say, in place of 9 in (6.3).)
Case 2. $r \leq p/9$. We replace $I_1$ by $I_3$ throughout the above argument. Let $\Phi_k = \bigcup_{v \in END} F_k(v)$ for $k = 1, 2$. Now by (6.6) there are at least $(\log n/76)|END_i|$ edges from $END_i$ to $\Phi_1$. Hence by Lemma 6.3 with $\alpha = \frac{1}{76}$ and by (4.2),

$$ \frac{\log n}{152}|END_i| \leq |\Phi_1| \leq (6 \log n)|END_i| \leq n/(76e^3 \log n). $$

But $|\Phi_2| = |\Phi_1|$ since $z \in \Phi_2$iff $z' \in \Phi_1$, where $z'$ is the successor of $z$ on $C_i(i - 1)$. Putting $\Phi_k = \bigcup_{v \in END} \bigcup_{z \in F_k(v)} F_k(z)$, for $k = 3, 4$, we see by (6.7) that there are at least $(\log n/76)|\Phi_2|$ edges from $\Phi_2$ to $\Phi_3$.

Hence, by Lemma 6.3,

$$ |\Phi_3| \geq (\log n/76)|\Phi_2|. $$

Also $|\Phi_4| = |\Phi_3|$ by the argument for $|\Phi_2| = |\Phi_1|$. Finally $\Phi_4 \subseteq END_{i+1}$ by (6.8) and so (6.5) follows.

Thus there exists $\tau \leq T - 1$ such that $|END_{\tau}| \geq n/(456e^3(\log n)^2)$. Choose a subset $END$ of $END_r$ of exactly this size and apply the preceding argument using $END$ in place of $END_r$ to show that $|END_{\tau+1}| \geq \beta n$ for some constant $\beta > 0$. Now the existence of an edge in $H$ from $END_i$ to $x_{j+1}$ is independent of the previous history, at the time of checking (see Remark 2). Hence

$$ \Pr(\text{FINDCYCLE fails}) \leq (1 - p)^{\beta n} \leq n^{-\beta/7}. $$

and Theorem 2 has been proved, except for the claimed running time.

**Running time of DHAM.** We analyse this assuming that those events proved to occur a.a. actually occur. We can always put a time limit on DHAM to achieve the $O(n^{1.5})$ running time with certainty. We assume that $D_{m^*}$ is processed in adjacency list form.

**Phase 1.** It takes $O(n \log n)$ time to construct $BIP$ assuming $O(1)$ time to obtain a random bit. (To construct $\sigma$ we need to be able to find the $k$th largest from a set of integers where $k$ is randomly chosen.) This takes $O(\log n)$ time, using a height balanced tree—see, for example, Aho, Hopcroft, and Ullman [1]. The running time of the algorithm in [8] is $O(n^{1.5})$ since $BIP$ has at most $20n$ edges.

**Phase 2.** We make $O(\log n)$ executions of the loop beginning at $A$. It takes $O(1)$ time to see if an edge belongs to $F_2$ and $O(\log n)$ time to see if it belongs to $E_{m^*}$. The vertex sets of the cycles of $\Phi_2$ partition $V$, and the operations required on these sets is UNION and FIND—see, for example, [1]. It follows that $O(n(\log n)^3)$ time is sufficient for this phase.

**Phase 3.** Our assumptions imply that it takes at most 2 executions of FINDCYCLE to merge a $C_i$ into $C_1$. Each execution of FINDCYCLE
generates $O((6 \log n)^T) = O(n^{4/3+o(1)})$ paths altogether. There is a natural
tree structure to the set of paths produced where a path is the father of all
the paths produced from it by one double-rotation. The description of
the algorithm suggests a breadth-first search of this tree. It is, however, more
efficient to explore this tree by depth-first search. We can then keep a single
path, as a doubly linked list and create the next son in $O(1)$ time.
Backtracking takes $O(1)$ time per level and we obtain a running time of
$O(n^{4/3+o(1)})$ for the whole of Phase 3. □

It is easy to see that Theorem 1 follows from Theorem 2. Let
e_1, e_2, \ldots, e_{n(n-1)} \text{ be a random permutation of } E(DK_n) \text{ and } m = n \log n + c_n n. \text{ Theorem 1 states the limiting value of } \Pr(\text{DHAM finds a hamilton cycle in } D_m). \text{ Now given } m < m^* \text{ then this is zero and given } m \geq m^* \text{ then the preceding proof shows that this probability tends to one (or only allow DHAM to use the first } m^* \text{ edges). The value of } \lim_{n \to \infty} \Pr(m \geq m^*) \text{ is easy to compute and is in fact (implicitly) done in [7].}

Outline Proof of Theorem 3. The idea is to (i) construct $k$ disjoint sets
$F_1, F_2, \ldots, F_k$ where each $F_i$ is the set of edges of a set of vertex disjoint
cycles covering $V_n$, (ii) split the edges of $(E_{m_0} - E_{m_1}) \cap \text{ELARGE}$ into $k$
sets of roughly equal size $E_1, E_2, \ldots, E_k$ and use $E_i$ to patch up the cycles
in $F_i$ as above, for $i = 1, 2, \ldots, k$.

To construct $F_1, F_2, \ldots, F_k$ we apply Phase 1 $k$ times. It is straightforward to show that we a.a. succeed in constructing $k$ permutations
$\phi_1, \phi_2, \ldots, \phi_k$. If we consider a fixed $i$, then our randomising method (i.e.,
use of $\sigma$) ensures that $\phi_i$ is sampled uniformly and so a.a. has at most
$2 \log n$ cycles. As $k$ is constant $\phi_1, \phi_2, \ldots, \phi_k$ will a.a. all have at most
$2 \log n$ cycles. The remaining proof goes through much as before with only
trivial changes to constants.

7. TRAVELLING SALESMAN PROBLEMS

Proof of Theorem 4. Given an instance of this problem let the edges of
$DK_n$ be ordered $e_1, e_2, \ldots, e_{n(n-1)}$, where $w(e_i) \leq w(e_{i+1})$ for $i = 1, 2, \ldots, n(n-1)$ and $w(e)$ is the weight of edge $e$. Note that our assumption implies that each such ordering is equally likely. If $E_m, D_m, m^*$ are as
before then we have

ALGORITHM DBOT.

begin
  apply DHAM to $D_{m^*}$
end

It is clear that if DHAM succeeds then it solves BTSP and Theorem 2 states
that it a.a. does. To obtain \( O(n^2) \) running time we do not sort all the edges but instead build a heap out of them and generate them in increasing order of weight until \( m^* \) is reached. This does require \( O(n^2) \) time a.a. □

**Outline Proof of Theorem 5.** We shall assume throughout the \( B(n) = [an/\log n] \) for some constant \( \alpha > 0 \). We can assume \( \alpha \geq 1 \) for if not then it is easy to see that we can use a.a. DHAM to construct a Hamilton cycle using edges of length zero only.

We shall describe DTSPSOLVE part formally and part informally. DTSPSOLVE first, independently, randomly colours each element of \( V_n \times V_n \) red or blue with probability \( \frac{1}{2} \). \( E^r \) is the set of red edges and \( E^b \) the set of blue edges produced. Also let \( E^{rk} = \{ e \in E^r : w(e) \leq k \} \) for \( k = 0, 1, \ldots, B(n) - 1 \). Define \( E^{bk} \) similarly and let \( E^k = E^{rk} \cup E^{bk} \). Note that \( D^k = (V_n, E^k) \) is distributed as \( \tilde{D}_{n,(k+1)p} \), where \( p = 1/B \).

The strategy now is similar to that of [9]. We let

\[
X = \{ v \in V_n : d^+(v, E_{r_0}) \leq l_0 \text{ or } d^-(v, E_{r_0}) \leq l_0 \}. \]

Now let

\[
\mathcal{F} = \{ F \subseteq E(DK_n) : \text{ (i) } F \text{ induces a set of vertex disjoint paths}, \]

\[
\text{ (ii) each } x \in X \text{ is an interior vertex of one such path}, \]

\[
\text{ (iii) each edge of } F \text{ is incident with a vertex of } X. \}

We first try to compute a minimum total weight member of \( \mathcal{F} \) and then extend it, using zero-length red edges, to a set of \( O(\log n) \) vertex disjoint cycles covering \( V_n \). More formally,

**Phase 1.**

\[
\begin{align*}
\sigma &:= \text{ a random permutation of } V_n; \\
\text{for } e \in V_n \times V_n &\text{ do } \hat{w}(e) := w(e) \text{ where } (v, w) = (v, \sigma(w)) \text{ as before;} \\
\text{let } e_1, e_2, \ldots, e_{n^2} &\text{ be random ordering of } V_n \times V_n; \\
E^{1+} &:= E^{1-} := \varnothing; \\
A: \text{ for } i = 1 \text{ to } n^2 &\text{ do } \\
&\text{ begin } \\
&\quad \text{ if } d^+(v_i, E^{1+}) \leq \log \log n \text{ and } e_i \in E^{r_0} \text{ then } E^{1+} := E^{1+} \cup \{ \hat{e}_i \}; \\
&\quad \text{ where } e_i = (v_i, w_i) \\
&\text{ end; } \\
B: \text{ for } i = 1 \text{ to } n^2 &\text{ do } \\
&\text{ begin } \\
&\quad \text{ if } e_i \in E^{1+} \text{ and } d^-(\sigma(w_i), E^{1-}) \leq \log \log n \text{ and } e_i \in E^{r_0} \text{ then } E^{1-} := E^{1-} \cup \{ \hat{e}_i \} \\
&\text{ end }
\end{align*}
\]
\[ E^1 := E^{1+} \cup E^{1-}; \]
compute \( F^* \) where \( F^* = \min\{ \hat{w}(F) : F \in \mathcal{F} \} \) and \( \hat{w}(F) = \sum_{e \in F} \hat{w}(e) \);
\[ Y_0 := \{ v \in V_n : 3e = (v, w) \in F^* \} \quad \text{and} \quad Y_1 := \{ w \in V_n : 3e = (v, w) \in F^* \}; \]
\[ \text{BIP} := \text{bipartite graph} \ (V_n - Y_0, V_n - Y_1, A) \] where \((v, w) \in A\) iff \((v, w) \in E^{1};\)
apply the algorithm of Even and Tarjan \[8\] to find a maximum matching \( M \) of BIP;
if \( M \) is not a perfect matching then DTSPSOLVE has failed else \( F' := \{ e : e^* \in F^* \cup F^* \} \); \( \text{output} \ F = \{ e : \hat{e} \in F' \cup F^* \} = \{ (u, \phi(u)) : u \in V_n \}, \text{say.} \)

Assuming Phase 1 is successful we then (Phase 2) randomly choose half the edges of \( E^{n_0} \) and use them to try to patch together cycles as in Phase 2 of DHAM. We then (Phase 3) use the remaining edges of \( E^{n_0} \) to try to finish the patching as in Phase 3 of DHAM.
We now state as a lemma, some properties which can be proved by straightforward calculation.

**Lemma 7.1.** The following hold a.a.:

\[ |X| \leq n^{1-1/2\alpha}. \]  
(7.1a)

If \( \mu = [2\alpha] \) then

\[ \delta^+(D^\mu), \delta^-(D^\mu) \geq \log n/3, \]  
(7.1b)

\[ \Delta^+(D^{2\mu}), \Delta^-(D^{2\mu}) \leq 12 \log n, \]  
(7.1c)

\( GD^{2\mu} \) does not contain any cycle \( C \) with \( |V(C) \cap X| \geq |C|/10 \), \( GD^{2\mu} \) does not contain a tree with more than \( 6\alpha \) vertices at least one third of which are in \( X \).  
(7.1d)

Now (7.1a) implies that \( \mathcal{F} \neq \emptyset \). If we succeed in constructing a Hamilton cycle \( C^* \) then \( w(C^*) = w(F^*) \) and for any Hamilton cycle \( C \), \( w(C) \geq w(F^*) \)—just delete all edges of \( C \) not incident with a vertex in \( X \) to obtain a member of \( \mathcal{F} \).

(7.1b) and (7.1e) imply that \( F^* \subseteq E^{2\mu} \), for any \( e \notin F^* \cap E^{2\mu} \) could be replaced by 2 edges of \( E^\mu \) to yield an \( F \) of smaller weight. Thus to compute \( F^* \) we need only look at the edges \( E^* \) of \( E^{2\mu} \) which are incident with \( X \).

(7.1d) implies that \( E^* \) induces a forest in \( GD^{2\mu} \). Let \( T \) be a tree in this forest. All its leaves are in \( V_n - X \), by (7.1b). If \( x \in X \) is adjacent to \( t \geq 3 \) leaves in \( T \) then there are either \( \geq 2 \) edges of \( T \) directed into \( x \) from a leaf or \( \geq 2 \) edges of \( T \) directed from \( x \) to a leaf. But we can then remove the larger weight edges to leave at most one edge directed from a leaf to a given
$x \in X$ or from a given $x \in X$ to a leaf. Assume this done. Then (7.1e) implies that the remaining tree has at most $12\alpha$ vertices—if it has $k$ nonleaf vertices then at least $\lfloor k/2 \rfloor + 1$ are in $X$. We can deal with each such tree in $O(1)$ time and hence compute $F^*$ in $o(n)$ time.

We henceforth assume that the search for $F^*$ is restricted to the subgraph induced by the vertices $Y = X \cup N(X, GD^{2\mu})$.

This leaves the lengths of blue edges joining 2 vertices in $V_n - Y$ unconditioned by Phase 1. (7.2)

We must now show that Phase 1 a.a. succeeds. Now if BIP does not have a perfect matching then by Hall's theorem there exists $K \subseteq V_n - Y$, $L \subseteq V_n - Y$, $|L| = |K| - 1$, and $N(K, BIP) \subseteq L$. Suppose first as in the proof of Lemma 4.2 that $|K| \leq n/2$. Then either (7.1e) fails or, since $K \cap X = \emptyset$ each $v \in K$ makes at least $\log \log n - 3\alpha$ choices in the $L$ in the construction of $E'$. The probability of the latter event $\to 0$ by a similar calculation to that in Lemma 4.2(a). For $|K| > n/2$ we note that the equivalent of (4.5) holds and we can finish as we did in Lemma 4.2(a).

We can argue as in Lemma 4.2(b) that the permutation $\phi$ output is equally likely to be any permutation of $V_n$—it does not matter to the theorem if we allow loops in the construction of $E'$.

The arguments that Phase 2 and Phase 3 work a.a. are as for DHAM except for trivial changes in constants, which now depend on $\alpha$, the use of (7.1d) in place of Lemma 6.4, and $V_n - Y$ in place of LARGE (see 7.2).

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