

Maximum Matchings in Random Bipartite Graphs and the Space Utilization of Cuckoo Hashtables

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Abstract

We study the the following question in Random Graphs. We are given two disjoint sets L, R with $|L| = n = \alpha m$ and $|R| = m$. We construct a random graph G by allowing each $x \in L$ to choose d random neighbours in R . The question discussed is as to the size $\mu(G)$ of the largest matching in G . When considered in the context of Cuckoo Hashing, one key question is as to when is $\mu(G) = n$ **whp**? We answer this question exactly when d is at least three. We also establish a precise threshold for when Phase 1 of the Karp-Sipser Greedy matching algorithm suffices to compute a maximum matching **whp**.

1 Introduction

For a graph G we let $\mu(G)$ denote the size of the maximum matching in G . In essence this paper provides an analysis of $\mu(G)$ in the following model of a random bipartite graph. We have two disjoint sets L, R where $L = [n], R = [m]$ where $n = \alpha m$. Each $v \in L$ independently chooses d random vertices of R as neighbours. Our assumptions are that $\alpha > 0$, $d \geq 3$ are fixed and $n \rightarrow \infty$. One motivation for this study comes from Cuckoo Hashing.

Briefly each one of n items $x \in L$ has d possible locations $h_1(x), h_2(x), \dots, h_d(x) \in R$, where d is typically a small constant and the h_i are hash functions, typically assumed to behave as independent fully random hash functions. (See [21] for some justification of this assumption.) We are thus led to consider the bipartite graph G which has vertex set $L \cup R$ and edge set $\{(x, h_j(x)) : x \in L, j = 1, 2, \dots, d\}$. Under the assumption that the hash functions are completely random we see that G has the same distribution as the random graph defined in the previous paragraph.

We assume each location can hold only one item. When an item x is inserted into the table, it can be placed immediately if one of its d locations is currently empty. If not, one of the items in its d locations must be displaced and moved to another of its d choices to make room for x . This item in turn may need to displace another item out of one its d locations. Inserting an item may require a sequence of moves, each maintaining the invariant that each item remains in one of its d

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potential locations, until no further evictions are needed. Thus having inserted k items, we have constructed a matching M of size k in G . Adding a $(k + 1)$ 'th item is tantamount to constructing an augmenting path with respect to M . All n items will be insertable in this way iff G contains a matching of size n .

The case of $d = 2$ choices is notably different from that for other values of d and the theory for the case where there are $d = 2$ bucket choices for each item is well understood at this point [9, 20, 22]. We will therefore assume that $d \geq 3$.

We will now revert to the abstract question posed in first paragraph of the paper.

2 Definitions and Results

This question was studied to some extent by Fotakis, Pagh, Sanders and Spirakis [15]. They show in the course of their analysis of Cuckoo hashing that the following holds:

Lemma 1 *Suppose that $0 < \varepsilon < 1$ and $d \geq 2(1 + \varepsilon) \log(e/\varepsilon)$. Suppose also that $m = (1 + \varepsilon)n$. Then **whp** G contains a matching of size n i.e. a matching of L into R .*

□

In particular, if $d = 3$ and $m \approx 1.57n$ then Lemma 1 shows that there is a matching of L into R **whp**.

This lemma is not tight and recently Mitzenmacher et al [12] observed a connection with a result of Dubois and Mandler on Random 3-XORSAT [10] that enables one to essentially answer the question as to when $\mu(G) \geq n$ for the case $d = 3$. More recently, Fountoulakis and Panagiotou [11] have established thresholds for when there is a matching of L into R **whp**, for all $d \geq 3$.

We begin with a simple observation that is the basis of the Karp-Sipser Algorithm [16, 2]. If v is a vertex of degree one in G and e is its unique incident edge, then there exists a maximum matching of G that includes e . Karp and Sipser exploited this via a simple greedy algorithm:

Algorithm 1 Karp-Sipser Algorithm

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1: procedure KSGREEDY( $G$ )
2:    $M \leftarrow \emptyset$ 
3:   while  $\Gamma \neq \emptyset$  do
4:     if  $\Gamma$  has vertices of degree one then
5:       Select a vertex  $\xi$  uniformly at random from the set of vertices of degree one
6:       Let  $e = (\xi, \eta)$  be the edge incident to  $\xi$ 
7:     else
8:       Select an edge  $e = (v, u)$  uniformly at random
9:     end if
10:     $M \leftarrow M \cup \{e\}$ 
11:     $\Gamma \leftarrow \Gamma \setminus \{\xi, \eta\}$ 
12:  end while
13:  return  $M$ 
14: end procedure

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Phase 1 of the Karp-Sipser Algorithm ends and Phase 2 begins when the graph remaining has minimum degree at least two. So if Γ_1 denotes the graph Γ remaining at the end of Phase 1 and τ_1 is the number of iterations involved in Phase 1 then

$$\mu(G) = \tau_1 + \mu(\Gamma_1). \quad (1)$$

Our approach to estimating $\mu(G)$ is to (i) obtain an asymptotic expression for τ_1 that holds **whp** and then (ii) show that **whp** Γ_1 has a (near) perfect matching and then apply (1).

We summarise our results as follows: Let z_1 satisfy

$$z_1 = \frac{e^{z_1} - 1}{d - 1} \quad (2)$$

and let

$$\alpha_1 = \frac{z_1}{d(1 - e^{-z_1})^{d-1}}. \quad (3)$$

Theorem 2 *If $\alpha \leq \alpha_1$ then whp $\mu(G) = \tau_1 = n$.*

Thus **whp** Phase 1 of the Karp-Sipser Algorithm finds a (near) maximum matching if $\alpha \leq \alpha_1$. In particular, if $d = 3$ then $z_1 \approx 1.251$ and $\alpha_1 \approx .818$ and thus $m \approx 1.222n$ is enough for a matching of L into R .

Andrea Montanari has pointed out that our proof of Theorem 2 via the differential equations method is not new and already appears in Luby, Mitzenmacher, Shokrollahi and Spielman [13] and also in Dembo and Montanari [8]. We will prune this from the final version of the paper, but leave it in here for now.

Now consider larger α . Let z^* be the largest non-negative solution to

$$\left(\frac{z}{\alpha d}\right)^{\frac{1}{d-1}} + e^{-z} - 1 = 0.$$

Theorem 3 *If $\alpha > \alpha_1$ then whp*

(a) $z^* > 0$.

(b) $\tau_1 \sim n \left(1 - \left(\frac{z^*}{\alpha d}\right)^{\frac{d}{d-1}}\right)$.

(c) *If $d \geq 3$ then*

$$\mu(\Gamma_1) = \min\{|L_1|, |R_1|\} = \min\left\{n - \tau_1, (1 - (1 + z^*)e^{-z^*})m + o(m)\right\}. \quad (4)$$

Here $L_1 \subseteq L, R_1 \subseteq R$ are the two sides of the bipartition of Γ_1 , after deleting any isolated vertices from the R -side.

3 Structure of the paper

We first prove Theorem 2. This involves studying Phase 1 of the Karp-Sipser Algorithm. For this we first describe the distribution of the graph G . This is done in Section 4. The distribution of Γ

is determined by a few parameters and these evolve as a Markov chain. To study this chain, we introduce and solve a set of differential equations. This is done in Section 5. We show that the chains trajectory and the solution to the equations are close. By analysing the equations we can tell when Phase 1 is sufficient to solve the problem. This is done in Section 6. If Phase 1 is not sufficient then the graph Γ_1 that remains has degree d on the L -side and minimum degree at least two on the R -side. We show that **whp** Γ_1 has a matching of size equal to the minimum set size of the partition. [12] and [9] and [11].

4 Probability Model for Phase 1

We will represent G and more generally Γ by a random sequence $\mathbf{x} \in \Omega_{L,R} = (R^d \cup \{\star\})^L$. A sequence $\mathbf{x} \in \Omega_{L,R}$ is to be viewed as n subsequences $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ where $\mathbf{x}_j = (x_{j,1}, x_{j,2}, \dots, x_{j,d}) \in R^d$ or $\mathbf{x}_j = \sigma = (\star, \star, \dots, \star)$. The \star 's represent edges that have been deleted by the Karp-Sipser algorithm. For $\mathbf{x} \in \Omega_{L,R}$ we define the bipartite (multi-)graph $\Gamma_{\mathbf{x}}$ as follows: Its vertex set consists of a bipartition $L_{\mathbf{x}} = \{j \in L : \mathbf{x}_j \neq \sigma\}$ (the *left side*) and R (the *right side*). The edges incident with $j \in L_{\mathbf{x}}$ are $(j, x_{j,i}), i \in [d]$ i.e. we read the sequence \mathbf{x} from left to right and add edges to $\Gamma_{\mathbf{x}}$ in blocks of size d . Each block being assigned to a unique vertex of $L_{\mathbf{x}}$.

We should be clear now that our probability space is $\Omega_{L,R}$ with uniform measure and not $\mathcal{G}_{L,R} = \{\Gamma_{\mathbf{x}} : \mathbf{x} \in \Omega_{L,R}\}$.

Given a graph $\Gamma_{\mathbf{x}}$ we let $v_j = v_j(\mathbf{x}) = |R_j(\mathbf{x})|$ where $R_j(\mathbf{x})$ is the set of vertices in R that have degree $j \geq 0$. We let $v = v(\mathbf{x}) = |R_{\mathbf{x}}|$ where $R_{\mathbf{x}} = \bigcup_{j \geq 2} R_j(\mathbf{x})$.

For the graph G we choose $\mathbf{x}(0)$ uniformly at random from $(R^d)^L$ and put $G = \Gamma_{\mathbf{x}(0)}$. Next let $\Gamma(0) = G$ and let $\Gamma(t) = \Gamma_{\mathbf{x}(t)}$ be the graph Γ that we have after t steps of Phase 1 of the Karp-Sipser algorithm. The sequence $\mathbf{x}(t)$ is defined as follows: Observe first that the vertex ξ of degree one is always in R . Suppose that it is incident to the unique edge $(\eta, \xi), \eta \in L$. Then we simply replace \mathbf{x}_{η} in $\mathbf{x}(t-1)$ by σ to obtain $\mathbf{x}(t)$. We should thus think of the Karp-Sipser Algorithm as acting on sequences \mathbf{x} and not on graphs. We write $\mathbf{x} \rightarrow \mathbf{y}$ to mean that \mathbf{y} can be obtained from \mathbf{x} by a single Phase 1 step of the Karp-Sipser Algorithm.

Let $\vec{v}(t) = (w(t), v_1(t), v(t))$ where $w(t) = |L_{\mathbf{x}(t)}|$ is the number of vertices on the left side of the bipartition of $\Gamma(t)$. Assuming that we have only run the Karp-Sipser algorithm up to the end of Phase 1, we have $w(t) = n - t$. Also

$$\vec{v}(0) \sim (n, \alpha d e^{-\alpha d} m, (1 - e^{-\alpha d} - \alpha d e^{-\alpha d}) m) \quad \mathbf{whp}. \quad (5)$$

We will omit the parameter t from $\vec{v}(t)$ when it is clear from the context. Let $\mathcal{X}_{\vec{v}}$ be the set of all $\mathbf{x} \in \Omega_{L,R}$ with parameters \vec{v} .

Lemma 4 *Suppose $\mathbf{x}(0)$ is a random member of $\mathcal{X}_{\vec{v}(0)}$. Then given $\vec{v}(0), \dots, \vec{v}(t)$, $\mathbf{x}(t)$ is a random member of $\mathcal{X}_{\vec{v}(t)}$ for all $t \geq 0$.*

Proof: We prove this by induction on t . It is true for $t = 0$ by assumption and so assume it is true for some $t \geq 0$. Let $\vec{v}(t) = (w, v_1, v)$ and now fix a triple $\vec{v}' = (w' = w - 1, v'_1, v')$ as a possible value for $\vec{v}(t+1)$. Fix $\mathbf{y} \in \mathcal{X}_{\vec{v}'}$. We first compute the number of $\mathbf{x} \in \mathcal{X}_{\vec{v}(t)}$ such that $\mathbf{x} \rightarrow \mathbf{y}$. Let $b = v - v'$ be the number of vertices in $R_{\mathbf{x}} \setminus R_{\mathbf{y}}$. Some of these will be in $R_0(\mathbf{y})$ and some will be in $R_1(\mathbf{y})$. So we choose non-negative integers b_0, b_1 such that $b_0 + b_1 = b$. Next let $a = v_1 - v'_1 + b_1$

be the number of vertices in $R_1(\mathbf{x}) \cap R_0(\mathbf{y})$. We can choose a vertex η so that $L_{\mathbf{x}} \setminus L_{\mathbf{y}} = \{\eta\}$ in t ways and now let us enumerate the ways of choosing $\mathbf{x}_\eta = (\zeta_1, \zeta_2, \dots, \zeta_d)$. Our choices for ζ_i are (i) distinctly from $R_0(\mathbf{y})$ (i.e. ζ_i is distinct from rest of the ζ_j), (ii) non-distinctly from $R_0(\mathbf{y})$ (i.e. $\zeta = \zeta_i$ is chosen more than once in the construction), (iii) from $R_1(\mathbf{y})$ and (iv) from $R_{\mathbf{y}}$. We must exercise choice (i) exactly a times, choice (ii) at least twice for each of b_0 distinct values, choice (iii) at least once for each of b_1 distinct values and choice (iv) the remaining times. ****

The number of choices for \mathbf{x}_η depends only on \vec{v} and \vec{v}' , i.e. for each $\mathbf{y} \in \mathcal{X}_{\vec{v}'}$ we have that $D(\vec{v}, \vec{v}') = |\{\mathbf{x} \in \mathcal{X}_{\vec{v}} : \mathbf{x} \rightarrow \mathbf{y}\}|$ is independent of \mathbf{y} , given \vec{v} and \vec{v}' .

Similarly given \mathbf{x} there is a unique $i \in R_1(\mathbf{x})$, which when removed determines \mathbf{y} . Thus $N(\vec{v}) = |\{\mathbf{y} : \mathbf{x} \rightarrow \mathbf{y}\}|$ is fixed given \vec{v} . Thus if $\mathbf{x}(t)$ is a random member of $\mathcal{X}_{\vec{v}}$ then

$$\begin{aligned} & \mathbb{P}(\mathbf{x}(t+1) = \mathbf{y} | \vec{v}(0), \dots, \vec{v}(t)) \\ &= \sum_{\mathbf{x} \in \mathcal{X}_{\vec{v}(t)}} \mathbb{P}(\mathbf{x}(t) = \mathbf{x} | \vec{v}(0), \dots, \vec{v}(t)) \cdot \mathbb{P}(\mathbf{x}(t+1) = \mathbf{y} | \vec{v}(0), \dots, \vec{v}(t-1), \mathbf{x}(t) = \mathbf{x}) \\ &= \sum_{\mathbf{x} \in \mathcal{X}_{\vec{v}(t)}} \mathbb{P}(\mathbf{x}(t+1) = \mathbf{y} | \mathbf{x}(t) = \mathbf{x}) \cdot |\mathcal{X}_{\vec{v}(t)}|^{-1} \\ &= \sum_{\mathbf{x} \in \mathcal{X}_{\vec{v}(t)}} \frac{D(\vec{v}(t), \vec{v}(t+1))}{N(\vec{v}(t))} \cdot |\mathcal{X}_{\vec{v}(t)}|^{-1} \end{aligned}$$

which is independent of \mathbf{y} given $\vec{v}(t)$ and so \mathbf{y} is a random member of $\mathcal{X}_{\vec{v}(t+1)}$. \square

Lemma 5 *The random sequence $\vec{v}(t), t = 0, 1, 2, \dots$ is a Markov chain.*

Proof: As in [2],

$$\begin{aligned} \mathbb{P}(\vec{v}(t+1) | \vec{v}(0), \vec{v}(1), \dots, \vec{v}(t)) &= \sum_{\mathbf{x}' \in \mathcal{X}_{\vec{v}(t+1)}} \mathbb{P}(\mathbf{x}' | \vec{v}(0), \vec{v}(1), \dots, \vec{v}(t)) \\ &= \sum_{\mathbf{x}' \in \mathcal{X}_{\vec{v}(t+1)}} \sum_{\mathbf{x} \in \mathcal{X}_{\vec{v}(t)}} \mathbb{P}(\mathbf{x}', \mathbf{x} | \vec{v}(0), \vec{v}(1), \dots, \vec{v}(t)) \\ &= \sum_{\mathbf{x}' \in \mathcal{X}_{\vec{v}(t+1)}} \sum_{\mathbf{x} \in \mathcal{X}_{\vec{v}(t)}} \mathbb{P}(\mathbf{x}' | \vec{v}(0), \vec{v}(1), \dots, \vec{v}(t-1), \mathbf{x}) \\ &\quad \times \mathbb{P}(\mathbf{x} | \vec{v}(0), \vec{v}(1), \dots, \vec{v}(t)) \\ &= \sum_{\mathbf{x}' \in \mathcal{X}_{\vec{v}(t+1)}} \sum_{\mathbf{x} \in \mathcal{X}_{\vec{v}(t)}} \mathbb{P}(\mathbf{x}' | \mathbf{x}) |\mathcal{X}_{\vec{v}(t)}|^{-1}, \end{aligned}$$

which depends only on $\vec{v}(t), \vec{v}(t+1)$. \square

Lemma 6 *Conditional on \vec{v} if \mathbf{x} is selected uniformly at random from $\mathcal{X}_{\vec{v}}$ then each vertex $i \in R_{\mathbf{x}}$ has degree Y_i where $Y_i = \text{Poi}(z; \geq 2)$, a Poisson random variable conditioned to take a value at least two, and z satisfies*

$$\frac{z(e^z - 1)}{f(z)} = \frac{dw - v_1}{v} \quad (6)$$

where $f(z) = e^z - z - 1$.

The Y_i are also conditioned to satisfy $\sum_{i=1}^v Y_i = wd - v_1$.

Proof: Suppose we first fix the edges incident with vertices of degree one in \mathbf{x} . Then we randomly fill in the remaining $dw - v_1$ non- \star positions in \mathbf{x} with values from some fixed v -subset $R_{\mathbf{x}}$ of R , subject to each of these v vertices having degree at least two. The degrees Y_i of these vertices will have the description described in the lemma (for a proof see Lemma 4 of [2]). \square

From [2] we can use the following lemma

Lemma 7 [2]

(a) Assume that $\log n = O((vz)^{\frac{1}{2}})$. For every $j \in R_{\mathbf{x}}$ and $2 \leq k \leq \log n$,

$$\mathbb{P}(Y_j = k|\vec{v}) = \frac{z^k}{k!f(z)} \left(1 + O\left(\frac{k^2 + 1}{vz}\right) \right) \quad (7)$$

(b) For all $k \geq 2, j \in R_{\mathbf{x}}$

$$\mathbb{P}(Y_j = k|\vec{v}) = O\left(\left(vz\right)^{\frac{1}{2}} \frac{z^k}{k!f(z)}\right)$$

\square

5 Differential Equations

Let \vec{v} be the current parameter tuple and \vec{v}' be the tuple after one step of the Karp-Sipser algorithm. The following lemma gives $E[\vec{v}' - \vec{v}|\vec{v}]$ for each step of Phase 1 of the Karp-Sipser algorithm.

Lemma 8 Assuming $\log v = O((vz)^{\frac{1}{2}})$ and $v_1 > 0$ we have

$$\begin{aligned} E[v'_1 - v_1|\vec{v}] &= -1 - \frac{d-1}{dw}v_1 + \frac{d-1}{dw} \frac{vz^2}{f} + O\left(\frac{1}{vz}\right) \\ E[v' - v|\vec{v}] &= -\frac{d-1}{dw} \frac{vz^2}{f} + O\left(\frac{1}{vz}\right) \end{aligned}$$

Proof: First note that $v_1 > 0$, one vertex $\xi \in R$ with $\deg(\xi) = 1$ will be picked and ξ and its neighbor $\eta \in L$ will be removed from G_t . This implies that w decreases by 1 and the number of edges removed is d , i.e. all edges incident to η . Let δ be the number of multiple edges incident to η . Then we have

$$\begin{aligned} E[\delta|\vec{v}] &\leq d \cdot \mathbb{P}(\eta \text{ is incident to parallel edges}) \leq d \binom{d}{2} \frac{1}{(wd)_2} \sum_j E[\deg(j)_2] \\ &\leq \frac{d^3}{2w^2} v E[Y(Y-1)|\vec{v}] = O\left(\frac{1}{w}\right) \end{aligned} \quad (8)$$

where Y has distribution (7).

Explanation: The dw choices of neighbours for the remaining vertices in L form a list with v_1 unique names and $wd - v_1$ non-unique names and where the number of times a vertex appears among the $wd - v_1$ has distribution (7). Also, if we construct this list vertex by vertex, it will appear in a random order. So the probability that j appears in two of the choices for η is bounded by $\frac{E[\deg(j)_2]}{(wd)_2}$ and this justifies (8).

The change in v_1 comes from ξ being removed, minus the number of other degree one vertices adjacent to η and plus the number of vertices adjacent to η of degree exactly two. Any change from vertices of degree three or more is absorbed by the $O\left(\frac{1}{w}\right)$ term for multiple edges.

The expected change is then

$$\begin{aligned}\mathbb{E}[v'_1 - v_1|\vec{v}] &= -1 - \frac{d-1}{dw-1}(v_1-1) + \frac{2(d-1)}{dw-1}\mathbb{E}[v_2|\vec{v}] + O\left(\frac{1}{w}\right) \\ &= -1 - \frac{d-1}{dw}v_1 + \frac{d-1}{dw}\frac{z^2}{f}v + O\left(\frac{1}{vz}\right)\end{aligned}\tag{9}$$

Similarly for v , the change is only due to vertices adjacent to y of degree exactly two, modulo multiple edges. Thus

$$\mathbb{E}[v' - v|\vec{v}] = -\frac{d-1}{dw}\frac{z^2}{f}v + O\left(\frac{1}{vz}\right)$$

□

Lemma 8 suggests that we consider the following pair of differential equations

$$\frac{dy_1}{dt} = -1 - \frac{d-1}{dw}y_1 + \frac{d-1}{dw}\frac{y\zeta^2}{f(\zeta)}\tag{10}$$

$$\frac{dy}{dt} = -\frac{d-1}{dw}\frac{y\zeta^2}{f(\zeta)}\tag{11}$$

where $w = n - t$ and ζ satisfies

$$\frac{\zeta(e^\zeta - 1)}{f(\zeta)} = \frac{dw - y_1}{y}.\tag{12}$$

The boundary conditions are (see (5))

$$\zeta(0) = \alpha d, \quad y_1(0) = m\alpha d e^{-\alpha d}, \quad y(0) = m(1 - (1 + \alpha d)e^{-\alpha d}).\tag{13}$$

The y_1, y, ζ are of course the deterministic counterparts of v_1, v, z respectively.

Lemma 9 *The solution to (10), (11) and (13) is*

$$w = \left(\frac{\zeta}{\alpha d}\right)^{\frac{d}{d-1}} n.\tag{14}$$

$$t = n \left(1 - \left(\frac{\zeta}{\alpha d}\right)^{\frac{d}{d-1}}\right)\tag{15}$$

$$y = e^{-\zeta} f(\zeta) m\tag{16}$$

$$y_1 = m\zeta \left(\left(\frac{\zeta}{\alpha d}\right)^{\frac{1}{d-1}} + e^{-\zeta} - 1 \right)\tag{17}$$

Proof: We take the derivative of (6) with respect to t . The RHS becomes, using (10) and (11)

$$\begin{aligned}
\frac{d}{dt} \left(\frac{dw - y_1}{y} \right) &= \frac{1}{y} \left(-d - \left(-1 - \frac{d-1}{d} \frac{y_1}{w} + \frac{d-1}{d} \frac{y}{w} \frac{\zeta^2}{f(\zeta)} \right) - \frac{dw - y_1}{y} \frac{1}{y} \left(-\frac{d-1}{d} \frac{y}{w} \frac{\zeta^2}{f(\zeta)} \right) \right) \\
&= -\frac{d-1}{dw} \left(\frac{dw - y_1}{y} + \frac{\zeta^2}{f(\zeta)} - \frac{dw - y_1}{y} \frac{\zeta^2}{f(\zeta)} \right) \\
&= -\frac{d-1}{dw} \left(\frac{\zeta(e^\zeta - 1)f(\zeta) + \zeta^2 f(\zeta) - \zeta^3(e^\zeta - 1)}{f(\zeta)^2} \right) \\
&= -\frac{d-1}{dw} \left(\frac{\zeta(e^\zeta - 1)^2 - \zeta^3 e^\zeta}{f(\zeta)^2} \right)
\end{aligned} \tag{18}$$

On the other hand, on differentiating the LHS of (6) (with z replaced by ζ) we get

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\zeta(e^\zeta - 1)}{f(\zeta)} \right) &= \frac{(\zeta e^\zeta + e^\zeta - 1)f(\zeta) - \zeta(e^\zeta - 1)^2}{f(\zeta)^2} \frac{d\zeta}{dt} \\
&= \frac{(e^\zeta - 1)^2 - \zeta^2 e^\zeta}{f(\zeta)^2} \frac{d\zeta}{dt}.
\end{aligned} \tag{19}$$

Comparing (18) and (19) we see that

$$\frac{1}{\zeta} \frac{d\zeta}{dt} = -\frac{d-1}{dw} \tag{20}$$

Integrating yields

$$\frac{\zeta^d}{w^{d-1}} = \text{constant}$$

Plugging in $\zeta(0) = \alpha d$ we see that

$$\frac{\zeta^d}{w^{d-1}} = \frac{(\alpha d)^d}{n^{d-1}}.$$

This verifies (14) and (15).

Going back to (11) and (20) we have

$$\frac{dy}{d\zeta} \frac{d\zeta}{dt} = \frac{dy}{dt} = -\frac{(d-1)y}{dw} \frac{\zeta^2}{f(\zeta)} = \frac{1}{\zeta} \frac{d\zeta}{dt} \frac{y\zeta^2}{f(\zeta)}.$$

So

$$\frac{1}{y} \frac{dy}{d\zeta} = \frac{\zeta}{f(\zeta)}$$

and integrating yields

$$\ln y = -\zeta + \ln f(\zeta) + C$$

Taking $y(0) = m(1 - (1 + \alpha d)e^{-\alpha d})$ gives $C = \ln m$ and we see that (16) holds.

We now solve for y_1 in terms of ζ as a function of d . It follows from (12) and (16) that

$$\begin{aligned}
y_1 &= dw + \zeta e^{-\zeta} m - m\zeta \\
&= \left(\frac{\zeta}{\alpha d} \right)^{\frac{d}{d-1}} nd + \zeta e^{-\zeta} m - m\zeta \\
&= m\zeta \left(\left(\frac{\zeta}{\alpha d} \right)^{\frac{1}{d-1}} + e^{-\zeta} - 1 \right)
\end{aligned}$$

□

At this point we wish to show that **whp** the sequence $\vec{v}(t), t \geq 0$ closely follows the trajectory $\vec{y}(t) = (w, y_1, y), t \geq 0$ described in Lemma 9. One possibility is to use Theorem 5.1 of Wormald [24], but there is a problem with an “unbounded” Lipschitz coefficient. One can allow for this in [24], but it is unsatisfactory to ask the reader to check this. We have decided to use an approach suggested in Bohman [4].

Next let K be a large positive constant and let $\gamma \ll 1/K$. Then let

$$g(x) = (1-x)^{-K} + K(1-x)^{-1}$$

and

$$Err(t) = n^{2/3}g(t/n).$$

Then define the event

$$\mathcal{E}(t) = \left\{ v(\tau)z(\tau) \geq n^{1/2} \text{ and } \zeta(\tau) \geq \zeta(t_1) + n^{-\gamma} \text{ and } |\vec{v}(\tau) - \vec{y}(\tau)|_\infty \leq 2Err(\tau) \text{ for } \tau \leq t \right\}$$

where

$$t_1 = \min \{t > 0 : y_1(t) = 0\}. \quad (21)$$

Now define four sequences of random variables:

$$\begin{aligned} X_1^\pm(t) &= \begin{cases} v_1(t) - y_1(t) \pm Err(t) & \mathcal{E}(t-1) \text{ holds} \\ X_1^\pm(t-1) & \text{otherwise} \end{cases} \\ X^\pm(t) &= \begin{cases} v(t) - y(t) \pm Err(t) & \mathcal{E}(t-1) \text{ holds} \\ X^\pm(t-1) & \text{otherwise} \end{cases} \end{aligned}$$

Because $(1-x)^{-L}$ is convex we have

$$\frac{1}{(1-(x+h))^L} \geq \frac{1}{(1-x)^L} + \frac{hL}{(1-x)^{L+1}}$$

for $L > 0$ and $0 < x < x+h < 1$. So,

$$g((t+1)/n) - g(t/n) \geq \frac{K}{n-t} g(t/n). \quad (22)$$

Suppose that $\mathcal{E}(t)$ holds. We write

$$\begin{aligned} \left| \frac{dw - y_1}{y} - \frac{dw - v_1}{v} \right| &= \left| \frac{(dw - y_1)(v - y)}{yv} + \frac{v_1 - y_1}{v} \right| \\ &= O\left(\frac{Err(t)}{v}\right). \end{aligned} \quad (23)$$

(For this we need $(dw - y_1)/y = O(1)$. But this follows from (12) and the fact that ζ is decreasing – see (20)).

Putting $F(x) = \frac{x(e^x-1)}{f(x)}$ we have (see (19)) $F'(x) = \frac{(e^x-1)^2 - x^2e^x}{f(x)^2}$ and since

$$(e^x - 1)^2 - x^2e^x = \sum_{k=4}^{\infty} (2^k - 2 - k(k-1)) \frac{x^k}{k!} \quad (24)$$

we see that $F'(x) = \Omega(1)$ in any bounded interval $[0, L]$.

Hence from (23) we have

$$O\left(\frac{Err(t)}{v}\right) = F(\zeta) - F(z) = \Omega(|\zeta - z|)$$

or

$$|\zeta - z| = O\left(\frac{Err(t)}{v}\right). \quad (25)$$

Using (25) we obtain

$$\begin{aligned} \left| \frac{vz^2}{f(z)} - \frac{y\zeta^2}{f(\zeta)} \right| &\leq \frac{|v-y|z^2}{f(z)} + y \left| \frac{z^2}{f(z)} - \frac{\zeta^2}{f(\zeta)} \right| \\ &\leq K_1 Err(t) \end{aligned} \quad (26)$$

for some $K_1 = K_1(\alpha, d) > 0$.

For the second term we use

$$\left(\frac{x^2}{f(x)}\right)' = \frac{2xf(x) - x^2(e^x - 1)}{f(x)^2} = -\frac{1}{f(x)^2} \sum_{k=4}^{\infty} \frac{k-3}{(k-1)!} x^k.$$

This implies that $\left(\frac{x^2}{f(x)}\right)' = O(1)$ for $x \geq 0$.

Now with $f = f(\zeta)$,

$$\begin{aligned} y_1''(t) &= \frac{d-1}{dw} \left(1 - \frac{y_1}{w} + \frac{y\zeta^2}{wf} + \frac{d-1}{dw} \left(y_1 - \frac{y\zeta^2}{f} - \frac{y\zeta^4}{f^2} - y\zeta^2 \left(\frac{2}{f} - \frac{e^\zeta - 1}{f^2} \right) \right) \right) \\ &= O(\zeta^{-O(1)} n^{-1}). \\ y''(t) &= \frac{d-1}{dw} \left(\frac{y\zeta^2}{wf} + \frac{d-1}{dw} \left(\frac{y\zeta^4}{f^2} + y\zeta \left(\frac{2\zeta}{f} - \frac{\zeta(e^\zeta - 1)}{f^2} \right) \right) \right) \\ &= O(\zeta^{-O(1)} n^{-1}). \end{aligned}$$

If $\mathcal{E}(t)$ holds then, where $\rho_t = \frac{d-1}{dw} = \frac{d-1}{d(n-t)}$,

$$\begin{aligned} &\mathbb{E}(X_1^+(t+1) - X_1^+(t) \mid \vec{v}(t)) = \\ &\mathbb{E}(v_1(t+1) - v_1(t) \mid \vec{v}(t)) - (y_1(t+1) - y_1(t)) + n^{2/3}(g((t+1)/n) - g(t/n)) \geq \\ &- \rho_t(v_1(t) - y_1(t)) + \rho_t \left(\frac{v(t)z(t)^2}{f(z(t))} - \frac{y(t)\zeta(t)^2}{f(\zeta(t))} \right) + O\left(\frac{1}{v(t)z(t)}\right) - y_1''(t+\theta) + \frac{Kn^{2/3}}{n-t} g(t/n) \\ &\geq \frac{n^{2/3}g(t/n)}{n-t} (-K_1 - 2 + K) + O\left(\frac{1}{v(t)z(t)}\right) - y_1''(t+\theta) \\ &\geq 0. \end{aligned}$$

This shows that $X_1^+(t), t \geq 0$ is a sub-martingale. Also,

$$\begin{aligned} |X_1^+(t+1) - X_1^+(t)| &\leq \\ &|v_1(t+1) - v_1(t)| + \sup_{0 \leq \theta \leq 1} |y_1'(t+\theta)| + n^{2/3}(g((t+1)/n) - g(t/n)) = O(1). \end{aligned}$$

It follows from the Azuma-Hoeffding inequality that we can write

$$\mathbb{P}(\exists 1 \leq t \leq t_1 : X_1^+(t) \leq X_1^+(0) - n^{3/5}) \leq e^{-\Omega(n^{1/5})}.$$

By almost identical arguments we have

$$\mathbb{P}(\exists 1 \leq t \leq t_1 : X_1^-(t) \geq X_1^-(0) + n^{3/5}) \leq e^{-\Omega(n^{1/5})}.$$

$$\mathbb{P}(\exists 1 \leq t \leq t_1 : X^+(t) \leq X^+(0) - n^{3/5}) \leq e^{-\Omega(n^{1/5})}.$$

$$\mathbb{P}(\exists 1 \leq t \leq t_1 : X^-(t) \geq X^-(0) + n^{3/5}) \leq e^{-\Omega(n^{1/5})}.$$

It follows that **whp**, when $\mathcal{E}(t)$ holds, we have

$$|v_1(t) - y_1(t)| \leq Err(t) + n^{3/5} + |v_1(0) - y_1(0)| < 2Err(t). \quad (27)$$

$$|v(t) - y(t)| \leq Err(t) + n^{3/5} + |v(0) - y(0)| < 2Err(t). \quad (28)$$

Now by construction, $\mathcal{E}(t)$ will fail at some time $t_2 \leq t_1$. It follows from (27), (28) that **whp** it will fail either because (i) $v(t_2)z(t_2) < n^{1/2}$ or $\zeta(t_2) < \zeta(t_1) + n^{-\gamma}$. We claim the latter. Observe that if $\zeta(\tau) \geq \zeta(t_1) + n^{-\gamma}$ then (14)–(17) imply $w, y_1 = \Omega(n^{1-d\gamma/(d-1)})$ and $y = \Omega(n^{1-2\gamma})$. Together with (27), (28), this implies that $v(t_2)z(t_2) = \Omega(n^{1-3\gamma}) \geq n^{1/2}$.

In summary then, **whp** the process satisfies

$$|\vec{v}(t) - \vec{y}(t)|_\infty < 2Err(t_2) = O\left(\frac{n^{2/3}}{(1 - t_2/n)^K}\right) = O(n^{2/3+Kd\gamma/(d-1)}) \quad \text{for } 1 \leq t \leq t_2 \quad (29)$$

and

$$v_1(t_2) = O(n^{1-\gamma}) \text{ where } t_2 = t_1 + O(n^{1-\gamma}). \quad (30)$$

We use (14) for (29) and (17), (25) for (30).

6 Analysis of Phase 1

We will first argue that **whp** Phase 1 is sufficient to find a matching from L to R when there is no solution $0 < \zeta \leq \alpha d$ to

$$\left(\frac{\zeta}{\alpha d}\right)^{\frac{1}{d-1}} + e^{-\zeta} - 1 = 0. \quad (31)$$

It follows from (17), (21) and (30) that in this case Phase 1 ends with there being at most $O(n^{1-\gamma})$ vertices of L left unmatched, **whp**. Furthermore at time t_2 we will have

$$w \sim \left(\frac{\zeta}{\alpha d}\right)^{\frac{d}{d-1}} n, v_1 \sim dw \text{ and } v = O(\zeta^{\frac{d-2}{d-1}} v_1)$$

where $\zeta = \zeta(t_2) = O(n^{-\gamma})$.

Lemma 10 *Suppose that $t_1 = 0$. Then **whp** at time t_2 , Γ is a forest.*

Proof: Let $R_{\geq 2}$ denote the set of vertices of degree at least two in the R -side of Γ . Let $P(d_1, \dots, d_k)$ denote $\mathbb{P}(X_1 = d_1, \dots, X_k = d_k)$ where X_1, \dots, X_v are truncated Poisson conditioned only to sum to $dw - v_1$. For large k we use the bound

$$\mathbb{P}(X_1 = d_1, \dots, X_k = d_k) \leq O(n^{1/2}) \prod_{i=1}^k \frac{z^{d_i}}{d_i! f(z)}. \quad (32)$$

For $k = O(1)$ and $d_1, \dots, d_k = O(\log n)$ we write

$$\mathbb{P}(X_1 = d_1, \dots, X_k = d_k) = \prod_{i=1}^k \mathbb{P}(X_i = d_i \mid X_j = d_j, j < i) = (1 + o(1)) \prod_{i=1}^k \frac{z^{d_i}}{d_i! f(z)}. \quad (33)$$

It is equation (7) that allows us to write the final equality in (33). The extra conditioning $X_j = d_j, j < i$ only changes the required sum.

Thus let B_k denote $O(n^{1/2})$ for $k \geq \frac{2(d-1)}{(d-2)\gamma}$ and $1 + o(1)$ otherwise. The expected number of cycles can be bounded by $o(1) = (\mathbb{P}(\exists \text{ vertex of degree } \geq \log n))$ plus

$$\sum_{k \geq 2} \sum_{\substack{S \subseteq R_{\geq 2} \\ |S|=k}} \sum_{2 \leq d_1, \dots, d_k \leq \log n} P(d_1, \dots, d_k) \times \binom{w}{k} (d(d-1))^k (k!)^2 \prod_{i=1}^k \frac{d_i(d_i-1)}{(dw-2i+2)(dw-2i+1)} \leq \quad (34)$$

$$\begin{aligned} & \sum_{k \geq 2} B_k \binom{v}{k} \sum_{2 \leq d_1, \dots, d_k \leq \log n} \prod_{i=1}^k \frac{z^{d_i}}{d_i! f(z)} \times \binom{w}{k} \binom{d}{2}^k (k!)^2 \prod_{i=1}^k \frac{d_i(d_i-1)}{(dw-2i+2)(dw-2i+1)} \leq \\ & \sum_{k \geq 2} B_k \left(\frac{vwz^2 d^2}{f(z)(dw-2k)^2} \right)^k \sum_{2 \leq d_1, \dots, d_k} \prod_{i=1}^k \frac{z^{d_i-2}}{(d_i-2)!} = \\ & \sum_{k \geq 2} B_k \left(\frac{vwz^2 e^z d^2}{f(z)(dw-2k)^2} \right)^k = \\ & \sum_{k \geq 2} B_k O(\zeta^{\frac{d-2}{d-1}})^k = \\ & o(1). \end{aligned}$$

□

Explanation of (34): We condition on the degree sequence. Having fixed the degree sequence, we swap to the configuration model [5]. Having chosen $S \subseteq R$ and k vertices W in L and their degrees, we can work within this model. We then choose a cycle through these vertices in $(k!)^2$ ways. We then choose the configuration points associated with our k -cycle in $(d(d-1))^k \prod_{i=1}^k d_i(d_i-1)$ ways. We then multiply by the probability $\prod_{i=1}^k \frac{1}{(dw-2i+2)(dw-2i+1)}$ of choosing the pairings associated with the edges of the cycle.

Corollary 11 *Suppose that $t_1 = 0$. Then **whp** at time t_2 , Γ contains a matching from L_Γ into R_Γ . Furthermore, such a matching will be constructed in Phase 1.*

Proof: We can assume from Lemma 10 that Γ is a forest. Each vertex of L_Γ has degree d and so Hall's theorem will show that the required matching exists. (Any Hall witness would induce a cycle). Finally note that Phase 1 of the Karp-Sipser algorithm is exact on a forest. □

6.1 Threshold for Phase 1 to be sufficient

Put $A = (\alpha d)^{-1/(d-1)}$ and $B = 1/(d-1)$ so that (31) can be written as

$$A\zeta^B + e^{-\zeta} - 1 = 0. \quad (35)$$

Assume B is fixed. We find a threshold for A in terms of B for there to be no positive solution to (35). We find the place ζ^* where the curve $y = 1 - e^{-\zeta}$ touches the curve $y = A\zeta^B$ i.e. where

$$\begin{aligned} A\zeta^B &= 1 - e^{-\zeta} \\ AB\zeta^{B-1} &= e^{-\zeta} \end{aligned}$$

In which case

$$\frac{\zeta^*}{B} = e^{\zeta^*} - 1 \text{ and } A^* = \frac{1 - e^{-\zeta^*}}{(\zeta^*)^B} \quad (36)$$

or

$$\zeta^* = \frac{e^{\zeta^*} - 1}{d-1} \text{ and } \alpha^* = \frac{\zeta^*}{d(1 - e^{-\zeta^*})^{d-1}}. \quad (37)$$

In general, keeping $B < 1$ fixed let

$$f_A(\zeta) = A\zeta^B + e^{-\zeta} - 1 \text{ and } L_A = A^{-1/B}.$$

We must show that if $A \geq A^*$ then the only solution to $f_A(\zeta) = 0$, $0 \leq L_A$ is $\zeta = 0$.

Observe that $f_A(L_A) = e^{-L_A} > 0$ and $f_A(0) = 0$. Also, if $A < 1$ then $1 < L_A$ and if $A < 1 - e^{-1}$ then $f_A(1) < 0$ and there must be a positive solution to $f_A(\zeta) = 0$.

Observe that $A' > A$ implies (i) $L_{A'} < L_A$ and that (ii) $f_{A'}(\zeta) > f_A(\zeta)$ for $\zeta \neq 0$. So if $f_A(\zeta)$ has no positive solution then neither has $f_{A'}$. We argue that $f'_A(\zeta) = 0$ has at most 2 solutions, which implies that $f_A(\zeta) = 0$ has at most two positive solutions. As we increase A to A^* these solutions must converge, by (ii).

Now

$$f'_A(\zeta) = 0 \text{ iff } \frac{e^\zeta}{\zeta^{1-B}} = \frac{1}{AB}.$$

But the function $g(\zeta) = e^\zeta/\zeta^\xi$ is convex for any $\xi > 0$. Indeed

$$g''(\zeta) = \frac{e^\zeta((\zeta - \xi)^2 + \xi)}{\zeta^{\xi+2}} > 0$$

and so $g(\zeta) = a$ has at most two solutions for any $a > 0$.

6.2 Finishing the proof of Theorem 2

We now have to relate the above results to the actual process. We know from our analysis of the differential equations that for some $A > 0$,

$$v_1(t_2 = t_1 + An^{-\gamma}) = O(n^{1-\gamma}).$$

When $\zeta_1 = \zeta(t_1) = 0$, Lemma 10 and Corollary 11 imply Theorem 2.

So assume that $\zeta_1 > 0$. Thus $\alpha > \alpha^*$ and $A < A^*$. We argue that if z_A is the solution to (35) then z_A decreases monotonically with A . Indeed, if $A' > A$ then $f_{A'}(\zeta) > f_A(\zeta)$ for $\zeta_A \leq \zeta \leq L_{A'} < L_A$. Now $\frac{(d-1)z}{e^z-1}$ is strictly monotone decreasing with z and so

$$\frac{(d-1)z_A}{e^{z_A}-1} < \frac{(d-1)z_{A^*}}{e^{z_{A^*}}-1} = 1. \quad (38)$$

The second equation in (38) is the first equation in (36).

At time t_2 we will have $v_1 = O(n^{1-\gamma})$ and $v = \Omega(n)$. For the next $o(n)$ steps we have from (9) that

$$\mathbb{E}(v'_1 - v_1 \mid \vec{v}) = -1 + (1 + o(1)) \frac{d-1}{dw} \frac{z^2}{f} v = -1 + (1 + o(1)) \frac{(d-1)z_A}{e^{z_A}-1} \leq -\varepsilon \quad (39)$$

for some small positive ε . In which case, **whp**, v_1 will become zero in $O(n^{1-\gamma} \log n)$ steps. Indeed (39) implies that the sequence

$$X_k = \begin{cases} v_1(t_2 + k) + \varepsilon k & \text{if } v_1(t_2 + k) > 0 \\ X_{k-1} & \text{otherwise} \end{cases}$$

is a supermartingale that cannot change by more than d in any step. The Azuma-Hoeffding inequality implies that for $T = 2X_0/\varepsilon$ we have

$$\mathbb{P}(v_1(t_2 + T) > 0) \leq \mathbb{P}(X_T - X_0 \geq \varepsilon T - X_0) \leq \exp \left\{ -\frac{(\varepsilon T - X_0)^2}{2Td^2} \right\} = o(1).$$

I.e. **whp** $v_1(t_2 + T) = 0$ and $v(t_2 + T) = v(t_2) - o(n) = \Omega(n)$.

7 Proof of Theorem 3

Let us summarize what we have to prove. We have a random bipartite graph Γ_1 with partition L_1, R_1 and $|L_1| = n_1 = \alpha_1 m_1, |R_1| = m_1$. Each vertex in L_1 has degree d and each vertex in R has degree at least 2. At this point it is convenient to drop the suffix 1. So from now on, m, n, α, Γ etc. refer to the graph left at the end of Phase 1.

The degrees of Γ satisfy, $d_L(a) = d$ for $a \in L$. The degrees of vertices in R are distributed as the box occupancies X_1, X_2, \dots, X_n in the following experiment. We throw dn balls randomly into n boxes and condition that each box gets at least two balls. In these circumstance the X_j 's are independent truncated Poisson, subject to the condition that $X_1 + X_2 + \dots + X_n = dn$, see Lemma 6 with $v_1 = 0$. Thus for any $S = \{b_1, b_2, \dots, b_s\} \subseteq R$ and any set of positive integers $k_i \geq 2, i \in S$ we have

$$\mathbb{P}(d_{R_1}(b_i) = k_i, i \in S) \leq O(n^{1/2}) \prod_{i \in S} \frac{z^{k_i}}{k_i! f(z)}$$

for $k \geq 2$ where z satisfies

$$\frac{z(e^z - 1)}{f(z)} = \frac{nd}{m}.$$

The $O(n^{1/2})$ term accounts for the conditioning $\sum_{b \in R} d_R(b) = dn$ We will prove

Theorem 12 Let Γ be a bipartite graph chosen uniformly from the sets of graphs with bipartition L, R , $|L| = n, |R| = m$ such that each vertex of L has degree $d \geq 4$ and each vertex of R has degree at least two. Then **whp**

$$\mu(\Gamma) = \min \{m, n\}.$$

7.1 Useful Lemmas

Define the function $\zeta(\gamma), \gamma > 0$ to be the unique solution to

$$\frac{u(e^u - 1)}{f(u)} = \gamma.$$

Let g be defined by

$$g(x) = (e^{\zeta(x)} - 1)^x f(\zeta(x))^{1-x}.$$

Observe that

$$\frac{f(\zeta(x))}{\zeta(x)^x} = \frac{g(x)}{x^x}. \quad (40)$$

Lemma 13 The function $g(x)$ is log-concave as a function of x

Proof: We will write ζ for $\zeta(x)$ and f for $f(\zeta)$ throughout this proof. Now $\frac{\zeta(e^\zeta - 1)}{f} = x$ from which we get

$$\frac{d\zeta}{dx} = \frac{f^2}{(e^\zeta - 1)^2 - \zeta^2 e^\zeta} \quad (41)$$

and note that $\frac{d\zeta}{dx} > 0$ for $\zeta > 0$. Taking the derivative of $\log(g(x))$ we get

$$\begin{aligned} \frac{d}{dx} \log(g(x)) &= \frac{d}{dx} \left(x \log(e^\zeta - 1) + (1 - x) \log(e^\zeta - \zeta - 1) \right) \\ &= \log \left(\frac{e^\zeta - 1}{f} \right) + \frac{d\zeta}{dx} \left(x \frac{e^\zeta}{e^\zeta - 1} + (1 - x) \frac{e^\zeta - 1}{f} \right) \end{aligned}$$

Now $x = \frac{\zeta(e^\zeta - 1)}{f}$ so

$$\begin{aligned} x \frac{e^\zeta}{e^\zeta - 1} + (1 - x) \frac{e^\zeta - 1}{f} &= \frac{\zeta e^\zeta}{f} + \frac{f - \zeta(e^\zeta - 1) e^\zeta - 1}{f} \\ &= \frac{\zeta e^\zeta (e^\zeta - \zeta - 1) + (e^\zeta - \zeta - 1 - \zeta e^\zeta + \zeta)(e^\zeta - 1)}{f^2} \\ &= \frac{(e^\zeta - 1)^2 - \zeta^2 e^\zeta}{f^2} \\ &= \frac{dx}{d\zeta} \end{aligned}$$

Thus we have

$$\frac{d}{dx} \log(g(x)) = \log \left(\frac{e^\zeta - 1}{f} \right) + 1 \quad (42)$$

Taking the second derivative we get

$$\begin{aligned}\frac{d^2}{dx^2} \log(g(x)) &= \frac{d}{dx} \left(\log \left(\frac{e^\zeta - 1}{f} \right) + 1 \right) \\ &= \frac{f}{e^\zeta - 1} \frac{e^\zeta (e^\zeta - \zeta - 1) - (e^\zeta - 1)^2}{f^2} \frac{d\zeta}{dx} \\ &= \frac{1}{(e^\zeta - 1)f} \frac{d\zeta}{dx} \left(-(\zeta - 1)e^\zeta - 1 \right)\end{aligned}$$

and since $-(\zeta - 1)e^\zeta - 1$ is strictly negative for $\zeta > 0$ we get that $g(x)$ is log-concave \square

Lemma 14 $\zeta(x)$ is concave as a function of x .

Proof: We begin with (41). We note from (24) that the denominator

$$(e^\zeta - 1)^2 - \zeta^2 e^\zeta \geq 0.$$

Then we have

$$\frac{d^2 \zeta}{dx^2} = \frac{2(1 + \zeta) + e^\zeta(-6 - e^{2\zeta}(2 + \zeta(\zeta - 4)) + \zeta^2(5 + \zeta(\zeta + 2)) + e^\zeta(6 - 2\zeta(2\zeta + 3)))}{((e^\zeta - 1)^2 - \zeta^2 e^\zeta)^2} \frac{d\zeta}{dx}.$$

Now let

$$\phi(u) = \sum_{n=0}^{\infty} \phi_n u^n = 2(1 + u) + \psi(u)$$

where

$$\psi(u) = \sum_{n=0}^{\infty} \psi_n u^n = e^u(-6 - e^{2u}(2 + u(u - 4)) + u^2(5 + u(u + 2)) + e^u(6 - 2u(2u + 3))).$$

We check that $\psi_0 = -2$ and $\psi_1 = 0$ which implies that $\phi_0 = \phi_1 = 0$. One can finish the argument by checking that

$$\psi_n = -\frac{3^{n-2}(n^2 - 13n + 18) + 2^n(n^2 + 2n - 6) - (n^4 - 4n^3 + 10n^2 - 7n - 6)}{n!} \leq 0$$

for $n \geq 2$. This is simply a matter of checking for small values until the 3^n term dominates. \square

Next let

$$H(u) = \log f(u) - u - 2 \log u = \log \left(\frac{e^u - u - 1}{u^2 e^u} \right)$$

Lemma 15 $H(u)$ is convex as a function of u .

Proof:

$$\begin{aligned}
\frac{d^2}{du^2}H(u) &= \frac{d}{du} \left(\frac{e^u - 1}{f(u)} - 1 - \frac{2}{u} \right) \\
&= \frac{e^u(e^u - 1 - u) - (e^u - 1)^2}{f^2(u)} + \frac{2}{u^2} \\
&= \frac{e^u - 1 - ue^u}{f^2(u)} + \frac{2}{u^2} \\
&= \frac{u^2(e^u - 1 - ue^u) + 2(e^u - 1 - u)^2}{u^2 f^2(u)} \\
&= \frac{2e^{2u} + u^2 e^u + u^2 + 4u + 2 - u^3 e^u - 4ue^u - 4e^u}{u^2 f^2(u)}
\end{aligned}$$

Let

$$\phi(u) = 2e^{2u} + u^2 e^u + u^2 + 4u + 2 - u^3 e^u - 4ue^u - 4e^u = \sum_{n=0}^{\infty} \phi_n u^n.$$

Direct computation gives $\phi_0 = \phi_1 = \phi_2 = 0$ and for $n \geq 3$

$$\phi_n = \frac{1}{n!} (2^{n+1} + n(n-1) - n(n-1)(n-2) - 4n - 4).$$

One can then check that $\phi_3 = \phi_4 = \phi_5 = 0 < \phi_n$ for $n \geq 6$. Thus $\frac{d^2}{du^2}H(u) \geq 0$ implying that $H(u)$ is convex. \square

7.2 The case $m \sim n$

We will first prove Theorem 12 under the assumption that $m = n$ and then in Sections 7.3 and 7.4 we will extend the result to arbitrary m . We will as usual prove that Hall's Condition holds **whp**. We will therefore estimate the probability of the existence of sets A, B where $|A| = k$ and $|B| \leq k - 1$ such that $N_\Gamma(A) \subseteq B$. Here $N_\Gamma(S)$ is the set of neighbours of S in Γ . We call such a pair of sets, a *witness* to the non-existence of a perfect matching. There are two possibilities to consider: (i) $A \subseteq L$ and $B \subseteq R$ or (ii) $A \subseteq R$ and $B \subseteq L$. We deal with both cases in order to help extend the results to $m \neq n$. We observe that if there exist a pair A, B then there exist a minimal pair and in this case each $b \in B$ has at least two neighbours in A . We deal first with the existence probability for a witness in Case (i) and leave Case (ii) until Section 7.2.2. We then combine these results to finish the case $m = n$ in Section 7.2.3. We will deal computationally with *minimal witnesses* where each vertex in B has at least 2 neighbours in A . If v has a unique neighbour w in A then $A \setminus \{w\}, B \setminus \{v\}$ is also a witness.

7.2.1 Case 1

We estimate

$$\begin{aligned}
\pi_L(k, \ell, D) &= \\
\text{P}(\exists A, B : |A| = k \leq n/2, |B| = \ell \leq k - 1, N_\Gamma(A) = B, d(B) = D, d_A(b) \geq 2, b \in B) &\leq
\end{aligned}$$

$$O(n^{1/2}) \binom{n}{k} \binom{m}{\ell} \sum_{\substack{2 \leq x_b \leq d_b, b \in [\ell] \\ \sum_b x_b = kd \\ \sum_{b \in [\ell]} d_b = D \\ \sum_{b \notin [\ell]} d_b = dn - D}} \prod_{b=1}^m \frac{z^{d_b}}{d_b! f(z)} \binom{d_b}{x_b} (kd)! \prod_{i=0}^{dk-1} \frac{1}{dn-i} = \quad (43)$$

$$O(n^{1/2}) \binom{n}{k} \binom{m}{\ell} \frac{(d(n-k))! (kd)! z^{dn}}{(dn)! f(z)^m} \times$$

$$\left(\sum_{\substack{2 \leq x_b, b \in [\ell] \\ \sum_b x_b = kd}} \prod_{b=1}^{\ell} \frac{1}{x_b!} \right) \left(\sum_{\substack{2 \leq d_b, b \notin [\ell] \\ \sum_b d_b = dn - D}} \prod_{b=k}^m \frac{1}{d_b!} \right) \left(\sum_{\substack{0 \leq y_b, b \in [\ell] \\ \sum_b y_b = D - kd}} \prod_{b=1}^{\ell} \frac{1}{y_b!} \right) =$$

$$O(n^{1/2}) \binom{n}{k} \binom{m}{\ell} \frac{(d(n-k))! (kd)! z^{dn}}{(dn)! f(z)^m} \times$$

$$\left([u^{kd}] (e^u - 1 - u)^\ell \right) \left([u^{dn-D}] (e^u - 1 - u)^{m-\ell} \right) \left([u^{D-kd}] e^{u\ell} \right) \leq \quad (44)$$

$$O(n^{1/2}) \binom{n}{k} \binom{m}{k-1} e^{2k|m-n|/n} \frac{(d(n-k))! (kd)! z^{dn}}{(dn)! f(z)^m} \frac{f(z)^\ell f(\zeta_1)^{m-k+1}}{z^{kd} \zeta_1^{dn-D}} \frac{\ell^{D-dk}}{(D-kd)!} \leq$$

where $\zeta_1 = \zeta(y) \leq z$ where $y = \frac{dn-D}{m-k+1} \geq 2$ due to our minimum degree assumption for R . The term $e^{2k|m-n|/n}$ accounts for replacing $\binom{m}{\ell}$ by $\binom{m}{k-1}$ for k exceeding $m/2$.

$$O(n^{1/2} e^{2k|m-n|/n}) \binom{n}{k} \binom{m}{k-1} \frac{(d(n-k))! (kd)! z^{dn}}{(dn)! f(z)^m} \frac{f(z)^{k-1} f(\zeta_1)^{m-k+1}}{z^{kd} \zeta_1^{dn-D}} \frac{(k-1)^{D-dk}}{(D-kd)!} \leq \quad (45)$$

$$O \left(\frac{ke^{|m-n|}}{m^{1/2}} \right) \frac{\binom{n}{k} \binom{m}{k}}{\binom{dn}{dk}} \left(\frac{z^d}{f(z)^{\frac{m-k}{n-k}}} \frac{f(\zeta_1)^{\frac{m-k}{n-k}}}{\zeta_1^{\frac{dn-D}{n-k}}} \right)^{n-k} \left(\frac{ek}{D-dk} \right)^{D-dk}.$$

Putting $k = an$ and $m = \beta n$ and $h(u) = u^u(1-u)^{1-u}$ and $x = d - y$ where $0 \leq x \leq d - 2$ we obtain, after substituting $\binom{n}{k} = O \left(\frac{1}{k^{1/2} h(a)^n} \right)$ etc.

$$\pi_L(k, \ell, D) \leq O \left(\frac{e^{|m-n|a}}{n^{1/2}} \right) \left(\frac{h(a)^{d-1}}{h(a/\beta)^\beta} \right)^n \left(\frac{z^d}{f(z)^{\frac{\beta-a}{1-a}}} \frac{f(\zeta_1)^{\frac{\beta-a}{1-a}}}{\zeta_1^{d-x}} \left(\frac{e^{\frac{a}{1-a}}}{x} \right)^x \right)^{n-k}. \quad (46)$$

Explanation of (43): Choose sets A, B in $\binom{n}{k} \binom{m}{\ell}$ ways. Choose degrees $d_b, b \in R$ with probability $O(n^{1/2}) \prod_{b=1}^m \frac{z^{d_b}}{d_b! f(z)}$ such that $\sum_{b \in B} d_b = D, \sum_{b \notin B} d_b = dn - D$ for some $D \geq 2(\ell)$. Choose the degrees $x_a, a \in A$ in the sub-graph induced by $A \cup B$. Having fixed the degree sequence, we swap to the configuration model. Choose the configuration points associated the $x_a, a \in A$ in $\prod_{a \in A} \binom{d}{x_a}$ ways. Assign these D choices of points points associated with A in $D!$ ways. Then multiply by the probability $(kd)! \prod_{i=0}^{kd-1} \frac{1}{dn-i}$ of a given pairing of points in A .

Explanation of (44) to (45): If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ where $a_n \geq 0$ for $n \geq 0$ then $a_n \leq A(\zeta)/\zeta^n$ for any positive ζ and $A(\zeta)/\zeta^n$ is minimised at ζ satisfying $\zeta A'(\zeta)/A(\zeta) = n$.

For the remainder of Section 7.2 we assume that

$$n \leq m \leq n + o(n^{7/8}). \quad (47)$$

It follows from (47) that we only need to consider

$$k \leq k_0 = m/2 = n/2 + o(n^{7/8}).$$

It also follows that

$$\left(\frac{h(a)}{h(a/\beta)^\beta} \right)^n \left(\frac{f(\zeta_1)^{\frac{\beta-1}{1-a}}}{f(z)^{\frac{\beta-1}{1-a}}} \right)^{n-k} = e^{o(n^{7/8}a)}.$$

Thus (46) becomes

$$\pi_L(k, \ell, D) = O \left(\frac{1}{n^{1/2}} \right) e^{o(n^{7/8}a)} h(a)^{(d-2)n} \left(\frac{z^d}{f(z)} \frac{f(\zeta_1)}{\zeta_1^{d-x}} \left(\frac{e^{\frac{a}{1-a}}}{x} \right)^x \right)^{n-k}. \quad (48)$$

Case 1.1: $0 \leq k \leq \min \{k_0, (1 - \frac{2}{d})n\}$.

Observe (see (40)) that

$$\frac{z^d}{f(z)} \frac{f(\zeta_1)}{\zeta_1^{d-x}} = \frac{d^d}{g(d)} \frac{g(d-x)}{(d-x)^{d-x}}$$

where $g(x)$ is as defined in Lemma 13.

It follows from (42) that

$$-\log \left(\frac{g(d-x)}{g(d)} \right) = \int_{d-x}^d \frac{d}{dt} \log(g(t)) dt \geq \int_{d-x}^d (\log(1 + \zeta e^{-\zeta}) + 1) dt. \quad (49)$$

Now $\zeta e^{-\zeta} \leq e^{-1}$ which implies that $\log(1 + \zeta e^{-\zeta}) \geq \zeta e^{-\zeta}/10$. Also,

$$\frac{\zeta(t)}{t} = 1 - \frac{\zeta}{e^\zeta - 1} \geq 1 - \frac{2}{\zeta + 2} = \frac{\zeta}{\zeta + 2}.$$

And so $\zeta \leq t \leq \zeta + 2$. Thus

$$\int_{d-x}^d (\log(1 + \zeta e^{-\zeta}) + 1) dt \geq \int_{d-x}^d \left(1 + \frac{t-2}{10e^d} \right) dt = x + \frac{x(2d-4-x)}{20e^d}.$$

This implies that $\frac{g(d-x)}{g(d)} \leq e^{-x} \psi(x)$ where $\psi(x) = e^{-\varepsilon_d(2d-4-x)x}$ and $\varepsilon_d = \frac{1}{20e^d}$. Note that $d-x \geq 2$ and so $\psi(x) \leq e^{-(d-2)\varepsilon_d x}$ in the range of interest. Plugging this into the last parenthesis of (48) gives

$$\pi_L(k, \ell, D) = O \left(\frac{1}{n^{1/2}} \right) e^{o(n^{7/8}a)} \psi(x)^n \left(h(a)^{d-2} \left(d^d \frac{1}{(d-x)^{d-x}} \left(\frac{a}{x} \right)^x \right)^{1-a} \right)^n \quad (50)$$

This immediately yields

$$A_{51} = \sum_{\ell < k = \varepsilon_L n}^{n(1-2/d)} \sum_{D=dk}^{dk+n^{1/10}} \pi_L(k, \ell, D) \leq \sum_{\ell < k = \varepsilon_L n}^{n(1-2/d)} \sum_{D=dk}^{dk+n^{1/10}} O \left(\frac{\log n}{n^{1/2}} \right) h(a)^{(d-2)n} e^{o(n)} = o(1). \quad (51)$$

We use the notation A_{51} so that the reader can easily refer back to the equation giving its definition.

We will work with $D \leq k \log n$ because it is easy to show that **whp** the maximum degree in Γ is $o(\log n)$. The bound for A_{51} comes from (50), using the fact that $h(a)$ is bounded away from 1 and $x = o(1)$ in this summation. A_{51} is the first of several sums that together show the unlikelihood chance of a witness. We will display them as they become available and use them in Sections 7.2.3, 7.3 and 7.4.

The main term $h(a)^{d-2} \left(d^d \frac{1}{(d-x)^{d-x}} \left(\frac{1-a}{x} \right)^x \right)^{1-a}$ in (50) is maximized when $x = ad$, provided $ad \leq d-2$ or $k \leq n(1 - \frac{2}{d})$. This in turn gives

$$\begin{aligned} \pi_L(k, D) &= O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8}a)} \psi(x)^n \left(h(a)^{d-2} \left(d^d \frac{1}{(d-ad)^{d-ad}} \frac{1}{((1-a)d)^{ad}} \right)^{1-a} \right)^n \\ &= O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8}a)} \psi(x)^n \left(h(a)^{d-2} \left(\frac{d^d}{d^{d-ad} d^{ad}} (1-a)^{-d} \right)^{1-a} \right)^n \\ &\leq O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8}a)} \psi(x)^n \left(a^{a(d-2)} (1-a)^{-2(1-a)} \right)^n \end{aligned} \quad (52)$$

The function $\rho_d(a) = a^{a(d-2)}(1-a)^{-2(1-a)}$ is at most 1 and is log-convex in a on $[0, 1 - \frac{2}{d}]$. Indeed, if $L_1(a) = \log \rho_d(a)$ then

$$\frac{dL_1}{da} = d + (d-2) \log a + 2 \log(1-a) \quad (53)$$

$$\frac{d^2 L_1}{da^2} = \frac{d-2-da}{a(1-a)} \quad (54)$$

We have $L_1(0) = 0$ and $L_1'(0) = -\infty$. It follows that for every $K > 0$ there exists a constant $\varepsilon_L(K, d) > 0$ such that

$$\rho_d(a) \leq e^{-Ka} \quad \text{for } a \leq \varepsilon_L(K). \quad (55)$$

We let $\varepsilon_L = \varepsilon_L(1, d)$.

We can immediately write

$$A_{56} = \sum_{\ell < k=2}^{n^{1/10}} \sum_{D=dk}^{k \log n} \pi_L(k, \ell, D) = \sum_{\ell < k=2}^{n^{1/10}} \sum_{D=dk}^{\log n} O\left(\frac{\log n}{n^{1/2}}\right) e^{o(kn^{-1/8})} = o(1). \quad (56)$$

The bound for A_{56} is derived from (52) using $\rho(a) \leq 1$.

Along the same lines we have

$$A_{57} = \sum_{\ell < k=n^{1/10}}^{\varepsilon_L n} \sum_{D=dk}^{k \log n} \pi_L(k, \ell, D) = \sum_{\ell < k=n^{1/10}}^{\varepsilon_L n} \sum_{D=dk}^{k \log n} O\left(\frac{\log n}{n^{1/2}}\right) e^{-k(1-o(n^{-1/8}))} = o(1). \quad (57)$$

The bound for A_{57} comes from (52) and (55).

Now $\psi(d) = (1 - \frac{2}{d})^{(d-2)^2/d} (\frac{2}{d})^{-4/d}$ decreases in d and is $< .9$ for $d \geq 5$. So if $d \geq 5$ then for some constant $0 < \xi < 1$ we have

$$A_{58} = \sum_{\ell < k=\varepsilon_L n}^{k_0} \sum_{D=dk+n^{1/10}}^{\log n} \pi_L(k, \ell, D) = \sum_{\ell < k=\varepsilon_L n}^{k_0} \sum_{D=dk+n^{1/10}}^{k \log n} O\left(\frac{\log n}{n^{1/2}}\right) e^{o(n^{7/8}a)} \xi^n = o(1). \quad (58)$$

The bound for A_{58} comes from (52) and using the fact that $\rho_d(a) \leq e^{-a}$.

Case 1.2: $d = 3, 4$.

Now consider the cases $d = 3, 4$. Putting $\beta_3 = .15$, $\beta_4 = .49$ we note that $\rho_d(\beta_d) \leq .995$ for $d = 3, 4$ and so arguing as above we have

$$A_{59} = \sum_{\ell < k = \varepsilon_L n}^{\beta_d n} \sum_{D = dk + n^{1/10}}^{k \log n} \pi_L(k, \ell, D) = \sum_{\ell < k = \varepsilon_L n}^{\beta_d n} \sum_{D = dk + n^{1/10}}^{\log n} O\left(\frac{\log n}{n^{1/2}}\right) e^{o(n^{7/8}a)} \xi^n = o(1). \quad (59)$$

Because we can choose any value for ζ_1 in the bound (48) we can simplify matters by choosing $\zeta_1 = \eta$ independent of x to get

$$\pi_L(k, D) = O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8}a)} h(a)^n \left(\frac{z^d}{f(z)} \frac{f(\eta)}{\eta^d} \left(\frac{e\eta a}{x(1-a)}\right)^x\right)^{n-k}. \quad (60)$$

Now

$$\left(\frac{\eta e a}{(1-a)x}\right)^x \leq \exp\left\{\frac{\eta a}{1-a}\right\} \quad (61)$$

and so

$$\pi_L(k, D) \leq O\left(\frac{k}{n^{1/2}}\right) \left(h(a) e^{\eta a} e^{o(n^{-1/8}a)} \left(\frac{z^d}{f(z)} \frac{f(\eta)}{\eta^d}\right)^{1-a}\right)^n. \quad (62)$$

Now the function $L_2(a) = h(a) e^{\eta a} \left(\frac{z^d}{f(z)} \frac{f(\eta)}{\eta^d}\right)^{1-a}$ is log-convex in a . Our initial choice of η will be 1.5 and we note that with this choice, when $d = 4$, $L_2(.49), L_2(.51) < .9$ and so

$$A_{63} = \sum_{\ell < k = \beta_4 n}^{k_0} \sum_{D = 4k}^{k \log n} \pi_L(k, \ell, D) \leq \sum_{\ell < k = \beta_4 n}^{k_0} \sum_{D = 4k}^{k \log n} O\left(\frac{\log n}{n^{1/2}}\right) e^{o(n^{-1/8}k)} (.9)^n = o(1). \quad (63)$$

When $d = 3$, with the same choice for η , we have $L_2(.15), L_2(2/5) < .98$ and so

$$A_{64} = \sum_{\ell < k = .15n}^{2n/5} \sum_{D = 3k + n^{1/10}}^{k \log n} \pi_L(k, \ell, D) \leq \sum_{\ell < k = .15n}^{2n/5} \sum_{D = 3k + n^{1/10}}^{k \log n} O\left(\frac{\log n}{n^{1/2}}\right) e^{o(n^{7/8}a)} (.98)^n = o(1). \quad (64)$$

We repeat this idea once more. Putting $\eta = .5$ we get $L_2(2/5), L_2(.51) < .995$ from which we deduce that

$$A_{65} = \sum_{\ell < k = 2n/5}^{k_0} \sum_{D = 3k + n^{1/10}}^{k \log n} \pi_L(k, \ell, D) \leq \sum_{\ell < k = 2n/5}^{k_0} \sum_{D = 3k + n^{1/10}}^{k \log n} O\left(\frac{\log n}{n^{1/2}}\right) e^{o(n^{7/8}a)} (.995)^n = o(1). \quad (65)$$

7.2.2 Case 2

Now let us estimate the probability of a violation of Hall's condition with $A \subseteq R$. We once again begin with arbitrary m . Let

$$\begin{aligned}
& \pi_R(k, \ell, D) = \\
& \mathbb{P}(\exists A \subseteq R, B \subseteq L : |A| = k, |B| = \ell \leq k-1, N_\Gamma(A) \subseteq B, d_B(b) \geq 2, b \in B, d_R(A) = D) \leq \\
& O(n^{1/2}) \binom{m}{k} \binom{n}{\ell} \sum_{\substack{2 \leq d_a, a \in [m] \\ 2 \leq x_b \leq d, b \in [\ell] \\ \sum_{a \in [k]} d_a = \sum_{b \in [\ell]} x_b = D \\ \sum_{a \notin [k]} d_a = dn - D}} \prod_{a=1}^m \frac{z^{d_a}}{d_a! f(z)} \prod_{b=1}^{\ell} \binom{d}{x_b} D! \prod_{i=0}^{D-1} \frac{1}{dn-i} = \tag{66} \\
& O(n^{1/2}) \binom{m}{k} \binom{n}{\ell} \frac{z^{dn} D! (dn-D)!}{f(z)^m (dn)!} \\
& \quad \times \left(\sum_{\substack{2 \leq d_a, a \in [k] \\ \sum_a d_a = D}} \prod_{a=1}^k \frac{1}{d_a!} \right) \left(\sum_{\substack{2 \leq d_a, a \notin [k] \\ \sum_a d_a = dn-D}} \prod_{a=k+1}^n \frac{1}{d_a!} \right) \left(\sum_{\substack{2 \leq x_b \leq d, b \in [\ell] \\ \sum_b x_b = D}} \prod_{b=1}^{\ell} \binom{d}{x_b} \right) = \\
& O(n^{1/2}) \binom{m}{k} \binom{n}{\ell} \frac{z^{dn}}{f(z)^m} \frac{1}{\binom{dn}{D}} \\
& \quad \times \left([u^D] (e^u - 1 - u)^k \right) \left([u^{dn-D}] (e^u - 1 - u)^{m-k} \right) \left([u^D] ((1+u)^d - (1+du))^\ell \right) \leq \\
& O(n^{1/2}) e^{2k|m-n|/n} \binom{m}{k} \binom{n}{k-1} \frac{z^{dn}}{f(z)^m} \frac{1}{\binom{dn}{D}} \\
& \quad \times \left([u^D] (e^u - 1 - u)^k \right) \left([u^{dn-D}] (e^u - 1 - u)^{m-k} \right) \left([u^D] ((1+u)^d - (1+du))^k \right) \leq \tag{67} \\
& O(n^{1/2}) e^{2k|m-n|/n} \binom{m}{k} \binom{n}{k-1} \frac{z^{dn}}{f(z)^m} \frac{1}{\binom{dn}{D}} \frac{f(\zeta_1)^k}{\zeta_1^D} \frac{f(\zeta_2)^{m-k}}{\zeta_2^{dn-D}} \binom{dk}{D} \leq
\end{aligned}$$

where $\zeta_1 = \zeta(D/k)$ and $\zeta_2 = \zeta\left(\frac{dn-D}{m-k}\right)$ – actually any value for ζ_1, ζ_2 is valid –

$$O(n^{1/2}) e^{2k|m-n|/n} \binom{m}{k} \binom{n}{k} \frac{\binom{dk}{D}}{\binom{dn}{D}} \left(\frac{f(\zeta_1)}{f(z)} \right)^k \left(\frac{f(\zeta_2)}{f(z)} \right)^{m-k} \left(\frac{z}{\zeta_1} \right)^D \left(\frac{z}{\zeta_2} \right)^{dn-D} \tag{68}$$

$$= O\left(\frac{1}{n^{1/2}} \right) e^{2a|m-n|} \left(\frac{h(\theta a/d)^d}{h(a)h(a/\beta)^\beta h(\theta/d)^{ad}} \left(\frac{f(\zeta_1)}{f(z)} \right)^a \left(\frac{f(\zeta_2)}{f(z)} \right)^{\beta-a} \left(\frac{z}{\zeta_1} \right)^{\theta a} \left(\frac{z}{\zeta_2} \right)^{d-\theta a} \right)^n \tag{69}$$

$$= O\left(\frac{1}{n^{1/2}} \right) e^{2a|m-n|} \left(\frac{h(\theta a/d)^d}{h(a)h(a/\beta)^\beta h(\theta/d)^{ad}} \frac{z^d}{f(z)^\beta} \frac{f(\zeta_1)^a}{\zeta_1^{\theta a}} \frac{f(\zeta_2)^{\beta-a}}{\zeta_2^{d-\theta a}} \right)^n \tag{70}$$

where $a = k/n$, $m = \beta n$ and $D = \theta k \leq dk$.

Explanation of (66): Choose sets A, B in $\binom{m}{k} \binom{n}{\ell}$ ways. Choose degrees $d_a, a \in R$ with probability $O(n^{1/2}) \prod_{a=1}^n \frac{z^{d_a}}{d_a! f(z)}$ such that $\sum_{a \in A} d_a = D$, $\sum_{a \notin A} d_a = dn - D$ for some $D \geq 2k$. Choose the degrees $x_b, b \in B$ in the sub-graph induced by $A \cup B$. Having fixed the degree sequence, swap to the

configuration model [5]. Choose the configuration points associated with the $x_b, b \in B$ in $\prod_{b \in B} \binom{d}{x_b}$ ways. Then multiply by the probability $D! \prod_{i=0}^{D-1} \frac{1}{dn-i}$ of a given pairing of points in A .

We assume that (47) holds for the remainder of the section. In which case we have

$$\frac{h(a)}{h(a/\beta)^\beta} \frac{f(\zeta_2)^{\beta-1}}{f(z)^{\beta-1}} = e^{o(n^{-1/8}a)}$$

and we only need to consider

$$k \leq k_0 = m/2 = n/2 + o(n^{7/8}).$$

Thus, (70) becomes

$$\pi_R(k, \ell, D) \leq O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8}a)} \frac{h(\theta a/d)^d}{h(a)^2 h(\theta/d)^{ad}} \frac{z^d}{f(z)} \frac{f(\zeta_1)^a}{\zeta_1^{\theta a}} \frac{f(\zeta_2)^{1-a}}{\zeta_2^{d-\theta a}} \right)^n \quad (71)$$

It follows from Lemma 13 that we can upper bound

$$\begin{aligned} \frac{z^d}{f(z)} \left(\frac{f(\zeta_1)}{\zeta_1^\theta} \right)^a \left(\frac{f(\zeta_2)}{\zeta_2^{\frac{d-a\theta}{1-a}}} \right)^{1-a} &= \frac{d^d}{g(d)} \frac{g(\theta)^a g\left(\frac{d-a\theta}{1-a}\right)^{1-a}}{\theta^{a\theta} \left(\frac{d-a\theta}{1-a}\right)^{d-a\theta}} \\ &\leq \frac{g(a\theta + (1-a)\frac{d-a\theta}{1-a})}{g(d)} \frac{d^d}{\theta^{a\theta} \left(\frac{d-a\theta}{1-a}\right)^{d-a\theta}} \\ &= \frac{a^{a\theta} (1-a)^{d-a\theta}}{\left(\left(\frac{a\theta}{d}\right)^{\frac{a\theta}{d}} (1-\frac{a\theta}{d})^{1-\frac{a\theta}{d}} \right)^d} \\ &= \frac{a^{a\theta} (1-a)^{d-a\theta}}{h\left(\frac{a\theta}{d}\right)^d} \end{aligned}$$

Plugging this into (71) gives

$$\pi_R(k, \ell, D) \leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8}a)} a^{a\theta} (1-a)^{d-a\theta}}{h(a)^2 h\left(\frac{\theta}{d}\right)^{ad}} \right)^n \quad (72)$$

Now let $R_1(\theta) = \log\left(\frac{a^{a\theta} (1-a)^{d-a\theta}}{h(a)^2 h\left(\frac{\theta}{d}\right)^{ad}}\right)$. Then

$$\begin{aligned} R_1'(\theta) &= a \log a - a \log(1-a) - a \log \theta + a \log(d-\theta). \\ R_1''(\theta) &= -\frac{ad}{\theta(d-\theta)} < 0. \end{aligned}$$

Thus $R_1(\theta)$ is concave and is maximized when $\theta = ad$. Because $\theta \geq 2$ we can only use this for $a \geq 2/d$.

Case 2.1: $k \geq 2n/d$.

$$\begin{aligned}
\pi_R(k, \ell, D) &\leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8}a)} a^{2d} (1-a)^{d-a^2d}}{h(a)^{2+ad}}\right)^n \\
&= O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8}a)} a^{2d-a(2+ad)} (1-a)^{d-a^2d-(1-a)(2+ad)}\right)^n \\
&= O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8}a)} a^{-2a} (1-a)^{(d-2)(1-a)}\right)^n \\
&= O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8}a)} \rho_d (1-a)^n
\end{aligned} \tag{73}$$

where the function ρ_d is defined following (52).

We find that

$$\rho_d(1-2/d) = \left(\frac{d^4}{16} \left(1 - \frac{2}{d}\right)^{(d-2)^2}\right)^{1/d} \leq .9 \text{ for } d \geq 5. \tag{74}$$

Now $\rho_d(1-2/d) < 9/10$ and $\rho_d(.5) < .8$ for $d \geq 5$. So, with the aid of (55),

$$B_{75} = \sum_{\ell < k = 2n/d}^{k_0} \sum_{D=2k}^{k \log n} \pi_R(k, \ell, D) \leq \sum_{\ell < k = 2n/d}^{k_0} \sum_{D=2k}^{k \log n} O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{7/8}a)} (.8)^n \quad \text{for } d \geq 5. \tag{75}$$

We will treat $d = 4$ under Case 2.2.

Case 2.2: $2 \leq k \leq 2n/d$.

In this case the expression in (72) (ignoring error terms) is maximized at $\theta = 2$. Then

$$\begin{aligned}
\pi_R(k, D) &\leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8}a)} a^{2a} (1-a)^{d-2a}}{h(a)^2 h\left(\frac{2}{d}\right)^{ad}}\right)^n \\
&= O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8}a)} (1-a)^{d-2a-2(1-a)}}{h\left(\frac{2}{d}\right)^{ad}}\right)^n \\
&= O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8}a)} (1-a)^{d-2}}{h\left(\frac{2}{d}\right)^{ad}}\right)^n
\end{aligned}$$

Let $R_2(a) = \log\left(\frac{(1-a)^{d-2}}{h\left(\frac{2}{d}\right)^{ad}}\right)$. Then

$$\begin{aligned}
R_2'(a) &= -\frac{d-2}{1-a} - d \log h(2/d) < 0 \quad \text{for } d \geq 6. \\
R_2''(a) &= -\frac{d-2}{(1-a)^2} < 0.
\end{aligned}$$

Thus $R_2(a)$ is strict concave and its maximum is taken at $a = 0$ and $R_2(a) \leq R_2(0)a$ for all

$a \in [0, \frac{2}{d}]$. Furthermore, $R_2(0) < -3/10$ for $d \geq 6$. It follows that if $d \geq 6$ then

$$B_{76} = \sum_{\ell < k=2}^{n^{1/10}} \sum_{D=2k}^{dk} \pi_R(k, \ell, D) \leq \sum_{\ell < k=2}^{n^{1/10}} \sum_{D=2k}^{dk} O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{-1/8}k)} = o(1). \quad (76)$$

$$B_{77} = \sum_{\ell < k=n^{1/10}}^{2n/d} \sum_{D=2k}^{dk} \pi_R(k, \ell, D) \leq \sum_{\ell < k=n^{1/10}}^{2n/d} \sum_{D=2k}^{dk} O\left(\frac{1}{n^{1/2}}\right) e^{-(3/10+o(n^{-1/8}))k} = o(1). \quad (77)$$

For $d = 3, 4, 5$ we use a better bound on $[u^D]((1+u)^d - 1 - du)^k$ in (67).

Case 2.2a: $d = 5$.

$$\begin{aligned} [u^D]((1+u)^5 - 1 - 5u)^k &= [u^D](10u^2 + 10u^4 + u^5)^k \\ &= [u^{D-2k}](10 + 10u + 5u^2 + u^3)^k \\ &= 10^k [u^{D-2k}] \left(1 + u + \frac{u^2}{2} + \frac{u^3}{10}\right)^k \\ &\leq 10^k [u^{D-2k}] \left(1 + \frac{u}{2}\right)^{3k} \\ &= 10^k \frac{\binom{3k}{D-2k}}{2^{D-2k}} \end{aligned}$$

Replacing the $\frac{1}{h(\frac{\theta}{d})^{ad}}$ factor in (72) which comes from $\binom{dk}{D}$ gives, for $d=5$,

$$\begin{aligned} \pi_R(k, D) &\leq O\left(\frac{k}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8}a)} a^{a\theta} (1-a)^{5-a\theta}}{h(a)^2}\right)^n \left(\frac{10}{2^{\theta-2} h\left(\frac{\theta-2}{3}\right)^3}\right)^k \\ &= O\left(\frac{k}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8}a)} 10^a a^{a(\theta-2)} (1-a)^{5-2-a(\theta-2)}}{\left(2^{\theta-2} h\left(\frac{\theta-2}{3}\right)^3\right)^a}\right)^n \\ &= O\left(\frac{k}{n^{1/2}}\right) \left(e^{o(n^{-1/8}a)} 10^a (1-a)^3 \left(\frac{\left(\frac{a}{1-a}\right)^{\frac{\theta-2}{3}}}{2^{\frac{\theta-2}{3}} h\left(\frac{\theta-2}{3}\right)}\right)^{3a}\right)^n \end{aligned} \quad (78)$$

Let $p(x) = \frac{q^x}{h(x)}$ for any $x \in [0, 1]$, note that if $P(x) = \log p(x)$ then

$$\begin{aligned} P'(x) &= \log q - \log x + \log(1-x) \\ P''(x) &= -\frac{1}{x} - \frac{1}{1-x} < 0 \end{aligned}$$

and so $p(x)$ is maximized when $\log q = \log\left(\frac{x}{1-x}\right)$ or $x = \frac{q}{1+q}$ and the maximum value is $1+q$

Thus from (78) we get

$$\pi_R(k, \ell, D) \leq O\left(\frac{k}{n^{1/2}}\right) \left(e^{o(n^{-1/8}a)} 10^a (1-a)^3 \left(1 + \frac{a}{2(1-a)}\right)^{3a}\right)^n$$

Let $R_3(a) = \log \left(10^a (1-a)^3 \left(1 + \frac{a}{2(1-a)} \right)^{3a} \right)$. Then

$$\begin{aligned} R'_3(a) &= \log 10 - \frac{6}{2-a} + 3 \log \left(\frac{2-a}{2-2a} \right) \\ R''_3(a) &= \frac{3a}{(2-a)^2(1-a)} > 0. \end{aligned}$$

So $R_3(a)$ is log-convex on $[0, \frac{2}{5}]$. We have $R_3(0) = 0$ and $R'_3(0) = \log 10 - 3 \leq -3/4$ and $R_3(2/5) < -1/4$. It follows that

$$\begin{aligned} B_{79} &= \sum_{\ell < k=2}^{2n/5} \sum_{D=2k}^{5k} \pi_R(k, \ell, D) \leq \\ &\sum_{\ell < k=2}^{n^{1/10}} \sum_{D=2k}^{5k} O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{-1/8}a)} + \sum_{\ell < k=n^{1/10}}^{2n/5} \sum_{D=2k}^{5k} O\left(\frac{1}{n^{1/2}}\right) e^{-(3/4+o(n^{-7/8}))k} = o(1). \quad (79) \end{aligned}$$

Case 2.2b: $d = 4$.

$$\begin{aligned} [u^D]((1+u)^4 - 1 - 4u)^k &= [u^{D-2k}](6 + 4u + u^2)^k \\ &= 6^k [u^{D-2k}] \left(1 + \frac{4}{6}u + \frac{u^2}{6} \right)^k \\ &\leq 6^k [u^{D-2k}] \left(1 + \frac{u}{2} \right)^k \\ &= 6^k \frac{\binom{2k}{D-2k}}{2^{D-2k}}. \end{aligned}$$

$$\begin{aligned} \pi_R(k, \ell, D) &\leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8}a)} a^{a\theta} (1-a)^{4-a\theta}}{h(a)^2} \right)^n \left(\frac{6}{2^{\theta-2} h\left(\frac{\theta-2}{2}\right)^2} \right)^k \\ &= O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8}a)} 6^a a^{a(\theta-2)} (1-a)^{2-a(\theta-2)} \left(\frac{1}{2^{\frac{\theta-2}{2}} h\left(\frac{\theta-2}{2}\right)} \right)^{2a} \right)^n \\ &= O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8}a)} 6^a (1-a)^2 \left(\frac{\left(\frac{a}{1-a}\right)^{\frac{\theta-2}{2}}}{2^{\frac{\theta-2}{2}} h\left(\frac{\theta-2}{2}\right)} \right)^{2a} \right)^n \\ &\leq O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8}a)} 6^a (1-a)^2 \left(1 + \frac{a}{2(1-a)} \right)^{2a} \right)^n. \end{aligned}$$

Now if $R_4(a) = \log \left(6^a (1-a)^2 \left(1 + \frac{a}{2(1-a)} \right)^{2a} \right)$ then

$$\begin{aligned} R'_4(a) &= \log 6 - \frac{4}{2-a} + 2 \log \left(\frac{2-a}{2-2a} \right) \\ R''_4(a) &= \frac{2a}{(2-a)^2(1-a)} > 0. \end{aligned}$$

Thus R_4 is log-convex on $[0, .51]$. We have $R_4(0) = 1$ and $R_4'(0) = \log 6 - 2 \leq -1/5$ and $R_4(.51) < -1/20$. It follows from this and (55) that

$$B_{80} = \sum_{\ell < k=2}^{k_0} \sum_{D=2k}^{4k} \pi_R(k, \ell, D) \leq \sum_{\ell < k=2}^{n^{1/10}} \sum_{D=2k}^{4k} O\left(\frac{1}{n^{1/2}}\right) e^{o(n^{-1/8}a)} + \sum_{\ell < k=n^{1/10}}^{k_0} \sum_{D=2k}^{4k} O\left(\frac{1}{n^{1/2}}\right) e^{-(1/5+o(n^{-1/8}))k} = o(1). \quad (80)$$

Case 2.2c: $d = 3$.

$$\begin{aligned} [u^D]((1+u)^3 - 1 - 3u)^k &= [u^{D-2k}](3+u)^k \\ &= 3^{3k-D} \binom{k}{D-2k}. \end{aligned}$$

$$\begin{aligned} \pi_R(k, \ell, D) &\leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{o(n^{-1/8}a)} a^{a\theta} (1-a)^{3-a\theta}}{h(a)^2}\right)^n \left(\frac{3^{3-\theta}}{h(\theta-2)}\right)^k \\ &= O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8}a)} a^{a(\theta-2)} (1-a)^{1-a(\theta-2)} \left(\frac{3^{3-\theta}}{h(\theta-2)}\right)^a\right)^n \\ &= O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8}a)} 3^a (1-a) \left(\frac{\left(\frac{a}{3(1-a)}\right)^{\theta-2}}{h(\theta-2)}\right)^a\right)^n \\ &\leq O\left(\frac{1}{n^{1/2}}\right) \left(e^{o(n^{-1/8}a)} 3^a (1-a) \left(1 + \frac{a}{3(1-a)}\right)^a\right)^n. \end{aligned}$$

Now if $R_5(a) = \log\left(3^a(1-a)\left(1 + \frac{a}{3(1-a)}\right)^a\right)$ then

$$\begin{aligned} R_5'(a) &= -\frac{3}{3-2a} + \log\left(\frac{3-2a}{1-a}\right) \\ R_5''(a) &= \frac{4a-3}{(3-2a)^2(1-a)}. \end{aligned}$$

Case 2.2c(ii): $0 \leq k \leq a_0 = k_0/n$.

(a) $\theta \geq 2.0005$ and $0 \leq a \leq a_0$.

We go back to (71) and make the choice $\zeta_1 = \zeta_2 = z$ and replace $h(\theta/d)^{-ad}$ by $\left(\frac{3^{3-\theta}}{h(\theta-2)}\right)^a$ and consider the function

$$F_1(\theta, a) = \frac{h(\theta a/3)^3 3^{3(3-\theta)a}}{h(a)^2 h(\theta-2)^a}$$

so that $\pi_R(k, \ell, D) \leq O\left(\frac{1}{n^{1/2}}\right) F_1(\theta, a)^n$. Let $G_1(\theta, a) = \log(F_1(\theta, a))$. Then

$$\frac{\partial G_1}{\partial a} = \log\left(\frac{27(1-a)^2(\theta-2)^{\theta-2}(a\theta)^\theta}{3^\theta a^2(3-\theta)^{3-\theta}(3-a\theta)^\theta}\right) \quad (81)$$

$$\frac{\partial G_1}{\partial \theta} = a\left(\log\left(\frac{3-\theta}{9}\right) - \log(\theta-2) + \log(a\theta) - \log\left(1 - \frac{a\theta}{3}\right)\right) \quad (82)$$

$$\frac{\partial^2 G_1}{\partial a^2} = \frac{(3-a)\theta - 6}{a(1-a)(3-a\theta)} \quad (83)$$

$$\frac{\partial^2 G_1}{\partial \theta^2} = \frac{((3-a)\theta^2 - 12\theta + 18)a}{\theta(3-a\theta)(\theta-3)(\theta-2)}. \quad (84)$$

It follows from (83) that

$$G_1(\theta, a) \text{ is a convex function of } a \text{ for } 0 \leq a \leq a_\theta = \frac{3\theta-6}{\theta}, \text{ for } \theta \text{ fixed, } 2 \leq \theta \leq 3 \quad (85)$$

and

$$G_1(\theta, a) \text{ is a concave function of } a \text{ for } a_\theta \leq a \leq 1, \text{ for } \theta \text{ fixed, } 2 \leq \theta \leq 3. \quad (86)$$

It follows from (84) that

$$G_1(\theta, a) \text{ is a concave function of } \theta \text{ on } [2, 3] \text{ for } a \text{ fixed, } 0 \leq a \leq 1. \quad (87)$$

A calculation shows that if $g_1(\theta) = G_1(\theta, a_\theta)$ then

$$g_1'(\theta) = \frac{3}{\theta^2} \log\left(\frac{144}{3^{\theta^2}(3-\theta)^2}\right) \quad (88)$$

$$g_1''(\theta) = -\frac{6}{\theta^3(3-\theta)} \left(-\theta + 2(3-\theta) \log\left(\frac{12}{3-\theta}\right)\right). \quad (89)$$

Furthermore, if $g_2(\theta) = \frac{\partial G_1}{\partial a} \big|_{a=a_\theta}$ then

$$g_2(\theta) = \log\left(\frac{12}{3^\theta(3-\theta)}\right). \quad (90)$$

(a) $2.0005 \leq \theta \leq 3$ and $0 \leq a \leq e^{-10000}$.

For $a \leq e^{-10000}$ we have $\frac{\partial G_1}{\partial a} \leq \log 10 - (\theta-2) \log 1/a \leq -2$. So,

$$F_1(\theta, a) \leq e^{-2a} \text{ for } 0 \leq a \leq e^{-10000}, 2.0005 \leq \theta \leq 3. \quad (91)$$

(b) $2.46 \leq \theta \leq 3$ and $e^{-10000} \leq a \leq a_0$.

Now $a_\theta > a_0$ for $\theta \geq 2.46$ and so (85) implies that $G_1(\theta, a) \leq \max\{G_1(\theta, e^{-10000}), G_1(\theta, a_0)\}$ for $2.46 \leq \theta \leq 3$ and $e^{-10000} \leq a \leq a_0$. Now (91) implies that $G_1(2.46, e^{-10000}) < -2e^{-10000}$ and (82) implies that $\frac{\partial G_1}{\partial \theta} \big|_{\theta=2.46, a=e^{-10000}} < 0$ and so (87) implies that $G_1(\theta, e^{-10000}) \leq -2e^{-10000}$ for $2.46 \leq \theta \leq 3$. Also, by direct calculation, we have $G_1(2.46, a_0) < -.002$ and $\frac{\partial G_1}{\partial \theta} \big|_{\theta=2.46, a=a_0} < 0$ and so $G_1(\theta, a_0) \leq -.002$ for $2.46 \leq \theta \leq 3$. Thus,

$$F_1(\theta, a) \leq e^{-2e^{-10000}} \text{ for } e^{-10000} \leq a \leq a_0 \text{ and } 2.46 \leq \theta \leq 3.$$

(c) $2.0005 \leq \theta \leq 2.25$ and $e^{-1000} \leq a \leq a_0$.

We take $\zeta_1 = .6$ and $\zeta_2 = 2.1$ in (71) and let

$$F_2(\theta, a) = F_1(\theta, a) \frac{z^3}{f(z)} \frac{f(\zeta_1)^a}{\zeta_1^{\theta a}} \frac{f(\zeta_2)^{1-a}}{\zeta_2^{3-\theta a}} = F_1(\theta, a) e^{\rho_2 + \sigma_2 a + \tau_2 a \theta}$$

where

$$e^{\rho_2} = \frac{z^3 f(\zeta_2)}{f(z) \zeta_2^3}, \quad e^{\sigma_2} = \frac{f(\zeta_1)}{f(\zeta_2)}, \quad e^{\tau_2} = \frac{\zeta_2}{\zeta_1}.$$

Let $G_2(\theta, a) = \log(F_2(\theta, a))$. $\frac{\partial^2 G_2}{\partial a^2} = \frac{\partial^2 G_1}{\partial a^2}$ and $\frac{\partial^2 G_2}{\partial \theta^2} = \frac{\partial^2 G_1}{\partial \theta^2}$ and so (85), (86) and (87) hold with G_1 replaced by G_2 . Putting $\gamma_2(\theta) = G_2(\theta, a_\theta)$ we see that $\gamma_2''(\theta) = g_1''(\theta) - \frac{12\sigma_2}{\theta^3} > 0$, using (89) ($\sigma_2 < -3.127$). Thus γ_2 is convex on $2.0005 \leq \theta \leq 2.25$. Furthermore $\gamma_2(2.0005), \gamma_2(2.25) < -.00003$ and so $\gamma_2(\theta) < -.00003$ for $\theta \in [2.0005, 2.25]$ and therefore $G_2(\theta, a) \leq -.00003a/a_\theta < -.00003a$ when $0 \leq a \leq a_\theta$ and $\theta \in [2.0005, 2.25]$. Next let $\phi_2(\theta) = \frac{\partial G_2}{\partial a} |_{a=a_\theta}$. We have $\phi_2(\theta) = g_2(\theta) + \sigma_2 + \tau_2 \theta < -.05$ for $2.0005 \leq \theta \leq 2.25$, using (90) ($\tau_2 < 1.253$). So, $G_2(\theta, a) \leq \phi_2(\theta) - .05(a - a_\theta)$ for $a \geq a_\theta$ when $\theta \in [2.0005, 2.25]$. Thus

$$F_2(\theta, a) < e^{-.00003a} \text{ for } e^{-1000} \leq a \leq a_0 \text{ and } 2.0005 \leq \theta \leq 2.25.$$

Now suppose that we repeat the idea of the previous paragraph, but this time we take $\zeta_1 = 1.4$ and $\zeta_2 = 3$ in (71) and use the same notation. Putting $\gamma_2(\theta) = G_2(\theta, a_\theta)$ we see that $\gamma_2''(\theta) = g_1''(\theta) - \frac{12\sigma_2}{\theta^3} > 0$, using (89) ($\sigma_2 < -2.27$). Thus γ_2 is convex on $2.25 \leq \theta \leq 2.46$. Furthermore $\gamma_2(2.25), \gamma_2(2.46) < -.05$ and so $\gamma_2(\theta) < -.05$ for $\theta \in [2.25, 2.46]$ and therefore $G_2(\theta, a) \leq -.05a/a_\theta < -.05a$ when $0 \leq a \leq a_\theta$ and $\theta \in [2.25, 2.46]$. Next let $\phi_2(\theta) = \frac{\partial G_2}{\partial a} |_{a=a_\theta}$. We have $\phi_2(\theta) = g_2(\theta) + \sigma_2 + \tau_2 \theta < -.2$ for $2.25 \leq \theta \leq 2.46$, using (90) ($\tau_2 < .763$). So, $G_2(\theta, a) \leq \phi_2(\theta) - .2(a - a_\theta)$ for $a \geq a_\theta$ when $\theta \in [2.25, 2.46]$. Thus

$$F_2(\theta, a) < e^{-.05a} \text{ for } e^{-1000} \leq a \leq a_0 \text{ and } 2.25 \leq \theta \leq 2.46.$$

(d) $2 \leq \theta \leq 2.0005$ and $e^{-1000} \leq a \leq a_0$.

For this we simplify our estimate of $\pi_R(k, \ell, D)$ by removing some terms involving β from (70).

$$\begin{aligned} \pi_R(k, \ell, D) &\leq \mathbb{P}(\exists A \subseteq R, B \subseteq L : |A| = k, |B| = k-1, N_\Gamma(A) \subseteq B, d_B(b) \geq 2, b \in B) \leq \\ &O(n^{1/2}) \binom{m}{k} \binom{n}{k-1} \sum_{\substack{2 \leq d_a, a \in [m] \\ 2 \leq x_b \leq d, b \in [k-1] \\ \sum_{a \in [k]} d_a = \sum_{b \in [k-1]} x_b = D}} \prod_{a=1}^k \frac{z^{d_a}}{d_a! f(z)} \prod_{b=1}^{k-1} \binom{d}{x_b} D! \prod_{i=0}^{D-1} \frac{1}{dn-i} = \\ &O\left(\frac{k}{m^{1/2}}\right) \binom{n}{k} \binom{m}{k} \frac{z^D D! (dn-D)!}{f(z)^k (dn)!} \left(\sum_{\substack{2 \leq d_a, a \in [k] \\ \sum_a d_a = D}} \prod_{a=1}^k \frac{1}{d_a!} \right) \left(\sum_{\substack{2 \leq x_b \leq d, b \in [k-1] \\ \sum_b x_b = D}} \prod_{b=1}^{k-1} \binom{d}{x_b} \right) = \\ &O\left(\frac{k}{m^{1/2}}\right) \binom{n}{k} \binom{m}{k} \frac{z^D}{f(z)^k} \frac{1}{\binom{dn}{D}} ([u^D](e^u - 1 - u)^k) ([u^D]((1+u)^d - (1+du)^k)) \leq \\ &O\left(\frac{k}{m^{1/2}}\right) \binom{n}{k} \binom{m}{k} \frac{z^D}{f(z)^k} \frac{1}{\binom{dn}{D}} \frac{f(\zeta_1)^k}{\zeta_1^D} \binom{dk}{D} = \end{aligned}$$

$$\begin{aligned}
& O\left(\frac{k}{m^{1/2}}\right) \binom{n}{k} \binom{m}{k} \frac{\binom{dk}{D}}{\binom{dn}{D}} \left(\frac{f(\zeta_1)}{f(z)}\right)^k \left(\frac{z}{\zeta_1}\right)^D \\
&= O\left(\frac{1}{m^{1/2}}\right) \left(\frac{h(\theta a/d)^d}{h(a)h(a/\beta)^\beta h(\theta/d)^{ad}} \left(\frac{f(\zeta_1)}{\zeta_1^\theta} \frac{z^\theta}{f(z)}\right)^a\right)^n \\
&= O\left(\frac{1}{m^{1/2}}\right) \left(e^{o(n^{-1/8}a)} \frac{h(\theta a/d)^d}{h(a)^2 h(\theta/d)^{ad}} \left(\frac{f(\zeta_1)}{\zeta_1^\theta} \frac{z^\theta}{f(z)}\right)^a\right)^n. \tag{92}
\end{aligned}$$

Now let

$$F_3(\theta, a) = \frac{h(\theta a/3)^3 3^{3(3-\theta)a}}{h(a)^2 h(\theta-2)^a} \left(\frac{f(\zeta_1)}{\zeta_1^\theta} \frac{z^\theta}{f(z)}\right)^a.$$

We take $\zeta_1 = .0001$ and then

$$\frac{f(\zeta_1)}{\zeta_1^\theta} \frac{z^\theta}{f(z)} < .e^{-.86}$$

for $2 \leq \theta \leq 2.0005$. Keeping some slack, we define

$$F_4(\theta, a) = \frac{h(\theta a/3)^3 3^{3(3-\theta)a} e^{-.85a}}{h(a)^2 h(\theta-2)^a}$$

and $G_4(\theta, a) = \log(F(\theta, a))$. Now let $\gamma_4(\theta) = G_4(\theta, a_\theta)$. We have $\gamma_4'(\theta) = g_1'(\theta) - \frac{5.1}{\theta^2}$ and $\gamma_4''(\theta) = g_1''(\theta) + \frac{10.2}{\theta^3}$ and we find from (89) that γ_4 is concave on $2 \leq \theta \leq 2.0005$. Furthermore $\gamma_4(2) = 0$ and using (88) we see that $\gamma_4'(2) < -.8$ and so $g_1(\theta) < -.8(\theta-2)$ for $\theta \in [2, 2.0005]$. So $G_4(\theta, a) \leq -.8a(\theta-2)/a_\theta \leq -.8a(\theta-2)$ for $0 \leq a \leq a_\theta$. Next let $\phi_4(\theta) = \frac{\partial G_4}{\partial a} |_{a=a_\theta} = g_2(\theta) - .85$. We see from (90) that $g_2(\theta) < -.5$ for $2 \leq \theta \leq 2.0005$ and thus $G_4(\theta, a) \leq -.5(a-a_\theta)$ for $a \geq a_\theta$ when $\theta \in [2, 2.0005]$. Replacing $e^{-.85}$ by $e^{-.86}$ in the definition of $F_4(\theta, a)$ we get $F_4(\theta, a) < e^{-(4(\theta-2)a/5+a/100)}$ for $0 \leq a \leq a_\theta$ when $2 \leq \theta \leq 2.0005$. So, for some small constant $c > 0$,

$$B_{93} = \sum_{\ell < k=2}^{k_0} \sum_{D=2k}^{3k} \pi_R(k, \ell, D) \leq \sum_{\ell < k=2}^{k_0} \sum_{D=2k}^{3k} e^{-ck} = o(1). \tag{93}$$

7.2.3 Finishing the case $m \sim n$

We repeat our observation that the maximum degree Δ in Γ is $o(\log n)$ **whp**. Therefore

Case 1: $m \geq n$.

$$\mathbb{P}(\mu(\Gamma) < n) \leq o(1) + \begin{cases} A_{51} + A_{56} + A_{57} + A_{58} + B_{75} + B_{76} + B_{77} & d \geq 6 \\ A_{51} + A_{56} + A_{57} + A_{58} + B_{75} + B_{79} & d = 5 \\ A_{51} + A_{56} + A_{57} + A_{58} + A_{59} + A_{63} + B_{80} & d = 4 \\ A_{51} + A_{56} + A_{57} + A_{59} + A_{64} + A_{65} + B_{93} & d = 3 \end{cases}$$

where the $o(1)$ term accounts for $\mathbb{P}(\Delta(\Gamma) > \log n)$. We use $B_{75} + B_{76} + B_{77}$ to account for witnesses $A \subseteq L, B$ with $|A| \geq n - n^{7/8}$. This is because if $A' = R \setminus B$ and $B' = L \setminus A$ then $|A'| = m - k + 1$ and $|B'| = n - k$ and $N_\Gamma(A') \subseteq B'$ and there will be a minimal witness A'', B'' with $A'' \subseteq A'$.

Case 2: $m \leq n$.

$$\mathbb{P}(\mu(\Gamma) < m) \leq o(1) + \begin{cases} B_{75} + B_{76} + B_{77} + A_{56} + A_{57} & d \geq 6 \\ B_{75} + B_{79} + A_{56} + A_{57} & d = 5 \\ B_{??} + B_{80} + A_{56} + A_{57} & d = 4 \\ B_{??} + B_{93} + A_{56} + A_{57} & d = 3 \end{cases}$$

We point out for use in the next section that our computations allow us to claim that we have

$$\sum_{\substack{k=n^{3/4} \\ \ell \leq \min\{k-1, m/2\}}}^{n-n^{3/4}} \sum_{D=dk}^{k \log n} \pi_L(k, \ell, D) = O(e^{-\Omega(n^{3/4})}). \quad (94)$$

Our computations also allow us to claim that

$$\sum_{\substack{k=n^{3/4} \\ \ell \leq \min\{k-1, n/2\}}}^{n-n^{3/4}} \sum_{D=2k}^{dk} \pi_R(k, \ell, D) = O(e^{-\Omega(n^{3/4})}). \quad (95)$$

7.3 The case $m \geq n + n^{4/5}$

Let $\mathcal{G}(n, m)$ denote the set of bipartite graphs with $|L| = n, |R| = m$ that are d -regular on L and degree at least 2 on R . Here $n + n^{4/5} \leq m \leq dn/2$. In fact suppose first that $m \leq \xi dn$ where $\xi < 1/2$ is a constant. Suppose that $G(n, m)$ is chosen uniformly at random from $\mathcal{G}(n, m)$.

If there is no matching from L to R , then let a minimal witness A, B be *small* if $|A| \leq n^{3/4}$ and *large* if $|A| \geq n - n^{3/4}$ and *medium* otherwise.

7.3.1 Small/Large Witnesses

We go back to (46). We can drop the $e^{(m-n)a}$ term since k is small. We see that $f(\zeta_1) < f(z)$ implies that the term $\frac{z^d}{f(z)^{\frac{\beta-a}{1-a}}} \frac{f(\zeta_1)^{\frac{\beta-a}{1-a}}}{\zeta_1^{d-x}} \left(\frac{e^{\frac{a}{1-a}}}{x}\right)^x$ is maximised over $\beta \geq 1$ when $\beta = 1$. Next let $H(\beta) = \beta \log h(a/\beta)$ then $H'(\beta) = \log(1 - a/\beta)$ and $H''(\beta) = \frac{a}{\beta(\beta-a)}$. Thus $h(a/\beta)^\beta$ is log-convex in β and so

$$h(a/\beta)^\beta \geq \exp \{H(1) + H'(1)(\beta - 1)\} = h(a)(1 - a)^{\beta-1}. \quad (96)$$

Going back to (52) we see that now we have

$$\pi_L(k, \ell, D) \leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{\rho_d(a)}{(1-a)^{\beta-1}}\right)^n. \quad (97)$$

We can drop the $e^{o(n^{7/8}a)}$ because we are dealing with m explicitly.

By taking $\varepsilon_L(\beta)$ in place of $\varepsilon_L(1)$ we can take $K = \beta$ in (55) and plugging this into (97) we see that

$$\sum_{\ell < k=2}^{n^{3/4}} \sum_{D=dk}^{k \log n} \pi_L(k, \ell, D) \leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{e^{-\beta a}}{(1-a)^{\beta-1}}\right)^n = o(1). \quad (98)$$

To deal with $k \geq n - n^{3/4}$ we treat this as $k \leq n^{3/4}$ in Section 7.2.2. Indeed, if there is such a witness A, B , let $A' = R \setminus B$ and $B' = L \setminus A$. Then $N_\Gamma(A') \subseteq B'$ and $|B'| < |A'|$ and so we can find a witness A'', B'' with $A'' \subseteq A', B'' \subseteq B'$ and $|B''| \leq n^{3/4}$.

We use (92) for this calculation. Now

$$\begin{aligned} \frac{h(\theta a/d)^d}{h(a)h(a/\beta)^\beta h(\theta/d)^{ad}} &= \frac{\left(\frac{\theta a}{d}\right)^{\theta a} \left(1 - \frac{\theta a}{d}\right)^{d-\theta a}}{a^a (1-a)^{1-a} \left(\frac{a}{\beta}\right)^a \left(1 - \frac{a}{\beta}\right)^{\beta-a} \left(\frac{\theta}{d}\right)^{\theta a} \left(1 - \frac{\theta}{d}\right)^{da-\theta a}} = \\ a^{(\theta-2)a} \exp &\left\{ -(d-\theta a) \sum_{k=1}^{\infty} \frac{\theta^k a^k}{k d^k} + a - \sum_{k=2}^{\infty} \frac{a^k}{k(k-1)} + \frac{a}{\beta} - \sum_{k=2}^{\infty} \frac{a^k}{\beta^{k-1} k(k-1)} + (da - \theta a) \sum_{k=1}^{\infty} \frac{\theta^k}{k d^k} \right\} \\ &= a^{(\theta-2)a} \exp \left\{ a \left(1 + \frac{1}{\beta} - (d-\theta) \log(1 - \theta/d) - \theta \right) + O(a^2) \right\} \end{aligned} \quad (99)$$

So from (92) we can write

$$\pi_R(k, \ell, D) \leq O\left(\frac{1}{m^{1/2}}\right) \left(\left(\frac{az}{\zeta_1}\right)^{\theta-2} \exp \left\{ 1 + \frac{1}{\beta} - (d-\theta) \log(1 - \theta/d) - \theta + O(a) \right\} \frac{f(\zeta_1)}{\zeta_1^2} \frac{z^2}{f(z)} \right)^{an}. \quad (100)$$

Now we claim that

$$\zeta_1^{\theta-2} \geq \frac{1}{2} \text{ and that } f(x)x^{-2} \text{ is monotone increasing in } x. \quad (101)$$

First notice that $f(x)x^{-2} = \sum_{i=2}^{\infty} \frac{x^{i-2}}{i!}$ which is clearly monotone increasing. Second note that $\zeta_1 = \zeta(\theta)$ and since $\frac{d\zeta(x)}{dx} > 0$ we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{d\zeta(x)}{dx} &= \lim_{x \rightarrow \infty} \frac{f(\zeta(x))^2}{(e^{\zeta(x)} - 1)^2 - \zeta(x)^2 e^{\zeta(x)}} \\ &= \lim_{\zeta \rightarrow \infty} \frac{f(\zeta)^2}{(e^\zeta - 1)^2 - \zeta^2 e^\zeta} = 1 \end{aligned}$$

and since $\zeta(x)$ is concave we have $\frac{d\zeta(x)}{dx} \geq 1$. This, along with $\lim_{x \rightarrow 2^-} \zeta(x) = 0$, implies that $\zeta(x) \geq x - 2$. We can then lower bound

$$\zeta_1^{\theta-2} = \zeta(\theta)^{\theta-2} \geq (\theta - 2)^{\theta-2} \geq e^{-e^{-1}} \geq 0.69$$

Using this we see from (100) that if

$$\theta \geq \theta_0 = 2 + \frac{4}{\log(1/az)}$$

then

$$\pi_R(k, \ell, D) \leq O\left(\frac{1}{m^{1/2}}\right) e^{-k}.$$

In which case we have

$$\sum_{\ell < k=2}^{n^{3/4}} \sum_{\theta \geq \theta_0} \pi_R(k, \ell, D) \leq O\left(\frac{1}{m^{1/2}}\right) e^{-k} = o(1). \quad (102)$$

When $\theta < \theta_0$ we have $\theta = 2 + o(1)$, $f(\zeta_1)/\zeta_1^2 = 1/2 + o(1)$. Therefore

$$\pi_R(k, \ell, D) \leq O\left(\frac{1}{m^{1/2}}\right) \left(\frac{z^2 e^{-(d-2)\log(1-2/d)+o(1)}}{2f(z)}\right)^k. \quad (103)$$

Now for $d \geq 4$ we have

$$\frac{z^2 e^{-(d-2)\log(1-2/d)+o(1)}}{2f(z)} \leq \frac{9}{10} \quad (104)$$

and so

$$\sum_{k=1}^{n^{3/4}} \sum_{\theta \leq \theta_0} \pi_R(k, \ell, D) \leq O\left(\frac{1}{m^{1/2}}\right) \left(\frac{9}{10}\right)^k = o(1). \quad (105)$$

When $d = 3$, the expression on the LHS of (104) is at most 1.26. So in this case we go back to (92) and replace $\frac{1}{h(\theta/d)^{ad}}$ by $\left(\frac{3^{\theta-2}}{h(\theta-2)}\right)^a = e^{o(a)}$. After this (99) is replaced by

$$a^{(\theta-2)a} \exp\left\{a\left(1 + \frac{1}{\beta} + \theta \log(\theta/d) - \theta\right) + o(a)\right\}.$$

And then (103) is replaced by

$$\pi_R(k, \ell, D) \leq O\left(\frac{1}{m^{1/2}}\right) \left(\frac{z^2 e^{-2\log(3/2)+o(1)}}{2f(z)}\right)^k \leq O\left(\frac{1}{m^{1/2}}\right) \frac{1}{2^k}$$

and so

$$\sum_{k=1}^{n^{3/4}} \sum_{\theta \leq \theta_0} \pi_R(k, \ell, D) \leq O\left(\frac{1}{m^{1/2}}\right) \frac{1}{2^k} = o(1). \quad (106)$$

7.3.2 Medium Witnesses

Let $d_i(n, m)$ denote the number of R -vertices of degree $i \geq 2$ in $G(n, m)$ and let $D_i(n, m) = \mathbb{E}(d_i(n, m))$.

We define three events:

$$\mathcal{A}_1(n, m-1) = \left\{G \in \mathcal{G}(n, m-1) : \exists i : |d_i(n, m-1) - D_i(n, m-1)| > n^{3/5}/i^3, 2 \leq i \leq \log^2 n\right\} \quad (107)$$

$$\mathcal{A}_2(n, m-1) = \{G \in \mathcal{G}(n, m-1) : \exists i : d_i(n, m-1) \neq 0, i > \log^2 n\} \quad (108)$$

$$\mathcal{B}(n, m) = \left\{G \in \mathcal{G}(n, m) : |d_2(n, m) - D_2(n, m)| > 2n^{3/5}\right\} \quad (109)$$

We argue next that if $\mathcal{A}(n, m) = \mathcal{A}_1(n, m-1) \cup \mathcal{A}_2(n, m)$ then

$$\mathbb{P}(\mathcal{A}(n, m) \cup \mathcal{B}(n, m)) = e^{-\Omega(\log^2 n)}. \quad (110)$$

For any $t > 0$ we have

$$\mathbb{P}(|d_i(n, m-1) - D_i(n, m-1)| > t) \leq O(n^{1/2})\mathbb{P}(\text{Bin}(n, q_i) > t)$$

where $q_i = \frac{z^i}{i!f(z)}$.

We will now use the following bounds (see for example [1])

$$\mathbb{P}(|\text{Bin}(n, p) - np| \geq t) \leq 2e^{-t^2/n}, \quad (111)$$

$$\mathbb{P}(\text{Bin}(n, p) \geq \alpha np) \leq (e/\alpha)^{\alpha np}. \quad (112)$$

If $i \leq \log^2 n$ then we can use (111) with $t = n^{3/5}/i^3$ to deal with $\mathcal{A}_1(n, m)$ and also with $\mathcal{B}(n, m)$. If $i \geq \log^2 n$ then $nq_i \leq e^{-\Omega(\log^2 n)}$. We can therefore use (112) with $\alpha = 1/nq_i$ to deal with $\mathcal{A}_2(n, m)$. This concludes the proof of (110).

Now consider a set of pairs $X \subseteq \mathcal{G}(n, m-1) \times \mathcal{G}(n, m)$. We place (G_1, G_2) into X if G_2 is obtained from G_1 in the following manner: Choose a vertex $x \in R$ of degree at least four in G_1 . Suppose that its neighbours are $y_i, i = 1, 2, \dots, k$ in any order. To create G_2 we (i) replace x by two vertices x and m and then (ii) let the neighbours of x in G_2 be y_1, y_2 and let the neighbours of m be y_3, \dots, y_k .

For $G \in \mathcal{G}^*(n, m-1)$ let

$$\pi_1(G) = |\{G_2 : (G, G_2) \in X\}|$$

and for $G \in \mathcal{G}^*(n, m)$ let

$$\pi_2(G) = |\{G_1 : (G_1, G) \in X\}|.$$

We note that if

$$\Sigma_1 = \sum_{i \geq 4} \binom{i}{2} D_i(n, m-1)$$

then

- $G \notin \mathcal{A}(n, m-1)$ implies that $|\pi_1(G) - \Sigma_1| \leq O(n^{3/5})$.
- $\pi_1(G) \leq \binom{m-1}{2}$ for all $G \in \mathcal{G}(n, m-1)$.
- $G \notin \mathcal{B}(n, m)$ implies that $|\pi_2(G) - D_2(n, m)| \leq n^{3/5}$.
- $\pi_2(G) \leq m$ for all $G \in \mathcal{G}(n, m)$.

We then note that

$$(\Sigma_1 - O(n^{3/5}))|\mathcal{G}(n, m-1)| \leq |X| \leq (D_2(n, m) + n^{3/5} + me^{-\Omega(\log^2 n)})|\mathcal{G}(n, m)|.$$

Now let \mathcal{P}, \mathcal{Q} be properties such that if $(G_1, G_2) \in X$ and $G_2 \in \mathcal{Q}$ then $G_1 \in \mathcal{P}$. Let (G_1, G_2) be chosen uniformly from X and let \mathbb{P}_X denote probabilities computed w.r.t. this choice. Then

$$\mathbb{P}_X(G_2 \in \mathcal{Q}) \leq \mathbb{P}_X(G_1 \in \mathcal{P}) \leq \frac{|\mathcal{P}|(\Sigma_1 + O(n^{3/5})) + m|\mathcal{A}(n, m-1)|}{|X|}$$

and

$$\mathbb{P}_X(G_2 \in \mathcal{Q}) \geq \frac{(|\mathcal{Q}| - |\mathcal{B}(n, m)|)(D_2(n, m) - n^{3/5})}{|X|}$$

So,

$$\frac{(|\mathcal{Q}| - |\mathcal{B}(n, m)|)(D_2(n, m) - n^{3/5})}{|\mathcal{G}(n, m)| (D_2(n, m) + n^{3/5} + me^{-\Omega(\log^2 n)})} \leq \frac{|\mathcal{P}|(\Sigma_1 + O(n^{3/5})) + m|\mathcal{A}(n, m-1)|}{|\mathcal{G}(n, m-1)| (\Sigma_1 - O(n^{3/5}))}.$$

So,

$$\frac{|\mathcal{Q}|}{|\mathcal{G}(n, m)|} \leq (1 + O(n^{-2/5})) \frac{|\mathcal{P}|}{|\mathcal{G}(n, m-1)|}.$$

So, if \mathcal{P}_j is a property of $\mathcal{G}(n, j)$ for $j = n, n+1, \dots, m$,

$$\frac{|\mathcal{P}_m|}{|\mathcal{G}(n, m)|} \leq (1 + O(n^{-2/5}))^{m-n} \frac{|\mathcal{P}_n|}{|\mathcal{G}(n, n+n^{4/5})|}. \quad (113)$$

We use (113) in the following way: First let \mathcal{B}_j , $n+n^{4/5} \leq j \leq m$ be the property that $G \in \mathcal{G}(n, j)$ contains a minimal witness A, B with $A \subseteq L$, $n^{3/4} \leq |A| \leq n/2$. If $(G_1, G_2) \in X$ and $G_2 \in \mathcal{B}_m$ then $G_1 \in \mathcal{B}_{m-1}$. Indeed $A, B \cap [m-1]$ is a minimal witness in G_1 . Applying (113) and (94) we see that **whp** \mathcal{B}_m fails to occur. Now let \mathcal{B}'_j be the property that $G \in \mathcal{G}(n, j)$ contains a minimal witness A, B with $A \subseteq R$, $n^{3/4} \leq |A|, |B| < \min\{|A| - (j-n), n/2\}$. If $(G_1, G_2) \in X$ and G_2 has a witness A, B with $A \subseteq L$ and $n/2 < |A| \leq n - n^{3/4}$ then $G_2 \in \mathcal{B}'_m$. Indeed $A' = R \setminus B, B' = L \setminus A$ is also a witness in G_2 . Now if $G_2 \in \mathcal{B}'_m$ with a witness A', B' then $A' \cap [m-1], B'$ is a witness in G_1 and so contains a minimal witness A'', B'' where $|A''| > |B''| + m - n > n^{3/4}$ i.e. $G_1 \in \mathcal{B}'_{m-1}$. Applying (113) and (95) we see that **whp** \mathcal{B}'_m fails to occur. This deals with medium witnesses.

It only remains to consider m close to $dn/2$ i.e. where ξ defined at the beginning of this section is close $1/2$. Observe first that the number of edges incident with vertices of degree greater than two is at most $3dn(1-2\xi)$. If there are d_i vertices of degree $i = 2, \geq 3$ then $d_2 + d_3 = m = \xi dn$ and $2d_2 + 3d_3 \leq dn$ which implies that $d_2 \geq dn(3\xi - 1)$. So the number of edges incident with vertices of degree greater than two is at most $dn - 2dn(3\xi - 1)$.

Now consider a witness A, B where $|A| = \gamma n$. We must have $\gamma dn \leq 2\gamma n + 3dn(1-2\xi)$ which implies that $\gamma \leq \frac{3d(1-2\xi)}{d-2}$ which can be made arbitrarily small. Now the estimate in (97) will suffice up to $k \leq \varepsilon_L n$ and so we only need to make ξ close enough to $1/2$ so that $\gamma < \varepsilon_L$ (which depends only on d and not γ).

7.4 The case $m \leq n - n^{4/5}$

We once again consider medium witnesses separately from small or large witnesses.

7.4.1 Small/Large Witnesses

We first go back to (92) and deal with $\pi_R(k, \ell, D)$ for $k \leq n^{3/4}$ as we did in Section 7.3. For $k \geq n - n^{3/4}$ we deal with $\pi_L(k, \ell, D)$ for $k \leq n^{3/4}$. We will go back to (46) and write

$$\pi_L(k, \ell, D) = O\left(\frac{1}{n^{1/2}}\right) \left(\frac{h(a)^{d-1}}{h(a/\beta)^\beta}\right)^n \left(\frac{z^d}{f(z)} \frac{f(\zeta_1)}{\zeta_1^{d-x}} \left(\frac{e^{\frac{a}{1-a}}}{x}\right)^x \left(\frac{f(z)}{f(\zeta_1)}\right)^{\frac{1-\beta}{1-a}}\right)^{n-k}$$

Now $x = \frac{d(m-n)+(D-dk)+1}{m-k+1} \geq 0$ implies that

$$1 - \beta \leq \frac{D - dk + 1}{n} = O\left(\frac{k \log n}{n}\right) \text{ and that } x = O\left(\frac{k \log n}{n}\right).$$

Also, $\zeta_1 = \zeta(d-x)$ implies that $f(\zeta_1) = f(z)(1 - O(x))$. Therefore,

$$\left(\frac{f(z)}{f(\zeta_1)}\right)^{\frac{1-\beta}{1-a}} = e^{O(a^2 \log^2 n)}.$$

Arguing as for (97) we get

$$\pi_L(k, \ell, D) \leq O\left(\frac{1}{n^{1/2}}\right) \left(\frac{\rho(a)e^{O(a^2 \log^2 n)}}{(1-a)^{\beta-1}}\right)^n.$$

Taking $\rho(a) \leq e^{-a}$ as in (57) and noting that $b \leq 1$ here we get Thus

$$\sum_{k=1}^{n^{3/4}} \sum_{D=dk}^{k \log n} \pi_L(k, \ell, D) \leq \sum_{k=1}^{n^{3/4}} \sum_{D=dk}^{k \log n} O\left(\frac{1}{n^{1/2}}\right) \left(e^{-a+O(a^2 \log^2 n)}\right)^n = o(1). \quad (114)$$

7.4.2 Medium Witnesses

Now consider a set of pairs $Y \subseteq \mathcal{G}(n, m) \times \mathcal{G}(n+1, m)$. We place (G_1, G_2) into Y if G_2 is obtained from G_1 in the following manner: Choose $0 \leq k \leq n$. Replace edges (ℓ, y) by $(\ell+1, y)$ for all $\ell > k$ and all y . Add vertex $k+1$ and d edges $(k+1, y_j), j = 1, 2, \dots, d$.

Note that if $(G_1, G_2) \in Y$ and G_1 has a matching of R into L then so does G_2 .

For $G \in \mathcal{G}(n, m)$ let now

$$\pi_1(G) = |\{G_2 : (G, G_2) \in Y\}|$$

and for $G \in \mathcal{G}(n+1, m)$ let

$$\pi_2(G) = |\{G_1 : (G_1, G) \in Y\}|.$$

Let

$$\Sigma_2 = (n+1) \left(1 - \frac{z^2}{2f(z)}\right)^d$$

and for $G \in \mathcal{G}(n+1, m)$ let

$$L_3(G) = |\{v \in L : \text{all neighbours of } v \text{ have degree at least } 3\}|.$$

Let

$$\mathcal{C}(n+1, m) = \left\{G \in \mathcal{G}(n+1, m) : |L_3(G) - \Sigma_2| \leq n^{3/5}\right\}.$$

Let

We note that

- $G \in \mathcal{G}(n, m)$ implies that $\pi_1(G) = (n+1) \binom{m}{d}$.
- $G \notin \mathcal{C}(n, m+1)$ implies that $|\pi_2(G) - \Sigma_2| \leq n^{3/5}$.
- $\pi_2(G) \leq n+1$ for all $G \in \mathcal{G}(n+1, m)$.

We then note that

$$\frac{|Y|}{|\mathcal{G}(n, m)|} = (n+1) \binom{m}{d}.$$

$$\Sigma_2 - n^{3/5} \leq \frac{|Y|}{|\mathcal{G}(n+1, m)|} \leq \Sigma_2 + n^{3/5} + (n+1)e^{-\Omega(\log^2 n)}.$$

Now let \mathcal{P}, \mathcal{Q} be properties such that if $(G_1, G_2) \in Y$ and $G_2 \in \mathcal{Q}$ then $G_1 \in \mathcal{P}$. Let (G_1, G_2) be chosen uniformly from Y and let P_Y denote probabilities computed with respect to this choice. Then

$$P_Y(G_2 \in \mathcal{Q}) \leq P_Y(G_1 \in \mathcal{P}) = \frac{|\mathcal{P}|(n+1)\binom{m}{d}}{|Y|}$$

and

$$P_Y(G_2 \in \mathcal{Q}) \geq \frac{(|\mathcal{Q}| - |\mathcal{C}(n+1, m)|)(\Sigma_2 - n^{3/5})}{|Y|}$$

Arguing as in Section 7.3 we see that if \mathcal{P}_j is a property of $\mathcal{G}(j, m)$ for $j = m, m+1, \dots, n$,

$$\frac{|\mathcal{P}_m|}{\mathcal{G}(n, m)} \leq (1 + O(n^{-2/5}))^{n-m} \frac{|\mathcal{Q}|}{\mathcal{G}(m + n^{4/5}, m)}. \quad (115)$$

First let $\mathcal{B}_j, m + n^{4/5} \leq j \leq n$ be the property that $G \in \mathcal{G}(j, m)$ contains a minimal witness A, B with $A \subseteq R, n^{3/4} \leq |A| \leq m/2$. If $(G_1, G_2) \in X$ and $G_2 \in \mathcal{B}_{n+1}$ then $G_1 \in \mathcal{B}_n$. Indeed $A, B \cap [n]$ is a witness in G_1 . Applying (115) and (95) we see that **whp** \mathcal{B}_n fails to occur. Now let \mathcal{B}'_j be the property that $G \in \mathcal{G}(j, m)$ contains a minimal witness A, B with $A \subseteq R, n^{3/4} \leq |A|, |B| \leq \min\{|A| - (j - m), m/2\}$. If $(G_1, G_2) \in X$ and G_2 has a witness A, B with $A \subseteq R$ and $m/2 < |A| \leq m - n^{3/4}$ then $G_2 \in \mathcal{B}'_m$. Indeed $A' = L \setminus B, B' = R \setminus A$ is also a witness in G_2 . Now if $G_2 \in \mathcal{B}'_m$ with a witness A', B' then $A' \cap [m], B'$ is a witness in G_1 and so contains a minimal witness A'', B'' where $|A''| > |B''| + n - m > n^{3/4}$ i.e. $G_1 \in \mathcal{B}'_{m-1}$. Applying (115) and (94) we see that **whp** \mathcal{B}'_m fails to occur. This deals with medium witnesses.

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