

Maximum Matchings in Random Bipartite Graphs and the Space Utilization of Cuckoo Hash Tables

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Abstract

We study the the following question in Random Graphs. We are given two disjoint sets L, R with $|L| = n$ and $|R| = m$. We construct a random graph G by allowing each $x \in L$ to choose d random neighbours in R . The question discussed is as to the size $\mu(G)$ of the largest matching in G . When considered in the context of Cuckoo Hashing, one key question is as to when is $\mu(G) = n$ **whp**? We answer this question exactly when d is at least three.

1 Introduction

For a graph G we let $\mu(G)$ denote the size of a maximum matching in G . This paper provides an analysis of $\mu(G)$ in the following model of a random bipartite graph. We have two disjoint sets L, R where $L = [n], R = [m]$ where $n = \alpha m$. Each $v \in L$ independently chooses d random vertices of R as neighbours. Our assumptions are that $\alpha > 0, d \geq 3$ are fixed and $n \rightarrow \infty$. There is of course the issue as to whether a vertex is allowed to make the same choice twice. We allow this in the paper in order to keep the d choices independent. Keeping the choices distinct makes no essential difference to the final result. One can for example couple the two modes of construction so that the size of the maximum matching is always at least as large when no repetitions are allowed.

After some preliminary analysis we can reduce the question to the following: We have two disjoint sets L_1, R_1 of sizes n_1, m_1 respectively. Asymptotic expression for the values of m_1, n_1 that hold **whp** are given. Each vertex of L_1 chooses $d \geq 3$ random neighbours in R_1 . The choices are conditioned so that each vertex of R_1 is of degree at least two. Let G_1 denote the sub-graph of G induced by L_1, R_1 . We show that

$$\mu(G_1) = \min \{|L_1|, |R_1|\} \text{ whp.}$$

This amounts to proving the following theorem:

Theorem 1 *Let Γ be a bipartite graph chosen uniformly from the sets of graphs with bipartition $L, R, |L| = n, |R| = m$ such that each vertex of L has degree $d \geq 3$ and each vertex of R has degree*

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at least two. Then **whp**

$$\mu(\Gamma) = \min \{m, n\}.$$

The proof of this comprises the main technical challenge of the paper. This result has a similar flavour to some classical results. Walkup [28] considered the problem when $|L| = |R| = n$ and each vertex of L chooses d random neighbours in R and each vertex of R chooses d random neighbours in L . He showed that a random bipartite graph constructed in this way has a perfect matching **whp** iff $d \geq 2$. Karonski and Pittel [17] considered a refinement where in the first round each vertex of $L \cup R$ first chooses a single random neighbour. Then each vertex not chosen by another vertex in first round gets another random choice. In this way the graph has minimum degree at least two. They show that this graph has a perfect matching **whp**, thus improving on Walkup's theorem.

Another motivation for this study comes from Cuckoo Hashing, see for example Mitzenmacher [25]. Briefly each one of n items $x \in L$ has d possible locations $h_1(x), h_2(x), \dots, h_d(x) \in R$, where d is typically a small constant and the h_i are hash functions, typically assumed to behave as independent fully random hash functions. (See [24] for some justification of this assumption.) We are thus led to consider the bipartite graph G which has vertex set $L \cup R$ and edge set $\{(x, h_j(x)) : x \in L, j = 1, 2, \dots, d\}$. Under the assumption that the hash functions are completely random we see that G has the same distribution as the random graph defined in the previous paragraph.

We assume each location can hold only one item. Items are inserted consecutively and when an item x is inserted into the table, it can be placed immediately if one of its d locations is currently empty. If not, one of the items in its d locations must be displaced and moved to another of its d choices to make room for x . This item in turn may need to displace another item out of one its d locations. Inserting an item may require a sequence of moves, each maintaining the invariant that each item remains in one of its d potential locations, until no further evictions are needed. Thus having inserted k items, we have constructed a matching M of size k in G . Adding a $(k + 1)$ -st item is tantamount to constructing an augmenting path with respect to M . All n items will be insertable in this way iff G contains a matching of size n .

The case of $d = 2$ choices is notably different from that for other values of d and the theory for the case where there are $d = 2$ bucket choices for each item is well understood at this point [9, 22, 27]. We will therefore assume that $d \geq 3$.

We note finally that Theorem 1 can be interpreted in the context of random d -uniform hypergraphs for $d \geq 3$. The 2-core of a hypergraph H is the largest set of vertices that induce a sub-graph of minimum degree two. The density of a set of vertices S is the ratio of the number of edges contained entirely in S to the size of S itself. If we interpret the neighbours of a vertex $v \in L$ as an edge of a random d -uniform hypergraph then the theorem implies the following: A random d -uniform hypergraph contains a set of density at least one, only when the 2-core has density at least one, **whp**. This is a consequence of Hall's theorem for matchings and our result for the case where $n \leq m$. For results on the 2-core of a random hypergraph, see for example Cooper [7] and Molloy [26].

We will now turn to the matchings question posed in first paragraph of the paper.

2 Definitions and Results

This question was studied initially by Fotakis, Pagh, Sanders and Spirakis [14]. They show in the course of their analysis of Cuckoo hashing that the following holds:

Lemma 2 *Suppose that $0 < \varepsilon < 1$ and $d \geq 2(1 + \varepsilon) \log(e/\varepsilon)$. Suppose also that $m = (1 + \varepsilon)n$. Then **whp** G contains a matching of size n i.e. a matching of L into R .*

□

In particular, if $d = 3$ and $m \approx 1.57n$ then Lemma 2 shows that there is a matching of L into R **whp**.

This lemma is not tight and recently Dietzfelbinger, Goerdt, Mitzenmacher, Montanari, Pagh and Rink [10] observed a connection with a result of Dubois and Mandler on Random 3-XORSAT [12] that enables one to essentially answer the question as to when $\mu(G) \geq n$ for the case $d = 3$. The final version extends the result of [12] to $d \geq 3$. More recently, Fountoulakis and Panagiotou [15] have also established thresholds for when there is a matching of L into R **whp**, for all $d \geq 3$. It should perhaps be noted that the result of this paper is stronger in the sense that it gives the size of the largest matching when there is no matching of L into R .

We begin with a simple observation that is the basis of the Karp-Sipser Algorithm [18, 2]. If v is a vertex of degree one in G and e is its unique incident edge, then there exists a maximum matching of G that includes e . Karp and Sipser exploited this via a simple greedy algorithm:

Algorithm 1 Karp-Sipser Algorithm

```

1: procedure KSGREEDY( $G$ )
2:    $M \leftarrow \emptyset, \Gamma \leftarrow G$ ;
3:   while  $\Gamma \neq \emptyset$  do
4:     if  $\Gamma$  has vertices of degree one then
5:       Select a vertex  $\xi$  uniformly at random from the set of vertices of degree one
6:       Let  $e = (\xi, \eta)$  be the edge incident to  $\xi$ 
7:     else
8:       Select an edge  $e = (v, u)$  uniformly at random
9:     end if
10:     $M \leftarrow M \cup \{e\}$ 
11:     $\Gamma \leftarrow \Gamma \setminus \{\xi, \eta\}$ 
12:  end while
13:  return  $M$ 
14: end procedure

```

Phase 1 of the Karp-Sipser Algorithm ends and Phase 2 begins when the graph remaining has minimum degree at least two. So if Γ_1 denotes the graph Γ remaining at the end of Phase 1 and τ_1 is the number of iterations involved in Phase 1 then

$$\mu(G) = \tau_1 + \mu(\Gamma_1). \tag{1}$$

Our approach to estimating $\mu(G)$ is to (i) obtain an asymptotic expression for τ_1 that holds **whp** and then (ii) show that **whp** Γ_1 has a (near) perfect matching and then apply (1).

We summarise the known results pertaining to the Karp-Sipser algorithm. For proofs of Theorem 3 and parts (a), (b) of Theorem 4 see Luby, Mitzenmacher, Shokrollahi and Spielman [23] or Dembo and Montanari [8] or even an earlier version of this paper [16]: Let z_1 satisfy

$$z_1 = \frac{e^{z_1} - 1}{d - 1} \quad (2)$$

and let

$$\alpha^* = \frac{z_1}{d(1 - e^{-z_1})^{d-1}}. \quad (3)$$

Theorem 3 *If $\alpha \leq \alpha^*$ then whp $\mu(G) = \tau_1 = n$.*

Thus **whp** Phase 1 of the Karp-Sipser Algorithm finds a (near) maximum matching if $\alpha \leq \alpha^*$. In particular, if $d = 3$ then $z_1 \approx 1.251$ and $\alpha_1 \approx .818$ and thus $m \approx 1.222n$ is enough for a matching of L into R .

Now consider larger α . Let z^* be the largest non-negative solution to

$$\left(\frac{z}{\alpha d}\right)^{\frac{1}{d-1}} + e^{-z} - 1 = 0.$$

Theorem 4 *If $\alpha > \alpha^*$ then whp*

(a) $z^* > 0$.

(b) $\tau_1 = n \left(1 - \left(\frac{z^*}{\alpha d}\right)^{\frac{d}{d-1}}\right) + o(n)$.

(c) *If $d \geq 3$ then*

$$\mu(\Gamma_1) = \min\{|L_1|, |R_1|\} = \min\left\{n - \tau_1, (1 - (1 + z^*)e^{-z^*})m + o(m)\right\}. \quad (4)$$

Here $L_1 \subseteq L, R_1 \subseteq R$ are the two sides of the bipartition of Γ_1 , after deleting any isolated vertices from the R -side.

Only part (c) needs to be proved here. As already mentioned, parts (a), (b) have already been proven in [8], [23]. In fact, the remainder of the paper constitutes a proof of part (c).

3 Proof of Theorem 1 (and Theorem 4(c))

Let us summarize what we have to prove. We have a bipartite graph Γ_1 with partition L_1, R_1 and $|L_1| = n_1 = \alpha_1 m_1, |R_1| = m_1$. Each vertex $a \in L_1$ has degree $d_{L_1}(a) = d$ and each vertex $b \in R_1$ has degree $d_{R_1}(b)$ at least 2. The graph Γ_1 is chosen uniformly from the set of bipartite graphs with these degree properties.

At this point it is convenient to drop the suffix 1. So from now on, m, n, α, Γ etc. refer to the graph left at the end of Phase 1.

The degrees of Γ satisfy $d_L(a) = d$ for $a \in L$. The degrees of vertices in R are distributed as the box occupancies X_1, X_2, \dots, X_m in the following experiment. We throw dn balls randomly into

m boxes and condition that each box gets at least two balls. In these circumstance the X_j 's are independent truncated Poisson, subject to the condition that $X_1 + X_2 + \dots + X_m = dn$, see Lemma 4 of [2]. Thus for any $S = \{b_1, b_2, \dots, b_s\} \subseteq R$ and any set of positive integers $k_i \geq 2, i \in S$ we have

$$\mathbb{P}(d_R(b_i) = k_i, i \in S) \leq O(n^{1/2}) \prod_{i \in S} \frac{z^{k_i}}{k_i! f(z)} \quad (5)$$

for $k \geq 2$ where z satisfies

$$\frac{z(e^z - 1)}{f(z)} = \frac{nd}{m} \quad (6)$$

and

$$f(z) = e^z - 1 - z.$$

The $O(n^{1/2})$ term accounts for the conditioning $\sum_{b \in R} d_R(b) = dn$.

We remark that

$$z > 2.$$

It follows from (5) that

$$\mathbb{P}(\exists b \in R : d_R(b) \geq L = \log n) \leq O(n^{3/2}) \frac{z^L}{L! f(z)} = O(n^{-K}) \quad (7)$$

for any positive constant K .

3.1 Outline of the proof of Theorem 4(c)

At the top level this involves showing that **whp** Hall's Condition holds. We will estimate the probability of the existence of sets A, B where $|A| = k$ and $|B| \leq k - 1$ such that $N_\Gamma(A) \subseteq B$. Here $N_\Gamma(S)$ is the set of neighbours of S in Γ . We call such a pair of sets, a *witness* to the non-existence of a matching that covers the smaller of L, R . There are two possibilities to consider: (i) *L-witnesses*: $A \subseteq L$ and $B \subseteq R$ or (ii) *R-witnesses*: $A \subseteq R$ and $B \subseteq L$. We have to deal with both cases in order to deal with the asymmetry between L and R . We say this because in the classic case of a binomial random bipartite graph considered by Erdős and Rényi [13] we can get away with only dealing with Case (i). We observe that if there exist a pair A, B then there exist a minimal pair and in this case each $b \in B$ has at least two neighbours in A . If v has a unique neighbour w in A then $A \setminus \{w\}, B \setminus \{v\}$ is also a witness.

We first consider *L-witnesses* A where $|A| \leq k_0$ where

$$k_0 = \max\{m, n\} / 2.$$

We can restrict our attention to $k \leq k_0$ because if $n \leq m$ and we have an *L-witness* $A \subseteq L$, $|A| = k > k_0$ then $B = R \setminus N_\Gamma(A)$ will be an *R-witness* and $|B| \leq k_0$. The same idea holds for *R-witnesses* $B \subseteq R$.

We begin by proving some lemmas involving the properties of functions that occur throughout the proof. This is the content of Section 3.2. We start the verification of Hall's condition in Section 3.3. In this section we assume that m and n are close, in particular $|m - n| = o(n^{7/8})$. We begin the analysis with *L-witnesses* in Section 3.3.1. When $k = |A|$ is small i.e. at most $n / \log^4 n$ (Case

1.0) we can use a simple bipartite configuration model to prove the non-existence of a witness. This is useful, because it then enables us to “ignore” factors of the order $e^{O(|m-n|)}$ in the main body of the proof.

We then consider larger k . This involves a complicated expression (15) for the expected number of L -witnesses. The proof continues by making various simplifications to this expression that are valid within different ranges and for different values of d . This makes the calculations rather lengthy.

Having dealt with L -witnesses we turn to R -witnesses in Section 3.3.2. We first produce a complex expression (42). When k is small, now less than $n^{9/10}$ we can make some simplifications. This again allows us to “ignore” factors of the order $e^{O(|m-n|)}$ in the main body of the proof. We are again faced with estimating a rather complex expression which we do by breaking into various sub-cases.

Once we have dealt with $|m-n| = o(n^{7/8})$ we tackle arbitrary m and n . We do this by relating the probability of events for the cases $m+s, n$ and $m, n+s$ with the probability of events for the case m, n , see (97) and (98). In this way we can reduce the case of arbitrary m, n to the case of $|m-n| = o(n^{7/8})$.

3.2 Useful Lemmas

Define the function $\zeta(x), x > 0$ to be the unique solution to

$$\frac{u(e^u - 1)}{f(u)} = x. \quad (8)$$

Let g be defined by

$$g(x) = (e^{\zeta(x)} - 1)^x f(\zeta(x))^{1-x}. \quad (9)$$

Observe that

$$\frac{f(\zeta(x))}{\zeta(x)^x} = \frac{g(x)}{x^x}. \quad (10)$$

Lemma 5 *The function $g(x)$ is log-concave as a function of x .*

Proof: We will write ζ for $\zeta(x)$ and f for $f(\zeta)$ throughout this proof. Now $\frac{\zeta(e^\zeta - 1)}{f} = x$ from which we get

$$\frac{d\zeta}{dx} = \frac{f^2}{(e^\zeta - 1)^2 - \zeta^2 e^\zeta} \quad (11)$$

and note that $\frac{d\zeta}{dx} > 0$ for $\zeta > 0$, see (13) below. Taking the derivative of $\log(g(x))$ we get

$$\begin{aligned} \frac{d}{dx} \log(g(x)) &= \frac{d}{dx} \left(x \log(e^\zeta - 1) + (1-x) \log(e^\zeta - \zeta - 1) \right) \\ &= \log\left(\frac{e^\zeta - 1}{f}\right) + \frac{d\zeta}{dx} \left(x \frac{e^\zeta}{e^\zeta - 1} + (1-x) \frac{e^\zeta - 1}{f} \right). \end{aligned}$$

Now $x = \frac{\zeta(e^\zeta - 1)}{f}$ so

$$\begin{aligned} x \frac{e^\zeta}{e^\zeta - 1} + (1 - x) \frac{e^\zeta - 1}{f} &= \frac{\zeta e^\zeta}{f} + \frac{f - \zeta(e^\zeta - 1)}{f} \frac{e^\zeta - 1}{f} \\ &= \frac{\zeta e^\zeta (e^\zeta - \zeta - 1) + (e^\zeta - \zeta - 1 - \zeta e^\zeta + \zeta)(e^\zeta - 1)}{f^2} \\ &= \frac{(e^\zeta - 1)^2 - \zeta^2 e^\zeta}{f^2} \\ &= \frac{dx}{d\zeta}. \end{aligned}$$

Thus we have

$$\frac{d}{dx} \log(g(x)) = \log\left(\frac{e^\zeta - 1}{f}\right) + 1. \quad (12)$$

Taking the second derivative we get

$$\begin{aligned} \frac{d^2}{dx^2} \log(g(x)) &= \frac{d}{dx} \left(\log\left(\frac{e^\zeta - 1}{f}\right) + 1 \right) \\ &= \frac{f}{e^\zeta - 1} \frac{e^\zeta (e^\zeta - \zeta - 1) - (e^\zeta - 1)^2}{f^2} \frac{d\zeta}{dx} \\ &= \frac{1}{(e^\zeta - 1)f} \frac{d\zeta}{dx} \left(-(\zeta - 1)e^\zeta - 1 \right) \end{aligned}$$

and since $-(\zeta - 1)e^\zeta - 1$ is strictly negative for $\zeta > 0$ we get that $g(x)$ is log-concave \square

Lemma 6 $\zeta(x)$ is concave as a function of x .

Proof: We begin with (11). We note that the denominator

$$(e^\zeta - 1)^2 - \zeta^2 e^\zeta = \sum_{k=4}^{\infty} (2^k - 2 - k(k-1)) \frac{\zeta^k}{k!} \geq 0. \quad (13)$$

Then we have

$$\frac{d^2 \zeta}{dx^2} = \frac{2(1 + \zeta) + e^\zeta(-6 - e^{2\zeta}(2 + \zeta(\zeta - 4)) + \zeta^2(5 + \zeta(\zeta + 2)) + e^\zeta(6 - 2\zeta(2\zeta + 3)))}{((e^\zeta - 1)^2 - \zeta^2 e^\zeta)^2} \frac{d\zeta}{dx}.$$

Now let

$$\phi(u) = \sum_{n=0}^{\infty} \phi_n u^n = 2(1 + u) + \psi(u)$$

where

$$\psi(u) = \sum_{n=0}^{\infty} \psi_n u^n = e^u(-6 - e^{2u}(2 + u(u - 4)) + u^2(5 + u(u + 2)) + e^u(6 - 2u(2u + 3))).$$

We check that $\psi_0 = -2$ and $\psi_1 = 0$ which implies that $\phi_0 = \phi_1 = 0$. One can finish the argument by checking that

$$\psi_n = -\frac{3^{n-2}(n^2 - 13n + 18) + 2^n(n^2 + 2n - 6) - (n^4 - 4n^3 + 10n^2 - 7n - 6)}{n!} \leq 0$$

for $n \geq 2$. This is simply a matter of checking for small values until the 3^n term dominates. \square
Next let

$$H(u) = \log f(u) - u - 2 \log u = \log \left(\frac{e^u - u - 1}{u^2 e^u} \right)$$

Lemma 7 $H(u)$ is convex as a function of u .

Proof:

$$\begin{aligned} \frac{d^2}{du^2} H(u) &= \frac{d}{du} \left(\frac{e^u - 1}{f(u)} - 1 - \frac{2}{u} \right) \\ &= \frac{e^u(e^u - 1 - u) - (e^u - 1)^2}{f(u)^2} + \frac{2}{u^2} \\ &= \frac{e^u - 1 - ue^u}{f(u)^2} + \frac{2}{u^2} \\ &= \frac{u^2(e^u - 1 - ue^u) + 2(e^u - 1 - u)^2}{u^2 f(u)^2} \\ &= \frac{2e^{2u} + u^2 e^u + u^2 + 4u + 2 - u^3 e^u - 4ue^u - 4e^u}{u^2 f(u)^2} \end{aligned}$$

Let

$$\phi(u) = 2e^{2u} + u^2 e^u + u^2 + 4u + 2 - u^3 e^u - 4ue^u - 4e^u = \sum_{n=0}^{\infty} \phi_n u^n.$$

Direct computation gives $\phi_0 = \phi_1 = \phi_2 = 0$ and for $n \geq 3$

$$\phi_n = \frac{1}{n!} (2^{n+1} + n(n-1) - n(n-1)(n-2) - 4n - 4).$$

One can then check that $\phi_3 = \phi_4 = \phi_5 = 0 < \phi_n$ for $n \geq 6$. Thus $\frac{d^2}{du^2} H(u) \geq 0$ implying that $H(u)$ is convex. \square

3.3 The case $|n - m| = o(n^{7/8})$

We will first prove Theorem 1 under the assumption that $|n - m| = o(n^{7/8})$ and then in Sections 3.4 and 3.5 we will extend the result to arbitrary m . We deal first with the existence probability for a minimal L -witness and leave R -witnesses until Section 3.3.2. We then combine these results to finish the case $|n - m| = o(n^{7/8})$ in Section 3.3.3.

3.3.1 Case 1: L -witnesses

For

$$k \leq k_0 \text{ and } \ell \leq k - 1$$

define

$$\pi_L(k, \ell, D) = Pr(\exists A, B : |A| = k, |B| = \ell, N_\Gamma(A) = B, d(B) = D \geq dk, d_A(b) \geq 2, \forall b \in B)$$

where $d(B) = \sum_{b \in B} d_A(b)$. This is the probability of the existence of a minimal L -witness A of size of size k that has a neighbour set B of size ℓ of total degree D .

We observe that if we condition on the degrees $d_b, b \in R$ then we will be able to use a bipartite configuration model for the bipartite graph Γ . Let $\mathbf{d} = (d_1, d_2, \dots, d_m)$ be a sequence of non-negative integers with $d_1 + d_2 + \dots + d_m = M = nd$. Let W_L, W_R be two disjoint copies of $[M]$ and let $W_{i,R} = [d_1 + \dots + d_{i-1} + 1, d_1 + \dots + d_i], i \in [m]$, partition W into sets of size d_1, d_2, \dots, d_m . Let ϕ be a uniform random bijection between W_L and W_R . Given ϕ we define the bipartite (multi-)graph Γ_ϕ as follows: If $\phi(x) = y$ then we add the edge $(\lfloor (x-1)/d \rfloor + 1, i)$ where $b \in W_{i,R}$. This bipartite graph has the same distribution as Γ_1 conditional on the degrees of the vertices in R .

Given this model, we can easily deal with small k .

Case 1.0: $2 \leq k \leq n/\log^4 n$.

We have

$$\begin{aligned} A_{14} &= \sum_{\ell < k=2}^{n/\log^4 n} \sum_{D=dk}^{k \log n} \pi_L(k, \ell, D) \leq \sum_{D=dk}^{k \log n} \binom{n}{k} \binom{m}{\ell} \left(\frac{D}{dn}\right)^{dk} \\ &\leq \sum_{\ell < k=2}^{n/\log^4 n} \sum_{D=dk}^{k \log n} \left(\frac{ne}{k}\right)^k \left(\frac{me}{\ell}\right)^\ell \left(\frac{k \log n}{n}\right)^{dk} \leq \sum_{\ell < k=2}^{n/\log^4 n} \sum_{D=dk}^{k \log n} \left(\frac{e^{2+o(1)} k^{d-2} \log^d n}{n^{d-2}}\right)^k = o(1). \end{aligned} \quad (14)$$

The notation $\sum_{\ell < k=r}^s$ is short for $\sum_{k=r}^s \sum_{\ell=1}^{k-1}$. We use the notation A_{14} so that the reader can easily refer back to the equation giving its definition. A_{14} is the first of several sums that together show the unlikelihood of a witness. We will display them as they become available and use them in Sections 3.3.3, 3.4 and 3.5.

We can restrict our attention to $D \leq k \log n$ because of (7).

Case 1.1: $n/\log^4 n \leq k \leq k_0$.

We must work much harder when k is large. We now estimate, with z as defined in (6),

$$\begin{aligned} \pi_L(k, \ell, D) &\leq O(n^{1/2}) \binom{n}{k} \binom{m}{\ell} \sum_{\substack{2 \leq x_b \leq d_b, \forall b \in [\ell] \\ \sum_b x_b = kd \\ \sum_{b \in [\ell]} d_b = D, \sum_{b \notin [\ell]} d_b = dn - D}} \prod_{b=1}^m \frac{z^{d_b}}{d_b! f(z)} \prod_{b=1}^{\ell} \binom{d_b}{x_b} (kd)! \prod_{i=0}^{dk-1} \frac{1}{dn-i} \end{aligned} \quad (15)$$

Explanation of (15): Choose sets A, B in $\binom{n}{k} \binom{m}{\ell}$ ways. Choose degrees $d_b, b \in R$ with probability $O(n^{1/2}) \prod_{b=1}^m \frac{z^{d_b}}{d_b! f(z)}$ such that $\sum_{b \in B} d_b = D, \sum_{b \notin B} d_b = dn - D$ for some $D \geq 2\ell$. Choose the degrees $x_a, a \in A$ in the sub-graph induced by $A \cup B$. Having fixed the degree sequence, we swap to the configuration model

Choose the configuration points associated the $x_a, a \in A$ in $\prod_{a \in A} \binom{d}{x_a}$ ways. Assign these D choices of points associated with A in $D!$ ways. Then multiply by the probability $(kd)! \prod_{i=0}^{kd-1} \frac{1}{dn-i}$ of a given pairing of points in A .

So, after writing $d_b = x_b + y_b$ for $b \in [\ell]$ we get,

$$\pi_L(k, \ell, D) = O(n^{1/2}) \binom{n}{k} \binom{m}{\ell} \frac{(d(n-k))! (kd)! z^{dn}}{(dn)! f(z)^m} \times$$

$$\begin{aligned}
& \left(\sum_{\substack{2 \leq x_b, \forall b \in [\ell] \\ \sum_b x_b = kd}} \prod_{b=1}^{\ell} \frac{1}{x_b!} \right) \left(\sum_{\substack{2 \leq d_b, \forall b \in [\ell] \\ \sum_b d_b = dn - D}} \prod_{b=\ell+1}^m \frac{1}{d_b!} \right) \left(\sum_{\substack{0 \leq y_b, \forall b \in [\ell] \\ \sum_b y_b = D - kd}} \prod_{b=1}^{\ell} \frac{1}{y_b!} \right) \\
&= O(n^{1/2}) \binom{n}{k} \binom{m}{\ell} \frac{(d(n-k))! (kd)! z^{dn}}{(dn)! f(z)^m} \times \\
&\quad \left([u^{kd}] (e^u - 1 - u)^\ell \right) \left([u^{dn-D}] (e^u - 1 - u)^{m-\ell} \right) \left([u^{D-kd}] e^{u\ell} \right) \quad (16) \\
&\leq O(n^{1/2}) e^{O(|m-n|)} \binom{n}{k} \binom{m}{k-1} \frac{(d(n-k))! (kd)! z^{dn}}{(dn)! f(z)^m} \frac{f(z)^\ell}{z^{kd}} \frac{f(\zeta_1)^{m-\ell}}{\zeta_1^{dn-D}} \frac{\ell^{D-dk}}{(D-kd)!}.
\end{aligned}$$

Here we use the general notation that $[x^r]\phi(x)$ is the coefficient of x^r in the series expansion of ϕ around zero. Then if $A(x) = \sum_{n=0}^{\infty} a_n x^n$ where $a_n \geq 0$ for $n \geq 0$ we have $a_n \leq A(\zeta)/\zeta^n$ for any positive ζ and $A(\zeta)/\zeta^n$ is minimised at ζ satisfying $\zeta A'(\zeta)/A(\zeta) = n$.

Here we take

$$\zeta_1 = \zeta(y) \text{ where } y = \frac{dn - D}{n - k} \geq 2 - O\left(\frac{|m-n|}{n}\right)$$

due to our minimum degree assumption for R . Indeed, our minimum degree assumption implies that

$$dn - D \geq 2(m - \ell) \geq 2(m - k) = 2(n - k) + 2(m - n). \quad (17)$$

We get an upper bound for any choice of ζ_1 , although if $y_0 = (dn - D)/(m - \ell)$ then $\zeta_1(y_0)$ (see (8)) gives the smallest upper bound. It is convenient to use $\zeta_1(y)$ instead of this. Later in the proof we choose other values for ζ_1 in this bound, but we will always choose $1 < \zeta_1 < 2 < z$.

The term $e^{O(|m-n|)}$ accounts for replacing $\binom{m}{\ell}$ by $\binom{m}{k-1}$ for k exceeding $m/2$. So,

$$\pi_L(k, \ell, D) \leq O(n^{1/2} e^{O(|m-n|)}) \binom{n}{k} \binom{m}{k-1} \frac{(d(n-k))! (kd)! z^{dn}}{(dn)! f(z)^m} \frac{f(z)^\ell}{z^{kd}} \frac{f(\zeta_1)^{m-\ell}}{\zeta_1^{dn-D}} \frac{(k-1)^{D-dk}}{(D-kd)!}. \quad (18)$$

Observe next that

$$\frac{dn - D}{n - k} - \frac{dn}{m} = \frac{dn(m-n) + (dk - D)n + D(n-m)}{m(n-k)} \leq O\left(\frac{|m-n|}{n}\right).$$

Hence

$$f(\zeta_1) \leq f(z) + O\left(\frac{|m-n|}{n}\right) \text{ and therefore } f(z)^\ell f(\zeta_1)^{m-\ell} \leq e^{O(|n-m|)} f(z)^k f(\zeta_1)^{m-k}. \quad (19)$$

Continuing, we find that

$$\pi_L(k, \ell, D) \leq O\left(\frac{e^{O(|m-n|)}}{m^{1/2}}\right) \frac{\binom{n}{k} \binom{m}{k}}{\binom{dn}{dk}} \left(\frac{z^d}{f(z)^{\frac{m-k}{n-k}}} \frac{f(\zeta_1)^{\frac{m-k}{n-k}}}{\zeta_1^{\frac{dn-D}{n-k}}}\right)^{n-k} \left(\frac{ek}{D-dk}\right)^{D-dk}. \quad (20)$$

Putting $k = an$ and $m = \beta n$ and $h(u) = u^u(1-u)^{1-u}$ and $x = d - y = \frac{D-dk}{n-k}$ where $0 \leq x \leq d - 2 + O(|n-m|/n)$ (see (17)) we obtain, after substituting $\binom{n}{k} = O\left(\frac{1}{k^{1/2}h(a)^n}\right)$, $\binom{m}{k} = O\left(\frac{1}{k^{1/2}h(a/\beta)^n}\right)$ and $\binom{dn}{dk} = O\left(\frac{1}{k^{1/2}h(a)^{dn}}\right)$,

$$\pi_L(k, \ell, D) \leq O\left(\frac{e^{O(|m-n|)}}{n^{1/2}}\right) \left(\frac{h(a)^{d-1}}{h(a/\beta)^\beta}\right)^n \left(\frac{z^d}{f(z)^{\frac{\beta-a}{1-a}}} \frac{f(\zeta_1)^{\frac{\beta-a}{1-a}}}{\zeta_1^{d-x}} \left(\frac{e^{\frac{a}{1-a}}}{x}\right)^x\right)^{n-k}. \quad (21)$$

Because $\beta = 1 + o(n^{-1/8})$, we see that

$$\left(\frac{h(a)}{h(a/\beta)^\beta} \right)^n \left(\frac{f(\zeta_1)^{\frac{\beta-1}{1-a}}}{f(z)^{\frac{\beta-1}{1-a}}} \right)^{n-k} \quad (22)$$

$$= \left(\beta^a \left(1 + \frac{a(1-\beta)}{\beta-a} \right)^{1-a} \left(\frac{1}{1-a/\beta} \right)^{\beta-1} \right)^n \left(\frac{f(\zeta_1)}{f(z)} \right)^{(\beta-1)n} \quad (23)$$

$$= e^{o(n^{7/8})}, \quad (24)$$

after using (19).

Thus (21) becomes

$$\pi_L(k, \ell, D) = O \left(\frac{1}{n^{1/2}} \right) e^{o(n^{7/8})} h(a)^{(d-2)n} \left(\frac{z^d}{f(z)} \frac{f(\zeta_1)}{\zeta_1^{d-x}} \left(\frac{e^{-\frac{a}{1-a}}}{x} \right)^x \right)^{n-k}. \quad (25)$$

Case 1.1.1: $d \geq 5$.

Observe (see (10)) that

$$\frac{z^d}{f(z)} \frac{f(\zeta_1)}{\zeta_1^{d-x}} = \frac{d^d}{g(d)} \frac{g(d-x)}{(d-x)^{d-x}} e^{o(n^{7/8})}$$

where $g(x)$ is as defined in (9). The term $e^{o(n^{7/8})}$ is derived as follows: In (10) $z = \zeta(dn/m)$ and so $z^{dn/m}/f(z) = (dn/m)^{dn/m}/g(dn/m)$ and $e^{o(n^{7/8})}$ is the correction for replacing $z^d/f(z)$ by $d^d/g(d)$.

It follows from (12) that

$$-\log \left(\frac{g(d-x)}{g(d)} \right) = \int_{d-x}^d \frac{d}{dt} \log(g(t)) dt \geq \int_{d-x}^d 1 dt. \quad (26)$$

Plugging this into the last parenthesis of (25) gives

$$\pi_L(k, \ell, D) = O \left(\frac{1}{n^{1/2}} \right) e^{o(n^{7/8})} \left(h(a)^{d-2} \left(\frac{d^d}{(d-x)^{d-x}} \left(\frac{\frac{a}{1-a}}{x} \right)^x \right)^{1-a} \right)^n \quad (27)$$

We will use this to prove that

$$A_{28} = \sum_{\ell < k = \varepsilon_L n} \sum_{D = dk}^{n(1-2/d)dk + n^{1/10}} \pi_L(k, \ell, D) \leq \sum_{\ell < k = \varepsilon_L n} \sum_{D = dk}^{n(1-2/d)dk + n^{1/10}} O \left(\frac{1}{n^{1/2}} \right) h(a)^{(d-2)n} e^{o(n)} = o(1). \quad (28)$$

where ε_L is some small constant defined in (32) below.

The bound for A_{28} comes from (27), using the fact that $h(a)$ is bounded away from 1 and $x = o(1)$ in this summation.

The main term $h(a)^{d-2} \left(d^d \frac{1}{(d-x)^{d-x}} \left(\frac{\frac{a}{1-a}}{x} \right)^x \right)^{1-a}$ in (27) is maximized when $x = ad$, provided

$ad \leq d - 2$ or $k \leq n(1 - \frac{2}{d})$. This in turn gives

$$\begin{aligned}
& \pi_L(k, \ell, D) \\
&= O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8})} \left((1 + o(n^{-1/8}))h(a)^{d-2} \left((1 + o(n^{-1/8}))d^d \frac{1}{(d-ad)^{d-ad}} \frac{1}{((1-a)d)^{ad}} \right)^{1-a} \right)^n \\
&= O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8})} \left((1 + o(n^{-1/8}))h(a)^{d-2} \left(\frac{d^d}{d^{d-ad}d^{ad}}(1-a)^{-d} \right)^{1-a} \right)^n \\
&\leq O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8})} \left(a^{a(d-2)}(1-a)^{-2(1-a)} \right)^n \tag{29}
\end{aligned}$$

The function $\rho_d(a) = a^{a(d-2)}(1-a)^{-2(1-a)}$ is at most 1 and is log-convex in a on $[0, 1 - \frac{2}{d}]$. Indeed, if $L_1(a) = \log \rho_d(a)$ then

$$\frac{dL_1}{da} = d + (d-2)\log a + 2\log(1-a) \tag{30}$$

$$\frac{d^2L_1}{da^2} = \frac{d-2-da}{a(1-a)} \tag{31}$$

We have $L_1(0) = 0$ and $L_1'(0) = -\infty$. It follows that for every $K > 0$ there exists a constant $\varepsilon_L(K, d) > 0$ such that

$$\rho_d(a) \leq e^{-Ka} \quad \text{for } a \leq \varepsilon_L(K). \tag{32}$$

We let $\varepsilon_L = \varepsilon_L(1, d)$. This completes the proof of (28).

In truth we should put

$$\rho_d(a) \leq \max \{ e^{-Ka}, \psi(d) \}$$

where

$$\psi(d) = \left(1 - \frac{2}{d}\right)^{(d-2)^2/d} \left(\frac{2}{d}\right)^{-4/d}.$$

For small $a < \varepsilon_L(K, d)$, $e^{-Ka} > \psi(d)$ and so it suffices to use (32). Thus

$$A_{33} = \sum_{\ell < k = n/\log^4 n}^{\varepsilon_L n} \sum_{D = dk}^{k \log n} \pi_L(k, \ell, D) = \sum_{\ell < k = n/\log^2 n}^{\varepsilon_L n} \sum_{D = dk}^{k \log n} O\left(\frac{1}{n^{1/2}}\right) e^{-Kk + o(n^{7/8})} = o(1). \tag{33}$$

The bound for A_{33} comes from (29) and (32).

Now $\psi(d)$ decreases in d and is $< .9$ for $d \geq 5$. So if $d \geq 5$ then for some constant $0 < \xi < 1$ we have

$$A_{34} = \sum_{\ell < k = \varepsilon_L n}^{k_0} \sum_{D = dk + n^{1/10}}^{k \log n} \pi_L(k, \ell, D) = \sum_{\ell < k = \varepsilon_L n}^{k_0} \sum_{D = dk + n^{1/10}}^{k \log n} O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8})} \xi^n = o(1). \tag{34}$$

The bound for A_{34} comes from (29) and using the fact that $\rho_d(a) \leq e^{-a}$.

We have now completed the analysis for $d \geq 5$ and witnesses $A \subseteq L$.

Case 1.1.2: $d = 3, 4$.

Now consider the cases $d = 3, 4$. Putting $\beta_3 = .15$, $\beta_4 = .49$ we note that $\rho_d(\beta_d) \leq .995$ for $d = 3, 4$ and so arguing as above we have

$$A_{35} = \sum_{\ell < k = \varepsilon_L n}^{\beta_d n} \sum_{D = dk + n^{1/10}}^{k \log n} \pi_L(k, \ell, D) = \sum_{\ell < k = \varepsilon_L n}^{\beta_d n} \sum_{D = dk + n^{1/10}}^{\log n} O\left(\frac{\log n}{n^{1/2}}\right) e^{o(n^{7/8})} \xi^n = o(1). \quad (35)$$

Because we can choose any value for ζ_1 in the bound (25) we can simplify matters by choosing $\zeta_1 = \eta > 1$ independent of x to get

$$\pi_L(k, D) = O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8})} h(a)^{(d-2)n} \left(\frac{z^d}{f(z)} \frac{f(\eta)}{\eta^d} \left(\frac{e\eta a}{x(1-a)}\right)^x\right)^{n-k}. \quad (36)$$

Now

$$\left(\frac{\eta e a}{(1-a)x}\right)^x \leq \exp\left\{\frac{\eta a}{1-a}\right\} \quad (37)$$

and so

$$\pi_L(k, D) \leq O\left(\frac{k}{n^{1/2}}\right) \left(h(a)^{d-2} e^{\eta a} e^{o(n^{-1/8})} \left(\frac{z^d}{f(z)} \frac{f(\eta)}{\eta^d}\right)^{1-a}\right)^n. \quad (38)$$

Now the function $L_2(a) = h(a)^{d-2} e^{\eta a} \left(\frac{z^d}{f(z)} \frac{f(\eta)}{\eta^d}\right)^{1-a}$ is log-convex in a . Our initial choice of η will be 1.5 and we note that with this choice, when $d = 4$, $L_2(.49), L_2(.51) < .9$ and so

$$A_{39} = \sum_{\ell < k = \beta_{4n}}^{k_0} \sum_{D = 4k}^{k \log n} \pi_L(k, \ell, D) \leq \sum_{\ell < k = \beta_{4n}}^{k_0} \sum_{D = 4k}^{k \log n} O\left(\frac{\log n}{n^{1/2}}\right) e^{o(n^{7/8})} (.9)^n = o(1). \quad (39)$$

When $d = 3$, with the same choice for η , we have $L_2(.15), L_2(2/5) < .98$ and so

$$A_{40} = \sum_{\ell < k = .15n}^{2n/5} \sum_{D = 3k + n^{1/10}}^{k \log n} \pi_L(k, \ell, D) \leq \sum_{\ell < k = .15n}^{2n/5} \sum_{D = 3k + n^{1/10}}^{k \log n} O\left(\frac{\log n}{n^{1/2}}\right) e^{o(n^{7/8})} (.98)^n = o(1). \quad (40)$$

We repeat this idea once more. Putting $\eta = 1.1$ we get $L_2(2/5), L_2(.51) < .98$ from which we deduce that

$$A_{41} = \sum_{\ell < k = 2n/5}^{k_0} \sum_{D = 3k + n^{1/10}}^{k \log n} \pi_L(k, \ell, D) \leq \sum_{\ell < k = 2n/5}^{k_0} \sum_{D = 3k + n^{1/10}}^{k \log n} O\left(\frac{\log n}{n^{1/2}}\right) e^{o(n^{7/8})} (.98)^n = o(1). \quad (41)$$

3.3.2 Case 2: R -witnesses

Now let us estimate the probability of a violation of Hall's condition with $A \subseteq R$. We once again begin with arbitrary m .

For

$$k \leq k_0 = \max\{m, n\}/2 \text{ and } \ell \leq k - 1$$

define

$$\begin{aligned} \pi_R(k, \ell, D) = & \\ \mathbb{P}(\exists A \subseteq R, B \subseteq L : |A| = k, |B| = \ell, N_\Gamma(A) \subseteq B, d_A(a) \geq 2, \forall a \in A, d_R(A) = D) \leq & \\ O(n^{1/2}) \binom{m}{k} \binom{n}{\ell} \sum_{\substack{2 \leq d_a, \forall a \in [m] \\ 2 \leq x_b \leq d, \forall b \in [\ell] \\ \sum_{a \in [k]} d_a = \sum_{b \in [\ell]} x_b = D \\ \sum_{a \notin [k]} d_a = dn - D}} \prod_{a=1}^m \frac{z^{d_a}}{d_a! f(z)} \prod_{b=1}^{\ell} \binom{d}{x_b} D! \prod_{i=0}^{D-1} \frac{1}{dn-i} \end{aligned} \quad (42)$$

Explanation of (42): Choose sets A, B in $\binom{m}{k} \binom{n}{\ell}$ ways. Choose degrees $d_a, a \in R$ with probability $O(n^{1/2}) \prod_{a=1}^m \frac{z^{d_a}}{d_a! f(z)}$ such that $\sum_{a \in A} d_a = D, \sum_{a \notin A} d_a = dn - D$ for some $D \geq 2k$. Choose the degrees $x_b, b \in B$ in the sub-graph induced by $A \cup B$. Having fixed the degree sequence, swap to the configuration model [4]. Choose the configuration points associated with the $x_b, b \in B$ in $\prod_{b \in B} \binom{d}{x_b}$ ways. Then multiply by the probability $D! \prod_{i=0}^{D-1} \frac{1}{dn-i}$ of a given pairing of points in A .

So

$$\begin{aligned} \pi_R(k, \ell, D) = & \\ O(n^{1/2}) \binom{m}{k} \binom{n}{\ell} \frac{z^{dn} D! (dn - D)!}{f(z)^m (dn)!} & \\ \times \left(\sum_{\substack{2 \leq d_a, \forall a \in [k] \\ \sum_a d_a = D}} \prod_{a=1}^k \frac{1}{d_a!} \right) \left(\sum_{\substack{2 \leq d_a, \forall a \notin [k] \\ \sum_a d_a = dn - D}} \prod_{a=k+1}^m \frac{1}{d_a!} \right) \left(\sum_{\substack{2 \leq x_b \leq d, \forall b \in [\ell] \\ \sum_b x_b = D}} \prod_{b=1}^{\ell} \binom{d}{x_b} \right) = & \\ O(n^{1/2}) \binom{m}{k} \binom{n}{\ell} \frac{z^{dn}}{f(z)^m} \frac{1}{\binom{dn}{D}} & \\ \times \left([u^D](e^u - 1 - u)^k \right) \left([u^{dn-D}](e^u - 1 - u)^{m-k} \right) \left([u^D]((1+u)^d - (1+du))^\ell \right) \leq & \\ O(n^{1/2}) e^{2k|m-n|/n} \binom{m}{k} \binom{n}{k-1} \frac{z^{dn}}{f(z)^m} \frac{1}{\binom{dn}{D}} & \\ \times \left([u^D](e^u - 1 - u)^k \right) \left([u^{dn-D}](e^u - 1 - u)^{m-k} \right) \left([u^D]((1+u)^d - (1+du))^k \right) \leq \end{aligned} \quad (43)$$

$$O(n^{1/2}) e^{2k|m-n|/n} \binom{m}{k} \binom{n}{k-1} \frac{z^{dn}}{f(z)^m} \frac{1}{\binom{dn}{D}} \frac{f(\zeta_1)^k}{\zeta_1^D} \frac{f(\zeta_2)^{m-k}}{\zeta_2^{dn-D}} \binom{dk}{D}. \quad (44)$$

Our choice of ζ_1, ζ_2 will differ according to circumstances.

Case 2.0: $k \leq n^{9/10}$.

In this case we first use a crude bound in place of (43), (44):

$$\begin{aligned} \pi_R(k, \ell, D = \theta k) \leq \binom{m}{k} \binom{n}{k-1} \left(\frac{k}{n} \right)^{\theta k} & \\ \leq \frac{2k}{n} \left(\left(\frac{k}{n} \right)^{\theta-2} e^{2+o(n^{-1/8})} \right)^k. \end{aligned} \quad (45)$$

To get (45) we choose sets $A \subseteq R, B \subseteq L$ and then use $(k/n)^D$ as an upper bound on the probability that $N_\Gamma(A) \subseteq B$.

Thus if

$$\theta_0 = 2 + \frac{100}{\log n}$$

then

$$B_{46} = \sum_{\ell < k=2}^{n^{9/10}} \sum_{D \geq \theta_0 k} \pi_R(k, \ell, D) = o(1). \quad (46)$$

Case 2.0.1: $d \geq 5$.

When $2 \leq \theta \leq \theta_0$ we will use (43). In this case we take $\zeta_1 = \varepsilon$ for small ε which implies that $f(\zeta_1) = \varepsilon^2/2 + O(\varepsilon^3)$ and $\zeta_2 = z = \zeta(d) + o(1)$. With $\binom{dn}{D} = \left(\frac{den}{2k}\right)^{k+o(k)}$ this gives

$$\begin{aligned} & \pi_R(k, \ell, D) \\ & \leq O\left(\frac{k}{n^{1/2}}\right) \left((1 + O(\varepsilon)) \frac{n^2 e^2}{k^2} \left(\frac{2k}{den}\right)^2 \frac{\zeta(d)^2}{2f(\zeta(d))} \right)^k ([u^D]((1+u)^d - (1+du))^k) \\ & = O\left(\frac{k}{n^{1/2}}\right) \left((1 + O(\varepsilon)) \frac{2}{d^2} \frac{\zeta(d)^2}{f(\zeta(d))} \right)^k ([u^D]((1+u)^d - (1+du))^k). \end{aligned} \quad (47)$$

If we use the bound

$$([u^D]((1+u)^d - (1+du))^k) \leq \binom{dk}{D} \leq \left(\frac{de}{2}\right)^{(2+o(1))k}$$

then (47) becomes

$$\pi_R(k, \ell, D) \leq O\left(\frac{k}{n^{1/2}}\right) \left((1 + O(\varepsilon)) \frac{\zeta(d)^2 e^2}{2f(\zeta(d))} \right)^k. \quad (48)$$

Another calculation shows that if $x \geq 4.5$ then $x^2 e^2 / 2f(x) \leq .89$ and then we have

$$d \geq 5 \text{ implies } \zeta(d) \geq 4.5 \text{ which implies } \pi_R(k, \ell, D) \leq O\left(\frac{k}{n^{1/2}}\right) (.9)^k \quad (49)$$

So we are only left with $d = 3$ or 4 for the case of small k .

Case 2.0.2: $d = 4$

We use

$$\begin{aligned} [u^D]((1+u)^4 - 1 - 4u)^k &= [u^{D-2k}](6 + 4u + u^2)^k \\ &= 6^k [u^{D-2k}] \left(1 + \frac{4}{6}u + \frac{u^2}{6}\right)^k \\ &\leq 6^k [u^{D-2k}] \left(1 + \frac{u}{2}\right)^{2k} \\ &= 6^k \frac{\binom{2k}{D-2k}}{2^{D-2k}}. \end{aligned} \quad (50)$$

Equation (47) now becomes

$$\pi_R(k, \ell, D) \leq O\left(\frac{k}{n^{1/2}}\right) \left((1 + O(\varepsilon)) \frac{12}{16} \frac{\zeta(4)^2}{f(\zeta(4))} \right)^k = O\left(\frac{k}{n^{1/2} 2^k}\right) \quad (51)$$

since $3.5 \leq \zeta(4) \leq 3.6$ and $(3.6)^2/f(3.5) \leq 1/2$.

Case 2.0.3: $d = 3$

We use

$$\begin{aligned} [u^D]((1+u)^3 - 1 - 3u)^k &= [u^{D-2k}](3+u)^k \\ &= 3^{3k-D} \binom{k}{D-2k}. \end{aligned} \quad (52)$$

In this case (47) becomes

$$\pi_R(k, \ell, D) \leq O\left(\frac{k}{n^{1/2}}\right) \left((1 + O(\varepsilon)) \frac{6}{9} \frac{\zeta(3)^2}{f(\zeta(3))} \right)^k = O\left(\frac{k 3^k}{n^{1/2} 4^k}\right). \quad (53)$$

since $2.1 \leq \zeta(3) \leq 2.2$ and $(2.2)^2/f(2.1) < 1$.

It follows from (49), (51) and (52) that

$$B_{54} = \sum_{\ell < k=2}^{n^{9/10}} \sum_{2k \leq D \leq \theta_0 k} \pi_R(k, \ell, D) = o(1). \quad (54)$$

Case 2.1: $k \geq n^{9/10}$.

Going back to (43) we initially take $\zeta_1 = \zeta(D/k)$ and $\zeta_2 = \zeta\left(\frac{dn-D}{n-k}\right)$. It is convenient to use this definition of ζ_2 in place of the more natural $\zeta\left(\frac{dn-D}{m-k}\right)$.

So

$$\begin{aligned} &\pi_R(k, \ell, D) \\ &= O(n^{1/2}) e^{2k|m-n|/n} \binom{m}{k} \binom{n}{k} \left(\frac{dk}{D}\right) \left(\frac{f(\zeta_1)}{f(z)}\right)^k \left(\frac{f(\zeta_2)}{f(z)}\right)^{m-k} \left(\frac{z}{\zeta_1}\right)^D \left(\frac{z}{\zeta_2}\right)^{dn-D} \end{aligned} \quad (55)$$

$$= O\left(\frac{1}{n^{1/2}}\right) e^{2a|m-n|} \left(\frac{h(\theta a/d)^d}{h(a)h(a/\beta)^\beta h(\theta/d)^{ad}} \left(\frac{f(\zeta_1)}{f(z)}\right)^a \left(\frac{f(\zeta_2)}{f(z)}\right)^{\beta-a} \left(\frac{z}{\zeta_1}\right)^{\theta a} \left(\frac{z}{\zeta_2}\right)^{d-\theta a} \right)^n \quad (56)$$

$$= O\left(\frac{1}{n^{1/2}}\right) e^{2a|m-n|} \left(\frac{h(\theta a/d)^d}{h(a)h(a/\beta)^\beta h(\theta/d)^{ad}} \frac{z^d}{f(z)^\beta} \frac{f(\zeta_1)^a}{\zeta_1^{\theta a}} \frac{f(\zeta_2)^{\beta-a}}{\zeta_2^{d-\theta a}} \right)^n, \quad (57)$$

where $a = k/n$, $m = \beta n$ and $D = \theta k \leq dk$.

We argue as in (24) that

$$\frac{h(a)}{h(a/\beta)^\beta} \frac{f(\zeta_2)^{\beta-1}}{f(z)^{\beta-1}} = e^{o(n^{-1/8})}.$$

Thus, (57) becomes

$$\pi_R(k, \ell, D) \leq O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8})} \frac{h(\theta a/d)^d}{h(a)^2 h(\theta/d)^{ad}} \frac{z^d}{f(z)} \frac{f(\zeta_1)^a}{\zeta_1^{\theta a}} \frac{f(\zeta_2)^{1-a}}{\zeta_2^{d-\theta a}} \right)^n. \quad (58)$$

It follows from Lemma 5 that we can upper bound

$$\begin{aligned}
\frac{z^d}{f(z)} \left(\frac{f(\zeta_1)}{\zeta_1^\theta} \right)^a \left(\frac{f(\zeta_2)}{\zeta_2^{\frac{d-a\theta}{1-a}}} \right)^{1-a} &= \frac{d^d}{g(d)} \frac{g(\theta)^a g\left(\frac{d-a\theta}{1-a}\right)^{1-a}}{\theta^{a\theta} \left(\frac{d-a\theta}{1-a}\right)^{d-a\theta}} \\
&\leq \frac{g(a\theta + (1-a)\frac{d-a\theta}{1-a})}{g(d)} \frac{d^d}{\theta^{a\theta} \left(\frac{d-a\theta}{1-a}\right)^{d-a\theta}} \\
&= \frac{a^{a\theta} (1-a)^{d-a\theta}}{\left(\left(\frac{a\theta}{d}\right)^{\frac{a\theta}{d}} \left(1 - \frac{a\theta}{d}\right)^{1-\frac{a\theta}{d}} \right)^d} \\
&= \frac{a^{a\theta} (1-a)^{d-a\theta}}{h\left(\frac{a\theta}{d}\right)^d}
\end{aligned}$$

Plugging this into (58) gives

$$\pi_R(k, \ell, D) \leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8})} a^{a\theta} (1-a)^{d-a\theta}}{h(a)^2 h\left(\frac{\theta}{d}\right)^{ad}} \right)^n \quad (59)$$

Now let $R_1(\theta) = \log\left(\frac{a^{a\theta} (1-a)^{d-a\theta}}{h(a)^2 h\left(\frac{\theta}{d}\right)^{ad}}\right)$. Then

$$\begin{aligned}
R_1'(\theta) &= a \log a - a \log(1-a) - a \log \theta + a \log(d-\theta). \\
R_1''(\theta) &= -\frac{ad}{\theta(d-\theta)} < 0.
\end{aligned}$$

Thus $R_1(\theta)$ is concave and is maximized when $\theta = ad$. Because $\theta \geq 2$ we can only use this for $a \geq 2/d$.

Case 2.1.1: $k \geq 2n/d$ and $d \geq 5$.

$$\begin{aligned}
\pi_R(k, \ell, D) &\leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8})} a^{a^2 d} (1-a)^{d-a^2 d}}{h(a)^{2+ad}} \right)^n \\
&= O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8})} a^{a^2 d - a(2+ad)} (1-a)^{d-a^2 d - (1-a)(2+ad)} \right)^n \\
&= O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8})} a^{-2a} (1-a)^{(d-2)(1-a)} \right)^n \\
&= O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8})} \rho_d (1-a)^n \quad (60)
\end{aligned}$$

where the function ρ_d is defined following (29).

We find that

$$\rho_d(1-2/d) = \left(\frac{d^4}{16} \left(1 - \frac{2}{d}\right)^{(d-2)^2} \right)^{1/d} \leq .9 \text{ for } d \geq 5. \quad (61)$$

Now $\rho_d(1 - 2/d) < 9/10$ and $\rho_d(.5) < .8$ for $d \geq 5$. So, with the aid of (32),

$$B_{62} = \sum_{\ell < k=2n/d}^{k_0} \sum_{D=2k}^{k \log n} \pi_R(k, \ell, D) \leq \sum_{\ell < k=2n/d}^{k_0} \sum_{D=2k}^{k \log n} O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8})} (.8)^n \quad \text{for } d \geq 5. \quad (62)$$

We will treat $d = 4$ under Case 2.2.

Case 2.2: $n^{9/10} \leq k \leq 2n/d$.

Case 2.2.1: $d \geq 6$.

In this case the expression in (59) (ignoring error terms) is maximized at $\theta = 2$. Then

$$\begin{aligned} \pi_R(k, D) &\leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8})} a^{2a} (1-a)^{d-2a}}{h(a)^2 h\left(\frac{2}{d}\right)^{ad}}\right)^n \\ &= O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8})} (1-a)^{d-2a-2(1-a)}}{h\left(\frac{2}{d}\right)^{ad}}\right)^n \\ &= O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8})} (1-a)^{d-2}}{h\left(\frac{2}{d}\right)^{ad}}\right)^n \end{aligned}$$

Let $R_2(a) = \log\left(\frac{(1-a)^{d-2}}{h\left(\frac{2}{d}\right)^{ad}}\right)$. Then

$$\begin{aligned} R_2'(a) &= -\frac{d-2}{1-a} - d \log h(2/d) < 0 \quad \text{for } d \geq 6. \\ R_2''(a) &= -\frac{d-2}{(1-a)^2} < 0. \end{aligned}$$

Thus $R_2(a)$ is strict concave and its maximum is taken at $a = 0$ and $R_2(a) \leq R_2(0)a$ for all $a \in [0, \frac{2}{d}]$. Furthermore, $R_2(0) < -3/10$ for $d \geq 6$. It follows that if $d \geq 6$ then

$$B_{63} = \sum_{\ell < k=n^{9/10}}^{2n/d} \sum_{D=2k}^{dk} \pi_R(k, \ell, D) \leq \sum_{\ell < k=n^{9/10}}^{2n/d} \sum_{D=2k}^{dk} O\left(\frac{1}{n^{1/2}}\right) e^{-(3k/10 - o(n^{7/8}))} = o(1). \quad (63)$$

For $d = 3, 4, 5$ we use a better bound on $[u^D]((1+u)^d - 1 - du)^k$ in (43).

Case 2.2.2: $d = 5$.

We use

$$\begin{aligned} [u^D]((1+u)^5 - 1 - 5u)^k &= [u^D](10u^2 + 10u^2 + 5u^4 + u^5)^k \\ &= [u^{D-2k}](10 + 10u + 5u^2 + u^3)^k \\ &= 10^k [u^{D-2k}] \left(1 + u + \frac{u^2}{2} + \frac{u^3}{10}\right)^k \\ &\leq 10^k [u^{D-2k}] \left(1 + \frac{u}{2}\right)^{3k} \\ &= 10^k \frac{\binom{3k}{D-2k}}{2^{D-2k}} \end{aligned}$$

Replacing the $\frac{1}{h(\frac{\theta}{a})^{ad}}$ factor in (59) which comes from $\binom{dk}{D}$ gives, for $d = 5$,

$$\begin{aligned}
\pi_R(k, D) &\leq O\left(\frac{k}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8})} a^{a\theta} (1-a)^{5-a\theta}}{h(a)^2}\right)^n \left(\frac{10}{2^{\theta-2} h\left(\frac{\theta-2}{3}\right)^3}\right)^k \\
&= O\left(\frac{k}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8})} 10^a a^{a(\theta-2)} (1-a)^{5-2-a(\theta-2)}}{\left(2^{\theta-2} h\left(\frac{\theta-2}{3}\right)^3\right)^a}\right)^n \\
&= O\left(\frac{k}{n^{1/2}}\right) \left(e^{o(n^{-1/8})} 10^a (1-a)^3 \left(\frac{\left(\frac{a}{1-a}\right)^{\frac{\theta-2}{3}}}{2^{\frac{\theta-2}{3}} h\left(\frac{\theta-2}{3}\right)}\right)^{3a}\right)^n \tag{64}
\end{aligned}$$

Let $p(x) = \frac{q^x}{h(x)}$ for any $x \in [0, 1]$, note that if $P(x) = \log p(x)$ then

$$\begin{aligned}
P'(x) &= \log q - \log x + \log(1-x) \\
P''(x) &= -\frac{1}{x} - \frac{1}{1-x} < 0
\end{aligned}$$

and so $p(x)$ is maximized when $\log q = \log\left(\frac{x}{1-x}\right)$ or $x = \frac{q}{1+q}$ and the maximum value is $1+q$

Thus from (64) we get

$$\pi_R(k, \ell, D) \leq O\left(\frac{k}{n^{1/2}}\right) \left(e^{o(n^{-1/8})} 10^a (1-a)^3 \left(1 + \frac{a}{2(1-a)}\right)^{3a}\right)^n$$

Let $R_3(a) = \log\left(10^a (1-a)^3 \left(1 + \frac{a}{2(1-a)}\right)^{3a}\right)$. Then

$$\begin{aligned}
R_3'(a) &= \log 10 - \frac{6}{2-a} + 3 \log\left(\frac{2-a}{2-2a}\right) \\
R_3''(a) &= \frac{3a}{(2-a)^2(1-a)} > 0.
\end{aligned}$$

So $R_3(a)$ is convex on $[0, \frac{2}{5}]$. We have $R_3(0) = 0$ and $R_3'(0) = \log 10 - 3 \leq -3/4$ and $R_3(2/5) < -1/4$. It follows that

$$B_{65} = \sum_{\ell < k = n^{9/10}}^{2n/5} \sum_{D=2k}^{5k} \pi_R(k, \ell, D) \leq \sum_{\ell < k = n^{9/10}}^{2n/5} \sum_{D=2k}^{5k} O\left(\frac{1}{n^{1/2}}\right) e^{-(3k/4 - o(n^{7/8}))} = o(1). \tag{65}$$

Case 2.2.3: $d = 4$.

Using (50) we get

$$\begin{aligned}
\pi_R(k, \ell, D) &\leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8})} a^{a\theta} (1-a)^{4-a\theta}}{h(a)^2}\right)^n \left(\frac{6}{2^{\theta-2} h\left(\frac{\theta-2}{2}\right)^2}\right)^k \\
&= O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8})} 6^a a^{a(\theta-2)} (1-a)^{2-a(\theta-2)} \left(\frac{1}{2^{\frac{\theta-2}{2}} h\left(\frac{\theta-2}{2}\right)}\right)^{2a}\right)^n \\
&= O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8})} 6^a (1-a)^2 \left(\frac{\left(\frac{a}{1-a}\right)^{\frac{\theta-2}{2}}}{2^{\frac{\theta-2}{2}} h\left(\frac{\theta-2}{2}\right)}\right)^{2a}\right)^n \\
&\leq O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8})} 6^a (1-a)^2 \left(1 + \frac{a}{2(1-a)}\right)^{2a}\right)^n.
\end{aligned}$$

Now if $R_4(a) = \log\left(6^a (1-a)^2 \left(1 + \frac{a}{2(1-a)}\right)^{2a}\right)$ then

$$\begin{aligned}
R'_4(a) &= \log 6 - \frac{4}{2-a} + 2 \log\left(\frac{2-a}{2-2a}\right) \\
R''_4(a) &= \frac{2a}{(2-a)^2(1-a)} > 0.
\end{aligned}$$

Thus R_4 is convex on $[0, .51]$. We have $R_4(0) = 1$ and $R'_4(0) = \log 6 - 2 \leq -1/5$ and $R_4(.51) < -1/20$. It follows from this and (32) that

$$B_{66} = \sum_{\ell < k = n^{9/10}}^{k_0} \sum_{D=2k}^{4k} \pi_R(k, \ell, D) \leq + \sum_{\ell < k = n^{9/10}}^{k_0} \sum_{D=2k}^{4k} O\left(\frac{1}{n^{1/2}}\right) e^{-(k/5 - o(n^{7/8}))} = o(1). \quad (66)$$

Case 2.2.4: $d = 3$.

Using (53) we get

$$\begin{aligned}
\pi_R(k, \ell, D) &\leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8})} a^{a\theta} (1-a)^{3-a\theta}}{h(a)^2}\right)^n \left(\frac{3^{3-\theta}}{h(\theta-2)}\right)^k \\
&= O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8})} a^{a(\theta-2)} (1-a)^{1-a(\theta-2)} \left(\frac{3^{3-\theta}}{h(\theta-2)}\right)^a\right)^n \\
&= O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8})} 3^a (1-a) \left(\frac{\left(\frac{a}{3(1-a)}\right)^{\theta-2}}{h(\theta-2)}\right)^a\right)^n \\
&\leq O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8})} 3^a (1-a) \left(1 + \frac{a}{3(1-a)}\right)^a\right)^n.
\end{aligned}$$

Now if $R_5(a) = \log\left(3^a (1-a) \left(1 + \frac{a}{3(1-a)}\right)^a\right)$ then

$$\begin{aligned}
R'_5(a) &= -\frac{3}{3-2a} + \log\left(\frac{3-2a}{1-a}\right) \\
R''_5(a) &= \frac{4a-3}{(3-2a)^2(1-a)}.
\end{aligned}$$

Case 2.2.4.1: $\theta \geq 2.0005$ and $0 \leq a \leq a_0 = k_0/n$.

We go back to (58) and make the choice $\zeta_1 = \zeta_2 = z$ and replace $h(\theta/d)^{-ad}$ by $\left(\frac{3^{3-\theta}}{h(\theta-2)}\right)^a$ and consider the function

$$F_1(\theta, a) = \frac{h(\theta a/3)^3 3^{(3-\theta)a}}{h(a)^2 h(\theta-2)^a}$$

so that $\pi_R(k, \ell, D) \leq O\left(\frac{1}{n^{1/2}}\right) F_1(\theta, a)^n$. Let $G_1(\theta, a) = \log(F_1(\theta, a))$. Then

$$\frac{\partial G_1}{\partial a} = \log\left(\frac{27(1-a)^2(\theta-2)^{\theta-2}(a\theta)^\theta}{3^\theta a^2(3-\theta)^{3-\theta}(3-a\theta)^\theta}\right) \quad (67)$$

$$\frac{\partial G_1}{\partial \theta} = a \left(\log\left(\frac{3-\theta}{9}\right) - \log(\theta-2) + \log(a\theta) - \log\left(1 - \frac{a\theta}{3}\right) \right) \quad (68)$$

$$\frac{\partial^2 G_1}{\partial a^2} = \frac{(3-a)\theta - 6}{a(1-a)(3-a\theta)} \quad (69)$$

$$\frac{\partial^2 G_1}{\partial \theta^2} = \frac{((3-a)\theta^2 - 12\theta + 18)a}{\theta(3-a\theta)(\theta-3)(\theta-2)}. \quad (70)$$

It follows from (69) that

$$G_1(\theta, a) \text{ is a convex function of } a \text{ for } 0 \leq a \leq a_\theta = \frac{3\theta-6}{\theta}, \text{ for } \theta \text{ fixed, } 2 \leq \theta \leq 3 \quad (71)$$

and

$$G_1(\theta, a) \text{ is a concave function of } a \text{ for } a_\theta \leq a \leq 1, \text{ for } \theta \text{ fixed, } 2 \leq \theta \leq 3. \quad (72)$$

It follows from (70) that

$$G_1(\theta, a) \text{ is a concave function of } \theta \text{ on } [2, 3] \text{ for } a \text{ fixed, } 0 \leq a \leq 1. \quad (73)$$

A calculation shows that if $g_1(\theta) = G_1(\theta, a_\theta)$ then

$$g_1'(\theta) = \frac{3}{\theta^2} \log\left(\frac{144}{3^{\theta^2}(3-\theta)^2}\right) \quad (74)$$

$$g_1''(\theta) = -\frac{6}{\theta^3(3-\theta)} \left(-\theta + 2(3-\theta) \log\left(\frac{12}{3-\theta}\right) \right). \quad (75)$$

Furthermore, if $g_2(\theta) = \frac{\partial G_1}{\partial a} |_{a=a_\theta}$ then

$$g_2(\theta) = \log\left(\frac{12}{3^\theta(3-\theta)}\right). \quad (76)$$

Case 2.2.4.2: $2.0005 \leq \theta \leq 3$ and $0 \leq a \leq e^{-10000}$.

For $a \leq e^{-10000}$ we have $\frac{\partial G_1}{\partial a} \leq \log 10 - (\theta-2) \log 1/a \leq -2$. So,

$$F_1(\theta, a) \leq e^{-2a} \text{ for } 0 \leq a \leq e^{-10000}, 2.0005 \leq \theta \leq 3. \quad (77)$$

Case 2.2.4.3: $2.46 \leq \theta \leq 3$ and $e^{-10000} \leq a \leq a_0$.

Now $a_\theta > a_0$ for $\theta \geq 2.46$ and so (71) implies that $G_1(\theta, a) \leq \max\{G_1(\theta, e^{-10000}), G_1(\theta, a_0)\}$ for $2.46 \leq \theta \leq 3$ and $e^{-10000} \leq a \leq a_0$. Now (77) implies that $G_1(2.46, e^{-10000}) < -2e^{-10000}$ and

(68) implies that $\frac{\partial G_1}{\partial \theta} |_{\theta=2.46, a=e^{-10000}} < 0$ and so (73) implies that $G_1(\theta, e^{-10000}) \leq -2e^{-10000}$ for $2.46 \leq \theta \leq 3$. Also, by direct calculation, we have $G_1(2.46, a_0) < -.002$ and $\frac{\partial G_1}{\partial \theta} |_{\theta=2.46, a=a_0} < 0$ and so $G_1(\theta, a_0) \leq -.002$ for $2.46 \leq \theta \leq 3$. Thus,

$$F_1(\theta, a) \leq e^{-2e^{-10000}} \text{ for } e^{-10000} \leq a \leq a_0 \text{ and } 2.46 \leq \theta \leq 3.$$

Case 2.2.4.4: $2.0005 \leq \theta \leq 2.25$ and $e^{-1000} \leq a \leq a_0$.

We take $\zeta_1 = .6$ and $\zeta_2 = 2.1$ in (58) and let

$$F_2(\theta, a) = F_1(\theta, a) \frac{z^3}{f(z)} \frac{f(\zeta_1)^a}{\zeta_1^{\theta a}} \frac{f(\zeta_2)^{1-a}}{\zeta_2^{3-\theta a}} = F_1(\theta, a) e^{\rho_2 + \sigma_2 a + \tau_2 a \theta}$$

where

$$e^{\rho_2} = \frac{z^3 f(\zeta_2)}{f(z) \zeta_2^3}, \quad e^{\sigma_2} = \frac{f(\zeta_1)}{f(\zeta_2)}, \quad e^{\tau_2} = \frac{\zeta_2}{\zeta_1}.$$

Let $G_2(\theta, a) = \log(F_2(\theta, a))$. $\frac{\partial^2 G_2}{\partial a^2} = \frac{\partial^2 G_1}{\partial a^2}$ and $\frac{\partial^2 G_2}{\partial \theta^2} = \frac{\partial^2 G_1}{\partial \theta^2}$ and so (71), (72) and (73) hold with G_1 replaced by G_2 . Putting $\gamma_2(\theta) = G_2(\theta, a_\theta)$ we see that $\gamma_2''(\theta) = g_1''(\theta) - \frac{12\sigma_2}{\theta^3} > 0$, using (75) ($\sigma_2 < -3.127$). Thus γ_2 is convex on $2.0005 \leq \theta \leq 2.25$. Furthermore $\gamma_2(2.0005), \gamma_2(2.25) < -.00003$ and so $\gamma_2(\theta) < -.00003$ for $\theta \in [2.0005, 2.25]$ and therefore $G_2(\theta, a) \leq -.00003a/a_\theta < -.00003a$ when $0 \leq a \leq a_\theta$ and $\theta \in [2.0005, 2.25]$. Next let $\phi_2(\theta) = \frac{\partial G_2}{\partial a} |_{a=a_\theta}$. We have $\phi_2(\theta) = g_2(\theta) + \sigma_2 + \tau_2 \theta < -.05$ for $2.0005 \leq \theta \leq 2.25$, using (76) ($\tau_2 < 1.253$). So, $G_2(\theta, a) \leq \phi_2(\theta) - .05(a - a_\theta)$ for $a \geq a_\theta$ when $\theta \in [2.0005, 2.25]$. Thus

$$F_2(\theta, a) < e^{-.00003a} \text{ for } e^{-1000} \leq a \leq a_0 \text{ and } 2.0005 \leq \theta \leq 2.25.$$

Now suppose that we repeat the idea of the previous paragraph, but this time we take $\zeta_1 = 1.4$ and $\zeta_2 = 3$ in (58) and use the same notation. Putting $\gamma_2(\theta) = G_2(\theta, a_\theta)$ we see that $\gamma_2''(\theta) = g_1''(\theta) - \frac{12\sigma_2}{\theta^3} > 0$, using (75) ($\sigma_2 < -2.27$). Thus γ_2 is convex on $2.25 \leq \theta \leq 2.46$. Furthermore $\gamma_2(2.25), \gamma_2(2.46) < -.05$ and so $\gamma_2(\theta) < -.05$ for $\theta \in [2.25, 2.46]$ and therefore $G_2(\theta, a) \leq -.05a/a_\theta < -.05a$ when $0 \leq a \leq a_\theta$ and $\theta \in [2.25, 2.46]$. Next let $\phi_2(\theta) = \frac{\partial G_2}{\partial a} |_{a=a_\theta}$. We have $\phi_2(\theta) = g_2(\theta) + \sigma_2 + \tau_2 \theta < -.2$ for $2.25 \leq \theta \leq 2.46$, using (76) ($\tau_2 < .763$). So, $G_2(\theta, a) \leq \phi_2(\theta) - .2(a - a_\theta)$ for $a \geq a_\theta$ when $\theta \in [2.25, 2.46]$. Thus

$$F_2(\theta, a) < e^{-.05a} \text{ for } e^{-1000} \leq a \leq a_0 \text{ and } 2.25 \leq \theta \leq 2.46.$$

Case 2.2.4.5: $2 \leq \theta \leq 2.0005$ and $e^{-1000} \leq a \leq a_0$.

For this we simplify our estimate of $\pi_R(k, \ell, D)$ by removing some terms involving β from (57).

$$\begin{aligned} \pi_R(k, \ell, D) &\leq \mathbb{P}(\exists A \subseteq R, B \subseteq L : |A| = k, |B| = k - 1, N_\Gamma(A) \subseteq B, d_A(a) \geq 2, a \in A) \leq \\ &O(n^{1/2}) \binom{m}{k} \binom{n}{k-1} \sum_{\substack{2 \leq d_a, a \in [m] \\ 2 \leq x_b \leq d, b \in [k-1] \\ \sum_{a \in [k]} d_a = \sum_{b \in [k-1]} x_b = D}} \prod_{a=1}^k \frac{z^{d_a}}{d_a! f(z)} \prod_{b=1}^{k-1} \binom{d}{x_b} D! \prod_{i=0}^{D-1} \frac{1}{dn - i} = \\ &O\left(\frac{k}{m^{1/2}}\right) \binom{n}{k} \binom{m}{k} \frac{z^D D! (dn - D)!}{f(z)^k (dn)!} \left(\sum_{\substack{2 \leq d_a, a \in [k] \\ \sum_a d_a = D}} \prod_{a=1}^k \frac{1}{d_a!} \right) \left(\sum_{\substack{2 \leq x_b \leq d, b \in [k-1] \\ \sum_b x_b = D}} \prod_{b=1}^{k-1} \binom{d}{x_b} \right) = \end{aligned}$$

$$\begin{aligned}
& O\left(\frac{k}{m^{1/2}}\right) \binom{n}{k} \binom{m}{k} \frac{z^D}{f(z)^k} \frac{1}{\binom{dn}{D}} \left([u^D](e^u - 1 - u)^k\right) \left([u^D]((1+u)^d - (1+du))^k\right) \leq \\
& O\left(\frac{k}{m^{1/2}}\right) \binom{n}{k} \binom{m}{k} \frac{z^D}{f(z)^k} \frac{1}{\binom{dn}{D}} \frac{f(\zeta_1)^k}{\zeta_1^D} \binom{dk}{D} = \\
& O\left(\frac{k}{m^{1/2}}\right) \binom{n}{k} \binom{m}{k} \frac{\binom{dk}{D}}{\binom{dn}{D}} \left(\frac{f(\zeta_1)}{f(z)}\right)^k \left(\frac{z}{\zeta_1}\right)^D \\
& = O\left(\frac{1}{m^{1/2}}\right) \left(\frac{h(\theta a/d)^d}{h(a)h(a/\beta)^\beta h(\theta/d)^{ad}} \left(\frac{f(\zeta_1)}{\zeta_1^\theta} \frac{z^\theta}{f(z)}\right)^a\right)^n \tag{78}
\end{aligned}$$

$$= O\left(\frac{1}{m^{1/2}}\right) \left(e^{\rho(n^{-1/8}a)} \frac{h(\theta a/d)^d}{h(a)^2 h(\theta/d)^{ad}} \left(\frac{f(\zeta_1)}{\zeta_1^\theta} \frac{z^\theta}{f(z)}\right)^a\right)^n. \tag{79}$$

Now let

$$F_3(\theta, a) = \frac{h(\theta a/3)^3 3^{(3-\theta)a}}{h(a)^2 h(\theta-2)^a} \left(\frac{f(\zeta_1)}{\zeta_1^\theta} \frac{z^\theta}{f(z)}\right)^a.$$

We take $\zeta_1 = .0001$ and then

$$\frac{f(\zeta_1)}{\zeta_1^\theta} \frac{z^\theta}{f(z)} < .e^{-.86}$$

for $2 \leq \theta \leq 2.0005$. Keeping some slack, we define

$$F_4(\theta, a) = \frac{h(\theta a/3)^3 3^{(3-\theta)a} e^{-.85a}}{h(a)^2 h(\theta-2)^a}$$

and $G_4(\theta, a) = \log(F(\theta, a))$. Now let $\gamma_4(\theta) = G_4(\theta, a_\theta)$. We have $\gamma_4'(\theta) = g_1'(\theta) - \frac{5.1}{\theta^2}$ and $\gamma_4''(\theta) = g_1''(\theta) + \frac{10.2}{\theta^3}$ and we find from (75) that γ_4 is concave on $2 \leq \theta \leq 2.0005$. Furthermore $\gamma_4(2) = 0$ and using (74) we see that $\gamma_4'(2) < -.8$ and so $g_1(\theta) < -.8(\theta-2)$ for $\theta \in [2, 2.0005]$. So $G_4(\theta, a) \leq -.8a(\theta-2)/a_\theta \leq -.8a(\theta-2)$ for $0 \leq a \leq a_\theta$. Next let $\phi_4(\theta) = \frac{\partial G_4}{\partial a} |_{a=a_\theta} = g_2(\theta) - .85$. We see from (76) that $g_2(\theta) < -.5$ for $2 \leq \theta \leq 2.0005$ and thus $G_4(\theta, a) \leq -.5(a-a_\theta)$ for $a \geq a_\theta$ when $\theta \in [2, 2.0005]$. Replacing $e^{-.85}$ by $e^{-.86}$ in the definition of $F_4(\theta, a)$ we get $F_4(\theta, a) < e^{-(4(\theta-2)a/5+a/100)}$ for $0 \leq a \leq a_0$ when $2 \leq \theta \leq 2.0005$. So, for some small constant $c > 0$,

$$B_{80} = \sum_{\ell < k=2}^{k_0} \sum_{D=2k}^{3k} \pi_R(k, \ell, D) \leq \sum_{\ell < k=2}^{k_0} \sum_{D=2k}^{3k} e^{-ck} = o(1). \tag{80}$$

3.3.3 Finishing the case $|n-m| = o(n^{7/8})$

We repeat our observation that the maximum degree Δ in Γ is $o(\log n)$ **whp**. Therefore

$$\mathbb{P}(\mu(\Gamma) < \min\{m, n\}) \leq o(1) + \begin{cases} A_{14} + A_{28} + A_{33} + A_{34} + B_{46} + B_{54} + B_{62} + B_{63} & d \geq 6 \\ A_{14} + A_{28} + A_{33} + A_{34} + B_{46} + B_{54} + B_{62} + B_{65} & d = 5 \\ A_{14} + A_{28} + A_{33} + A_{35} + A_{39} + B_{46} + B_{54} + B_{66} & d = 4 \\ A_{14} + A_{28} + A_{33} + A_{35} + A_{40} + A_{41} + B_{46} + B_{54} + B_{80} & d = 3 \end{cases}$$

where the $o(1)$ term accounts for $\mathbb{P}(\Delta(\Gamma) > \log n)$. We use $A_{14} + A_{28} + A_{33} + A_{34}$ etc. to account for witnesses $A \subseteq L, B$ with $|A| \leq k_0$ and $B_{46} + B_{54} + B_{62} + B_{63}$ etc. to account for witnesses

$A \subseteq L, B$ with $|A| > k_0$. This is because if $A' = R \setminus B$ and $B' = L \setminus A$ then $|A'| = m - k + 1$ and $|B'| = n - k$ and $N_\Gamma(A') \subseteq B'$ and there will be a minimal witness A'', B'' with $A'' \subseteq A'$. Similarly, we use $B_{46} + B_{54} + B_{62} + B_{63}$ etc. to account for witnesses $A \subseteq R, B$ with $|A| \leq k_0$ and $A_{14} + A_{28} + A_{33} + A_{34}$ etc. to account for witnesses $A \subseteq R, B$ with $|A| > k_0$.

We point out for use in the next section that our computations allow us to claim that we have

$$\sum_{\substack{k=n^{9/10} \\ \ell \leq \min\{k-1, m/2\}}}^{n-n^{9/10}} \sum_{D=dk}^{k \log n} \pi_L(k, \ell, D) = O(e^{-\Omega(n^{9/10})}). \quad (81)$$

One can see this by noting that all of the upper bounds in question are of order $e^{O(|m-n|)\xi^k}$ where $0 < \xi < 1$ is some absolute constant and $k \geq n^{9/10}$.

The same argument allows us to claim that

$$\sum_{\substack{k=n^{9/10} \\ \ell \leq \min\{k-1, n/2\}}}^{n-n^{9/10}} \sum_{D=2k}^{dk} \pi_R(k, \ell, D) = O(e^{-\Omega(n^{9/10})}). \quad (82)$$

3.4 The case $n + n^{4/5} \leq m \leq dn/2$

Let $\mathcal{G}(n, m)$ denote the set of bipartite graphs with $|L| = n, |R| = m$ that are d -regular on L and degree at least 2 on R . The upper bound on m is due to the fact that there are precisely dn edges and the minimum degree in R is at least two. Suppose that $G(n, m)$ is chosen uniformly at random from $\mathcal{G}(n, m)$.

If there is no matching from L to R , then let a minimal witness A, B be *small* if $|A| \leq n^{3/4}$ and *large* if $|A| \geq n - n^{3/4}$ and *medium* otherwise.

3.4.1 Small Witnesses

The RHS of (14) remains $o(1)$ for all $C_1 n \leq m \leq C_2 n$, for any constants $0 < C_1 < C_2$.

3.4.2 Large Witnesses

To deal with $k \geq n - n^{9/10}$ we treat this as $k \leq n^{9/10}$ in Section 3.3.2. Indeed, if there is such a witness A, B , let $A' = R \setminus B$ and $B' = L \setminus A$. Then $N_\Gamma(A') \subseteq B'$ and $|B'| < \min\{|A'|, n^{3/4}\}$ and so we can find a witness A'', B'' with $A'' \subseteq A', B'' \subseteq B'$ and $|A''| \leq n^{9/10}$.

We use (78) for this calculation. Now

$$\begin{aligned} \frac{h(\theta a/d)^d}{h(a)h(a/\beta)^\beta h(\theta/d)^{ad}} &= \frac{\left(\frac{\theta a}{d}\right)^{\theta a} \left(1 - \frac{\theta a}{d}\right)^{d-\theta a}}{a^a (1-a)^{1-a} \left(\frac{a}{\beta}\right)^a \left(1 - \frac{a}{\beta}\right)^{\beta-a} \left(\frac{\theta}{d}\right)^{\theta a} \left(1 - \frac{\theta}{d}\right)^{da-\theta a}} = \\ &= a^{(\theta-2)a} \exp \left\{ -(d-\theta a) \sum_{k=1}^{\infty} \frac{\theta^k a^k}{k d^k} + a - \sum_{k=2}^{\infty} \frac{a^k}{k(k-1)} + \frac{a}{\beta} - \sum_{k=2}^{\infty} \frac{a^k}{\beta^{k-1} k(k-1)} + (da - \theta a) \sum_{k=1}^{\infty} \frac{\theta^k}{k d^k} \right\} \\ &= a^{(\theta-2)a} \exp \left\{ a \left(1 + \frac{1}{\beta} - (d-\theta) \log(1 - \theta/d) - \theta \right) + O(a^2) \right\} \end{aligned} \quad (83)$$

So from (78) we can write

$$\pi_R(k, \ell, D) \leq O \left(\frac{1}{m^{1/2}} \right) \left(\left(\frac{az}{\zeta_1} \right)^{\theta-2} \exp \left\{ 1 + \frac{1}{\beta} - (d-\theta) \log(1 - \theta/d) - \theta + O(a) \right\} \frac{f(\zeta_1)}{\zeta_1^2} \frac{z^2}{f(z)} \right)^{an}. \quad (84)$$

Now we claim that

$$\zeta_1^{\theta-2} \geq \frac{1}{2} \text{ and that } f(x)x^{-2} \text{ is monotone increasing in } x. \quad (85)$$

First notice that $f(x)x^{-2} = \sum_{i=2}^{\infty} \frac{x^{i-2}}{i!}$ which is clearly monotone increasing. Second note that $\zeta_1 = \zeta(\theta)$ and since $\frac{d\zeta(x)}{dx} > 0$ we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{d\zeta(x)}{dx} &= \lim_{x \rightarrow \infty} \frac{f(\zeta(x))^2}{(e^{\zeta(x)} - 1)^2 - \zeta(x)^2 e^{\zeta(x)}} \\ &= \lim_{\zeta \rightarrow \infty} \frac{f(\zeta)^2}{(e^\zeta - 1)^2 - \zeta^2 e^\zeta} = 1 \end{aligned}$$

and since $\zeta(x)$ is concave we have $\frac{d\zeta(x)}{dx} \geq 1$. This, along with $\lim_{x \rightarrow 2^-} \zeta(x) = 0$, implies that $\zeta(x) \geq x - 2$. We can then lower bound

$$\zeta_1^{\theta-2} = \zeta(\theta)^{\theta-2} \geq (\theta - 2)^{\theta-2} \geq e^{-e^{-1}} \geq 0.69$$

Using this we see from (84) that if

$$\theta \geq \theta_0 = 2 + \frac{4}{\log(1/az)}$$

then

$$\pi_R(k, \ell, D) \leq O \left(\frac{1}{m^{1/2}} \right) e^{-k}.$$

In which case we have

$$\sum_{\ell < k=2}^{n^{9/10}} \sum_{\theta \geq \theta_0} \pi_R(k, \ell, D) \leq O \left(\frac{1}{m^{1/2}} \right) \sum_{k \geq 1} e^{-k} = o(1). \quad (86)$$

When $\theta < \theta_0$ we have $\theta = 2 + o(1)$, $f(\zeta_1)/\zeta_1^2 = 1/2 + o(1)$. Therefore

$$\pi_R(k, \ell, D) \leq O\left(\frac{1}{m^{1/2}}\right) \left(\frac{z^2 e^{-(d-2)\log(1-2/d)+o(1)}}{2f(z)}\right)^k. \quad (87)$$

Now for $d \geq 4$ we have

$$\frac{z^2 e^{-(d-2)\log(1-2/d)+o(1)}}{2f(z)} \leq \frac{9}{10} \quad (88)$$

and so

$$\sum_{k=1}^{n^{9/10}} \sum_{\theta \leq \theta_0} \pi_R(k, \ell, D) \leq O\left(\frac{1}{m^{1/2}}\right) \left(\frac{9}{10}\right)^k = o(1). \quad (89)$$

When $d = 3$, the expression on the LHS of (88) is at most 1.26. So in this case we go back to (79) and replace $\frac{1}{h(\theta/d)^{ad}}$ by $\left(\frac{3^{\theta-2}}{h(\theta-2)}\right)^a = e^{o(a)}$. After this (83) is replaced by

$$a^{(\theta-2)a} \exp\left\{a\left(1 + \frac{1}{\beta} + \theta \log(\theta/d) - \theta\right) + o(a)\right\}.$$

And then (87) is replaced by

$$\pi_R(k, \ell, D) \leq O\left(\frac{1}{m^{1/2}}\right) \left(\frac{z^2 e^{-2\log(3/2)+o(1)}}{2f(z)}\right)^k \leq O\left(\frac{1}{m^{1/2}}\right) \frac{1}{2^k}$$

and so

$$\sum_{k=1}^{n^{9/10}} \sum_{\theta \leq \theta_0} \pi_R(k, \ell, D) \leq O\left(\frac{1}{m^{1/2}}\right) \frac{1}{2^k} = o(1). \quad (90)$$

3.4.3 Medium Witnesses

Let $d_i(n, m)$ denote the number of R -vertices of degree $i \geq 2$ in $G(n, m)$ and let $D_i(n, m) = \mathbb{E}(d_i(n, m))$.

We define three events:

$$\mathcal{A}_1(n, m-1) = \left\{G \in \mathcal{G}(n, m-1) : \exists i : |d_i(n, m-1) - D_i(n, m-1)| > n^{3/5}/i^3, 2 \leq i \leq \log^2 n\right\} \quad (91)$$

$$\mathcal{A}_2(n, m) = \{G \in \mathcal{G}(n, m) : \exists i : d_i(n, m) \neq 0, i > \log^2 n\} \quad (92)$$

$$\mathcal{B}(n, m) = \left\{G \in \mathcal{G}(n, m) : |d_2(n, m) - D_2(n, m)| > 2n^{3/5}\right\} \quad (93)$$

We argue next that if $\mathcal{A}(n, m) = \mathcal{A}_1(n, m-1) \cup \mathcal{A}_2(n, m)$ then

$$\mathbb{P}(\mathcal{A}(n, m) \cup \mathcal{B}(n, m)) = e^{-\Omega(\log^2 n)}. \quad (94)$$

For any $t > 0$ we have

$$\mathbb{P}(|d_i(n, m-1) - D_i(n, m-1)| > t) \leq O(n^{1/2})\mathbb{P}(\text{Bin}(m, q_i) > t)$$

where $q_i = \frac{z^i}{i!f(z)}$.

We will now use the following bounds (see for example [1])

$$\mathbb{P}(|\text{Bin}(N, p) - Np| \geq t) \leq 2e^{-t^2/N}, \quad (95)$$

$$\mathbb{P}(\text{Bin}(N, p) \geq \alpha Np) \leq (e/\alpha)^{\alpha Np}. \quad (96)$$

If $i \leq \log^2 n$ then we can use (95) with $t = n^{3/5}/i^3$ to deal with $\mathcal{A}_1(n, m)$ and also with $\mathcal{B}(n, m)$. If $i \geq \log^2 n$ then $nq_i \leq e^{-\Omega(\log^2 n)}$. We can therefore use (96) with $\alpha = 1/nq_i$ to deal with $\mathcal{A}_2(n, m)$. This concludes the proof of (94).

Now consider a set of pairs $X \subseteq \mathcal{G}(n, m-1) \times \mathcal{G}(n, m)$. We place (G_1, G_2) into X if G_2 is obtained from G_1 in the following manner: Choose a vertex $x \in R$ of degree at least four in G_1 . Suppose that its neighbours are $y_i, i = 1, 2, \dots, k$ in any order. To create G_2 we (i) replace x by two vertices x and m and then (ii) let the neighbours of x in G_2 be y_1, y_2 and let the neighbours of m be y_3, \dots, y_k .

For $G \in \mathcal{G}(n, m-1)$ let

$$\pi_1(G) = |\{G_2 : (G, G_2) \in X\}|$$

and for $G \in \mathcal{G}(n, m)$ let

$$\pi_2(G) = |\{G_1 : (G_1, G) \in X\}|.$$

We note that if

$$\Sigma_1 = \sum_{i \geq 4} \binom{i}{2} D_i(n, m-1)$$

then

- $G \notin \mathcal{A}(n, m-1)$ implies that $|\pi_1(G) - \Sigma_1| \leq O(n^{3/5})$.
- $\pi_1(G) \leq \binom{m-1}{2}$ for all $G \in \mathcal{G}(n, m-1)$.
- $G \notin \mathcal{B}(n, m)$ implies that $|\pi_2(G) - D_2(n, m)| \leq n^{3/5}$.
- $\pi_2(G) \leq m$ for all $G \in \mathcal{G}(n, m)$.

We then note that

$$(\Sigma_1 - O(n^{3/5}))|\mathcal{G}(n, m-1)| \leq |X| \leq (D_2(n, m) + n^{3/5} + me^{-\Omega(\log^2 n)})|\mathcal{G}(n, m)|.$$

Now let \mathcal{P}, \mathcal{Q} be properties such that if $(G_1, G_2) \in X$ and $G_2 \in \mathcal{Q}$ then $G_1 \in \mathcal{P}$. Let (G_1, G_2) be chosen uniformly from X and let \mathbb{P}_X denote probabilities computed w.r.t. this choice. Then

$$\mathbb{P}_X(G_2 \in \mathcal{Q}) \leq \mathbb{P}_X(G_1 \in \mathcal{P}) \leq \frac{|\mathcal{P}|(\Sigma_1 + O(n^{3/5})) + m|\mathcal{A}(n, m-1)|}{|X|}$$

and

$$\mathbb{P}_X(G_2 \in \mathcal{Q}) \geq \frac{(|\mathcal{Q}| - |\mathcal{B}(n, m)|)(D_2(n, m) - n^{3/5})}{|X|}$$

So,

$$\frac{(|\mathcal{Q}| - |\mathcal{B}(n, m)|)(D_2(n, m) - n^{3/5})}{|\mathcal{G}(n, m)| (D_2(n, m) + n^{3/5} + me^{-\Omega(\log^2 n)})} \leq \frac{|\mathcal{P}|(\Sigma_1 + O(n^{3/5})) + m|\mathcal{A}(n, m-1)|}{|\mathcal{G}(n, m-1)| (\Sigma_1 - O(n^{3/5}))}.$$

So,

$$\frac{|\mathcal{Q}|}{|\mathcal{G}(n, m)|} \leq (1 + O(n^{-2/5})) \frac{|\mathcal{P}|}{|\mathcal{G}(n, m-1)|}.$$

So, if \mathcal{P}_j is a property of $\mathcal{G}(n, j)$ for $j = n, n+1, \dots, m$,

$$\frac{|\mathcal{P}_m|}{|\mathcal{G}(n, m)|} \leq (1 + O(n^{-2/5}))^{m-n} \frac{|\mathcal{P}_{n+n^{4/5}}|}{|\mathcal{G}(n, n+n^{4/5})|}. \quad (97)$$

We use (97) in the following way: First let \mathcal{B}_j , $n + n^{4/5} \leq j \leq m$ be the property that $G \in \mathcal{G}(n, j)$ contains a minimal witness A, B with $A \subseteq L$, $n^{9/10} \leq |A| \leq n/2$. If $(G_1, G_2) \in X$ and $G_2 \in \mathcal{B}_m$ then $G_1 \in \mathcal{B}_{m-1}$. Indeed $A, B \cap [m-1]$ is a minimal witness in G_1 . Applying (97) and (81) we see that **whp** \mathcal{B}_m fails to occur. Now let \mathcal{B}'_j be the property that $G \in \mathcal{G}(n, j)$ contains a minimal witness A, B with $A \subseteq R$, $n^{9/10} \leq |A|, |B| < \min\{|A| - (j-n), n/2\}$. If $(G_1, G_2) \in X$ and G_2 has a witness A, B with $A \subseteq L$ and $n/2 < |A| \leq n - n^{9/10}$ then $G_2 \in \mathcal{B}'_m$. Indeed $A' = R \setminus B, B' = L \setminus A$ is also a witness in G_2 . Now if $G_2 \in \mathcal{B}'_m$ with a witness A', B' then $A' \cap [m-1], B'$ is a witness in G_1 and so contains a minimal witness A'', B'' where $|A''| > |B''| + m - n > n^{9/10}$ i.e. $G_1 \in \mathcal{B}'_{m-1}$. Applying (97) and (82) we see that **whp** \mathcal{B}'_m fails to occur. This deals with medium witnesses.

3.5 The case $m \leq n - n^{4/5}$

We once again consider medium witnesses separately from small or large witnesses.

3.5.1 Small Witnesses

We first observe that the RHS of (46) remains $o(1)$ for all $m \leq n$. Then to finish, we can argue as in cases 2.0.1 to 2.0.3 with $m \leq n$ making the calculations easier.

3.5.2 Large Witnesses

For $k \geq n - n^{9/10}$ we deal with $\pi_L(k, \ell, D)$ for $k \leq n^{9/10}$. As in Section 3.4.1, we use the fact that the RHS of (14) remains $o(1)$ for all $C_1 n \leq m \leq C_2 n$, for any constants $0 < C_1 < C_2$.

3.5.3 Medium Witnesses

Now consider a set of pairs $Y \subseteq \mathcal{G}(n, m) \times \mathcal{G}(n+1, m)$. We place (G_1, G_2) into Y if G_2 is obtained from G_1 in the following manner: Choose $0 \leq k \leq n$. Replace edges (ℓ, y) by $(\ell+1, y)$ for all $\ell > k$ and all y . Add vertex $k+1$ and d edges $(k+1, y_j), j = 1, 2, \dots, d$.

Note that if $(G_1, G_2) \in Y$ and G_1 has a matching of R into L then so does G_2 .

For $G \in \mathcal{G}(n, m)$ let now

$$\pi_1(G) = |\{G_2 : (G, G_2) \in Y\}|$$

and for $G \in \mathcal{G}(n+1, m)$ let

$$\pi_2(G) = |\{G_1 : (G_1, G) \in Y\}|.$$

Let

$$\Sigma_2 = (n+1) \left(1 - \frac{z^2}{2f(z)}\right)^d$$

and for $G \in \mathcal{G}(n+1, m)$ let

$$L_3(G) = |\{v \in L : \text{all neighbours of } v \text{ have degree at least } 3\}|.$$

Let

$$\mathcal{C}(n+1, m) = \left\{G \in \mathcal{G}(n+1, m) : |L_3(G) - \Sigma_2| \leq n^{3/5}\right\}.$$

We note that

- $G \in \mathcal{G}(n, m)$ implies that $\pi_1(G) = (n+1) \binom{m}{d}$.
- $G \notin \mathcal{C}(n, m+1)$ implies that $|\pi_2(G) - \Sigma_2| \leq n^{3/5}$.
- $\pi_2(G) \leq n+1$ for all $G \in \mathcal{G}(n+1, m)$.

We then note that

$$\frac{|Y|}{|\mathcal{G}(n, m)|} = (n+1) \binom{m}{d}.$$

$$\Sigma_2 - n^{3/5} \leq \frac{|Y|}{|\mathcal{G}(n+1, m)|} \leq \Sigma_2 + n^{3/5} + (n+1)e^{-\Omega(\log^2 n)}.$$

Now let \mathcal{P}, \mathcal{Q} be properties such that if $(G_1, G_2) \in Y$ and $G_2 \in \mathcal{Q}$ then $G_1 \in \mathcal{P}$. Let (G_1, G_2) be chosen uniformly from Y and let P_Y denote probabilities computed with respect to this choice. Then

$$P_Y(G_2 \in \mathcal{Q}) \leq P_Y(G_1 \in \mathcal{P}) = \frac{|\mathcal{P}|(n+1) \binom{m}{d}}{|Y|}$$

and

$$P_Y(G_2 \in \mathcal{Q}) \geq \frac{(|\mathcal{Q}| - |\mathcal{C}(n+1, m)|)(\Sigma_2 - n^{3/5})}{|Y|}$$

Arguing as in Section 3.4 we see that if \mathcal{P}_j is a property of $\mathcal{G}(j, m)$ for $j = m, m+1, \dots, n$,

$$\frac{|\mathcal{P}_m|}{\mathcal{G}(n, m)} \leq (1 + O(n^{-2/5}))^{n-m} \frac{|\mathcal{P}_{m+n^{4/5}}|}{\mathcal{G}(m+n^{4/5}, m)}. \quad (98)$$

First let $\mathcal{B}_j, m+n^{4/5} \leq j \leq n$ be the property that $G \in \mathcal{G}(j, m)$ contains a minimal witness A, B with $A \subseteq R, n^{9/10} \leq |A| \leq m/2$. If $(G_1, G_2) \in X$ and $G_2 \in \mathcal{B}_{n+1}$ then $G_1 \in \mathcal{B}_n$. Indeed $A, B \cap [n]$ is a witness in G_1 . Applying (98) and (82) we see that **whp** \mathcal{B}_n fails to occur. Now let \mathcal{B}'_j be the property that $G \in \mathcal{G}(j, m)$ contains a minimal witness A, B with $A \subseteq R, n^{9/10} \leq |A|, |B| \leq \min\{|A| - (j-m), m/2\}$. If $(G_1, G_2) \in X$ and G_2 has a witness A, B with $A \subseteq R$ and $m/2 < |A| \leq m - n^{9/10}$ then $G_2 \in \mathcal{B}'_m$. Indeed $A' = L \setminus B, B' = R \setminus A$ is also a witness in G_2 . Now if $G_2 \in \mathcal{B}'_m$ with a witness A', B' then $A' \cap [n], B'$ is a witness in G_1 and so contains a minimal witness A'', B'' where $|A''| > |B''| + n - m > n^{9/10}$ i.e. $G_1 \in \mathcal{B}'_{m-1}$. Applying (98) and (81) we see that **whp** \mathcal{B}'_m fails to occur. This deals with medium witnesses.

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