The effect of adding randomly weighted edges

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Abstract

We consider the following question. We have a dense regular graph $G$ with degree $\alpha n$, where $\alpha > 0$ is a constant. We add $m = o(n^2)$ random edges. The edges of the augmented graph $G(m)$ are given independent edge weights $X(e), e \in E(G(m))$. We estimate the minimum weight of some specified combinatorial structures. We show that in certain cases, we can obtain the same estimate as is known for the complete graph, but scaled by a factor $\alpha^{-1}$. We consider spanning trees, shortest paths and perfect matchings in (pseudo-random) bipartite graphs.

1 Introduction

It is often the case that adding some randomness to a combinatorial structure can lead to significant positive change. Perhaps the most important example of this and the inspiration for a lot of what has followed, is the seminal result of Spielman and Teng [36] on the performance of the simplex algorithm, see also Vershynin [39] and Dadush and Huiberts [13].

The paper [36] inspired the following model of Bohman, Frieze and Martin [10]. They consider adding random edges to an arbitrary member $G$ of $\mathcal{G}(\alpha)$. Here $\alpha$ is a positive constant and $\mathcal{G}(\alpha)$ is the set of graphs with vertex set $[n]$ and minimum degree $\alpha n$. They show that adding $O(n)$ random edges to $G$ is enough to create a Hamilton cycle w.h.p. This is in contrast to the approximately $\frac{1}{2}n \log n$ edges needed if we rely only on the random edges. Research on this model and its variations has been quite substantial, see for example [11], [31], [37], [27], [28], [12], [6], [30], [9], [22], [14], [35], [16], [33] or see Section 1.3 of [5] for a list of recent Arxiv papers on randomly perturbed graphs and hypergraphs.

Anastos and Frieze [4] introduced a variation on this theme by adding color to the edges. They consider rainbow Hamiltonicity and rainbow connection in the context of a randomly colored dense graph with the addition of randomly colored edges. Aigner-Horev and Hefetz [1] strengthened the Hamiltonicity result of [4].

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In this paper we introduce another variation. We start with a dense graph in which each edge $e$ has a random weight $X(e)$ and add randomly weighted random edges. We study the effect on the minimum value of various combinatorial structures. We will for simplicity restrict our attention to what we will call $G_{reg}(\alpha)$, the graphs in $G(\alpha)$ that are $\alpha n$ regular.

### 1.1 Spanning Trees

We start with spanning trees. Suppose that $G \in G_{reg}(\alpha)$ and each edge $e$ of $G$ is given an independent random weight $X(e)$ chosen uniformly from $[0,1]$. Let $mst(G)$ denote the expected minimum weight of a spanning tree of $G$, assuming it is connected. Beveridge, Frieze and McDiarmid [10] and Frieze, Ruszinko and Thoma [20] show that under certain circumstances

$$mst(G) \approx \frac{\zeta(3)}{\alpha} \text{ as } n \to \infty. \tag{1}$$

Here $A_n \approx B_n$ if $A_n = (1 + o(1))B_n$ as $n \to \infty$ and $A_n \gg B_n$ if $A_n/B_n \to \infty$.

Now let $G(m)$ be obtained from $G$ by adding $m$ randomly weighted random edges to $G$. Also, let $G(p)$ be obtained from $G$ by independently adding weighted copies of edges not in $G$, with probability $p$. We let $R_m, R_p$ denote the added edges. Our first theorem is a simple extension of (17).

**Theorem 1.** Suppose that $G \in G_{reg}(\alpha)$ and $n \log n \ll m \ll n^{5/3}$ and the edges of $G(m)$ have independent weights chosen uniformly from $[0,1]$. Then w.h.p.

$$mst(G(m)) \approx \frac{\zeta(3)}{\alpha} \text{ as } n \to \infty. \tag{2}$$

This theorem is very easy to prove. One simply verifies that certain conditions in [10] hold w.h.p. On the other hand it sets the stage for what we are trying to prove in other scenarios. The upper bound is not essential, we could most likely replace it by $o(n^2)$, but this would require us to re-do the calculations in [10].

Without the addition of random edges, all that can be claimed (assuming $G$ is connected) is that

$$\frac{\zeta(3)}{\alpha} \leq mst(G) \leq \frac{\zeta(3) + 1}{\alpha}. \tag{3}$$

See [20].

**Conjecture:** The $+1$ in (3) can be replaced by $+1/2$ (which is best possible).

The example giving $1/2$ is a collection of $n/r$ copies of $H = K_r - e, \ r = \alpha n$ where there is a perfect matching on the vertices of degree $r - 2$ added so that the copies of $H$ are connected in a cycle by bridges.

### 1.2 Shortest paths

We turn our attention next to shortest paths. Janson [23] considered the following scenario: the edges of $K_n$ are given independent exponential mean one random lengths. Let $d_{i,j}$ denote the shortest distance between vertex $i$ and vertex $j$. He shows that w.h.p.

$$d_{1,2} \approx \frac{\log n}{n}, \ \max_{j \in [n]} d_{1,j} \approx \frac{2 \log n}{n}, \ \max_{i,j} d_{i,j} \approx \frac{3 \log n}{n}.$$
Bhamidi and van der Hofstad [7] proved an equivalent expression for $d_{1,2}$ for a much wider class of distribution. They actually determined an asymptotic limiting distribution. (See also Bhamidi, van der Hofstad and Hooghiemstra [8].) We prove the following:

**Theorem 2.** Suppose that $n^2 / \log n \ll m \ll n^2$ and that $G \in G_{\text{reg}}(\alpha)$ and the edges of $G(m)$ are given independent exponential mean one random lengths. Let $d_{i,j}$ denote the shortest distance between vertex $i$ and vertex $j$. Then w.h.p.

$$d_{1,2} \approx \frac{\log n}{\alpha n}, \quad \max_{j \in [n]} d_{1,j} \approx \frac{2 \log n}{\alpha n}, \quad \max_{i,j \in [n]} d_{i,j} \approx \frac{3 \log n}{\alpha n}$$

### 1.3 Bipartite matchings

We turn our attention next to bipartite matchings. For background consider the following well-studied problem: each edge of the complete bipartite graph $K_{n,n}$ is given an independent edge weight $X(e)$. Let $C_n$ denote the minimum weight of a perfect matching in this context. Walkup [10] considered the case where $X(e)$ is uniform $[0, 1]$ and proved that $\mathbb{E}(C_n) \leq 3$. Later Karp [25] proved that $\mathbb{E}(C_n) \leq 2$. Aldous [2, 3] proved that if the $X(e)$ are independent exponential mean one random variables then $\lim_{n \to \infty} \mathbb{E}(C_n) = \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2}$. Parisi [34] conjectured that in fact $\mathbb{E}(C_n) = \sum_{k=1}^{\infty} \frac{1}{k^2}$. This was proved independently by Linusson and Wästlund [29] and by Nair, Prabhakar and Sharma [32]. A short elegant proof was given by Wästlund [42, 43].

We now consider $G(m)$. $G$ is an $\alpha n$ regular bipartite graph with vertex set $A \cup B$, $|A| = |B| = n$. Unfortunately, our proof only works if $G$ is pseudo-random, as defined by Thomason [38]. By this we mean that for some $0 < \eta < 1$ we have

$$|\text{co} - \text{degree}(u, v) - \alpha^2 n| \leq \mu = O(n^{1-\eta}) \quad \text{for all } u, v \in A. \tag{4}$$

Here, as usual, $\text{co} - \text{degree}(u, v) = |\{w \in B : (u, w), (v, w) \in E(G)\}|$.

**Theorem 3.** Let $G$ be a pseudo-random $\alpha n$-regular bipartite graph with vertex set $A \cup B$, $|A| = |B| = n$. Suppose that $\log^3 n \ll m = o(n^2)$. Let $C_n$ denote the minimum weight of a perfect matching when the weights of the edges of $G(m)$ are independent exponential mean one random variables, denoted by $E(1)$. Then

$$\mathbb{E}(C_n) \approx \frac{1}{\alpha} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6\alpha}. \tag{5}$$

**Conjecture:** equation (6) holds for $G(m)$, $m = o(n^2)$ growing sufficiently quickly, but without the assumption of pseudo-randomness.

Frieze and Johansson [18] showed that if $G$ is the random bipartite graph $K_{n,n,p}$ where $np \gg \log^2 n$ then $\mathbb{E}(C_n) \approx \frac{\pi^2}{6p}$. That paper also conjectured that if $G$ is an $r$-regular bipartite graph with $n + n$ vertices $r \to \infty$ that $\mathbb{E}(C_n) \approx \frac{nr^2}{6r}$. This conjecture is false. Instead we have:

**Conjecture:** $\mathbb{E}(C_n) \approx \frac{nr^2}{6r}$ if the connectivity of $G$ tends to infinity. Also, in general $\mathbb{E}(C_n) \lesssim \frac{\pi^2}{6} \left( \frac{\pi^2}{6} + \frac{1}{2} \right)$.

The $1/2$ here is best possible in general. We take $n/r$ copies of $H = K_r - e$ where there is a perfect matching on the vertices of degree $r - 1$ added so that the copies of $H$ are connected in a cycle by bridges.

In what follows we will sometimes treat large values as integer when strictly speaking we should round up or down. In all cases the choice of up or down has negligible effect on the proof.
1.4 Asymmetric Traveling Salesperson Problem

Theorem 3 can be used as a basis for the ATSP. Karp [24] showed if the entries of the $n \times n$ matrix are independent uniform $[0,1]$ random variables then w.h.p. the associated assignment problem and the ATSP have the same value asymptotically. (The assignment problem is just the problem of finding the minimum cost perfect matching in $K_{n,n}$ when the cost of edge $(i,j)$ is $A_{i,j}$.) The error term in Karp’s paper was improved by Karp and Steele [26], Dyer and Frieze [17] and by Frieze and Sorkin [21]. We will use Theorem 3 and the argument from [26] to prove the following. Let $D$ be an $\alpha n$-regular digraph with vertex set $[n]$. By this we mean that every vertex has in-degree and out-degree $\alpha n$. Associate $D$ with a bipartite graph $G$ with vertex set $A \cup B$ as in Section 1.3. An edge $(i,j)$ of $D$ corresponds to an edge $(a_i,b_j)$ of $G$ and vice-versa.

We say that $D$ is pseudo-random if $G$ is. We let $D(m), D(p)$ be the perturbed digraphs where we add $m$ random edges or edges independently with probability $p$. Let $T_n$ denote the minimum length of a Hamilton cycle (tour) in $D(m)$.

**Theorem 4.** Let $D$ be an $\alpha n$ regular pseudo-random digraph and suppose that $n^{15/8} \log n \leq m = o(n^2)$. Then w.h.p.

$$\mathbb{E}(T_n) \approx \frac{\pi^2}{6\alpha}. \quad (6)$$

2 Spanning Trees

Theorem 2 of Beveridge, Frieze and McDiarmid [10] yields the following. Suppose that $\beta = O(n^{-1/3})$ and

$$\alpha n \leq \delta(G) \leq \Delta(G) \leq \alpha(1+\beta)n. \quad (7)$$

Suppose also that where $S : \bar{S}$ is the set of edges of $G$ with exactly one endpoint in $S$,

$$\frac{|S : \bar{S}|}{|S|} \geq n^{2/3} \log^{3/2} n \text{ for all } S \subseteq [n], \frac{\alpha n}{2} \leq |S| \leq \frac{n}{2}, \quad (8)$$

then (18) holds.

Now if we add $m$ random edges satisfying the conditions of the theorem then all degrees will be $\alpha n + o(n^{-1/3})$ and this will satisfy (7).

So, to prove Theorem 4 all we need to do is to verify (8). Let now $p = \frac{m}{\binom{n}{2}} \gg \frac{\log n}{n}$. The probability that $G(p)$ contains a set failing to satisfy (8) can be bounded by

$$\sum_{s=\alpha n}^{n/2} \binom{n}{s} \mathbb{P}(Bin(sn/2,p) \leq sn^{2/3} \log^{3/2} n) \leq \sum_{s=\alpha n}^{n/2} \left(\frac{ne}{s}\right)^s e^{-snp/6} = o(1), \quad (9)$$

where we have just looked at the edges $R_p$ to satisfy (8). The property described in (8) is monotone increasing and so the $o(1)$ upper bound in (9) holds in $G(m)$ as well, see for example Lemma 1.3 of [19]. This completes the proof of Theorem 4.

3 Shortest Paths

We use the ideas of Janson [23]. Sometimes we make a small tweak and in one case we shorten his proof considerably.
3.1 \( d_{1,2} \)

We set \( S_1 = \{1\} \) and \( d_1 = 0 \) and consider running Dijkstra’s shortest path algorithm [13]. At the end of Step \( k \) we will have computed \( S_k = \{1 = v_1, v_2, \ldots, v_k\} \) and \( 0 = d_1, d_2, \ldots, d_k \) where \( d_i \) is the minimum length of a path from 1 to \( i, i = 1, 2, \ldots, k \). Let there be \( \nu_k \) edges from \( S_k \) to \([n] \setminus S_k\). Arguing as in [23] we see that \( d_{k+1} - d_k = Z_k \) where \( Z_k \) is the minimum of \( \nu_k \) independent exponential mean one random variables. We note that

\[
\mathbb{E}(Z_k) = \frac{1}{\nu_k} \quad \text{and} \quad \text{Var}(Z_k) = \frac{1}{\nu_k^2}.
\]

Also, the memoryless property of the exponential distribution implies that \( Z_k \) is independent of \( d_k \). Suppose now that

\[
m = \frac{\omega n^2}{\log n} \quad \text{where} \ 1 \ll \omega \ll \log n.
\]

It follows that w.h.p. \( \delta(G(m)) \approx \Delta(G(m)) \approx \alpha n \). Now

\[
k\delta - 2 \binom{k}{2} \leq \nu_k \leq k\Delta
\]

and so

\[
w.h.p. \ \nu_k \approx k\alpha n \quad \text{for} \ k = o(n). \quad (10)
\]

Conditioning on the set of added edges and taking expectations with respect to edge weights, we see that if \( k \gg 1 \) then

\[
\mathbb{E}(d_k) = \mathbb{E} \left( \sum_{i=1}^{k} \frac{1}{\nu_i} \right) = \sum_{i=1}^{k} \frac{1}{\iota \alpha n} \approx \frac{\log k}{\alpha n}. \quad (11)
\]

By the same token,

\[
\text{Var}(d_k) \approx \sum_{i=1}^{k} \frac{1}{\iota^2 \alpha^2 n^2} = O(n^{-2}). \quad (12)
\]

So, if \( k_0 = n^{1/2}/\omega^{1/2} \) then w.h.p. \( d_k \lesssim \log n \) for \( 0 \leq k \leq k_0 \). Now execute Dijkstra’s algorithm from vertex 2 and let \( d_k, T_k \) correspond to \( d_k, S_k \). If \( S_{k_0} \cap T_{k_0} \neq \emptyset \) then we already have \( d_{1,2} \lesssim \frac{\log n}{\alpha n} \). If \( S_{k_0}, T_{k_0} \) are disjoint then we use the random edges \( R_m \) or \( R_p \). Let \( p = m/\binom{n}{2} \approx 2\omega/\log n \). Then,

\[
\mathbb{P} \left( \bar{e} \in R_p \cap (S_{k_0} : T_{k_0}) : X(e) \leq \frac{\log n}{\omega n} \right) \leq \left( 1 - p \left( 1 - \exp \left\{ - \frac{\log n}{\omega n} \right\} \right) \right)^{k_0^2} = \left( 1 - (1 + o(1)) \frac{p \log n}{\omega n} \right)^{k_0^2} \leq \exp \left\{ - \frac{k_0^2 p \log n}{2\omega n} \right\} = e^{-\omega}.
\]

So, in this case we see too that w.h.p.

\[
d_{1,2} \leq (1 + o(1)) \left( \frac{\log n}{2\alpha n} + \frac{\log n}{2\alpha n} \right) + \frac{\log n}{\omega n} \approx \frac{\log n}{\alpha n}.
\]

We now consider a lower bound for \( d_{1,2} \). Let \( k_1 = n^{1/2}/\log n \). We observe that because all vertices have degree \( \approx \alpha n \) and because the edge joining \( v_{k+1} \) to \( S_k \) is uniform among \( S_k : S_k \) edges, we see that \( \mathbb{P}(2 \in S_{k_1}) = O(k_1/n) = o(1) \). By the same token, \( \mathbb{P}(T_{k_1} \cap S_{k_1} \neq \emptyset) = O(k_1^2/n) = o(1) \). It follows that w.h.p.

\[
d_{1,2} \geq 2\frac{\log k_1}{\alpha n} \approx \frac{\log n}{\alpha n}.
\]
3.2 \( \max_j d_{1,j} \)

For this we run Dijkstra’s algorithm until all vertices have been included in the shortest path tree. We can therefore immediately see that if \( k_2 = n/\log n \) then

\[
\mathbb{E}(\max_j d_{1,j}) \gtrsim \sum_{i=1}^{k_2} \frac{1}{i \alpha n} + \sum_{i=n-k_2+1}^{n} \frac{1}{(n-i) \alpha n} \approx \frac{2 \log n}{\alpha n}. \tag{13}
\]

The second sum in (13) is the contribution from adding the final \( k_2 \) vertices and uses \( \nu_{n-i} \approx (n-i) \alpha n \) w.h.p. for \( i = o(n) \).

For an upper bound we use the fact that w.h.p. there are approximately \( i(n-i)p \) edges between \( S_i \) and \( \bar{S}_i \) in order to show that if \( k_2 = n/\omega \) then

\[
\mathbb{E}(\max_j d_{1,j}) \leq (1 + o(1)) \left( \frac{2 \log n}{\alpha n} + \frac{\log n}{2 \omega n} \sum_{i=k_2+1}^{n-k_2} \left( \frac{1}{i} + \frac{1}{n-i} \right) \right) = \frac{2 \log n}{\alpha n} \left( 1 + \frac{(\alpha + o(1)) \log \omega}{2 \omega} \right) \approx \frac{2 \log n}{\alpha n}. \tag{14}
\]

Equations (13) and (14) imply that \( \mathbb{E}(\max_j d_{1,j}) \approx \frac{2 \log n}{\alpha n} \) and we can use equation (12) to get concentration around the mean.

3.3 \( \max_{i,j} d_{i,j} \)

We begin with a lower bound. Let \( Y_v = \min \{ X(e) : e = \{v, w\} \in G(m) \} \). Let \( A = \left\{ v : Y_v \geq \left( \frac{1-\varepsilon}{\alpha n} \right) \log n \right\} \).

Then, given that all vertex degrees are asymptotically equal to \( \alpha n \), we have that for \( v \in [n] \),

\[
\mathbb{P}(v \in A) = \exp \left\{ -\left( \alpha n + o(n) \right) \frac{(1-\varepsilon) \log n}{\alpha n} \right\} = n^{1+\varepsilon+o(1)}. \tag{15}
\]

An application of the Chebyshev inequality shows that \( |A| \approx n^{\varepsilon+o(1)} \) w.h.p. Now the expected number of paths from \( a_1 \in A \) to \( a_2 \in A \) of length at most \( \frac{(2-2\varepsilon+\beta) \log n}{\alpha n} \), where \( \beta < 1 \), can be bounded by

\[
n^{2\varepsilon+o(1)} \times n^2 \times n^{-1+\beta+o(1)} \times \frac{\log^2 n}{\alpha^2 n^2} = n^{2\varepsilon+\beta-1+o(1)}. \tag{16}
\]

**Explanation for (16):** The first factor \( n^{2\varepsilon+o(1)} \) is the expected number of pairs of vertices \( a_1, a_2 \in A \). The second factor is a bound on the number of choices \( b_1, b_2 \) for the neighbors of \( a_1, a_2 \) on the path. The third factor \( F_3 \) is a bound on the expected number of paths of length at most \( \frac{\beta \log n}{\alpha n} \) from \( b_1 \) to \( b_2 \). This factor comes from

\[
F_3 \leq \sum_{\ell \geq 0} \left( (\alpha + o(1))n \right)^\ell \left( \frac{\beta \log n}{\alpha n} \right)^{\ell+1} \frac{1}{(\ell+1)!}.
\]

Here \( \ell \) is the number of internal vertices on the path. There will be \( (\alpha + o(1))n^\ell \) choices for the sequence of vertices on the path. We then use the fact that the exponential mean one random variable stochastically
dominates the uniform $[0,1]$ random variable $U$. The final two factors are the probability that the sum of $\ell + 1$ independent copies of $U$ sum to at most $\frac{\beta \log n}{an}$. Continuing we have

$$F_3 \leq \sum_{\ell \geq 0} \frac{\beta \log n}{an(\ell + 1)} \left(\frac{e^{1+o(1)} \beta \log n}{\ell}\right)^\ell \leq \frac{\beta \log n}{an} \left(\sum_{\ell=0}^{10 \log n} n^{\beta + o(1)} + \sum_{\ell > 10 \log n} e^{-\ell}\right) = n^{-1+\beta + o(1)}.$$ 

The final factor in $[16]$ is a bound on the probability that $X_{a_i b_i}, i = 1, 2$ is distributed as $\frac{(1-\varepsilon) \log n}{an} + E_i$ where $E_1, E_2$ are independent exponential mean one. Now $\mathbb{P}(E_1 + E_2 \leq t) \leq (1 - e^{-t})^2 \leq t^2$ and taking $t = \frac{(1-\beta) \log n}{an}$ justifies the final factor of $[16]$.

It follows from $[16]$, with $\beta = 1 - 3\varepsilon$, that the shortest distance between a pair of vertices in $A$ is at least $\frac{(3-5\varepsilon) \log n}{an}$ w.h.p., completing our proof of the lower bound in Theorem 2.

We now consider the upper bound. Let $Y_1 = d_{k_3}$ where $d_k$ is from Section 3.1 and $k_3 = n^{1/2} \log n$. For $t < 1 - \frac{1+o(1)}{an}$ we have that w.h.p. over our choice of $R_m$, that

$$\mathbb{E}(\epsilon^t \tan Y_1) = \mathbb{E}\left(\exp\left\{\sum_{i=1}^{k_3} at_n Z_i\right\}\right) = \prod_{i=1}^{k_3} \left(1 - \frac{1+o(1)}{i} t\right)^{-1}$$

Then for any $\beta > 0$ and for we have

$$\mathbb{P}\left(Y_1 \geq \frac{\beta \log n}{an}\right) \leq \mathbb{E}(\epsilon^t \tan Y_1 - t\beta \log n) \leq e^{-t \beta \log n} \prod_{i=1}^{k_3} \left(1 - \frac{1+o(1)}{i} t\right)^{-1} \leq e^{-t \beta \log n} \exp\left\{\sum_{i=1}^{k_3} \frac{(1+o(1))t}{i} + O\left(\frac{1}{i^2}\right)\right\} = O(1) \times \exp\left\{\frac{1}{2} + o(1) - \beta\right\} t \log n \right\}. $$

It follows, on taking $\beta = 3/2 + o(1)$ that w.h.p.

$$Y_j \leq \frac{(3 + o(1)) \log n}{2an}$$

for all $j \in [n]$.

Letting $T_j$ be the set corresponding to $S_{k_3}$ when we execute Dijkstra’s algorithm starting at $j$, then we have that for $j \neq k$ where $T_j \cap T_k = \emptyset$,

$$\mathbb{P}\left(\exists e \in R_p \cap (T_j : T_k) : X(e) \leq \frac{\log n}{\omega n}\right) \leq \exp\left\{-\frac{(1+o(1))k_3^2 p \log n}{\omega n}\right\} = e^{-2+o(1)) \log^2 n} = o(n^{-2})$$

and this is enough to complete the proof of Theorem 2.

4 Bipartite matchings

We find, just as in [18], that the proofs in [42, 43] can be adapted to our current situation. Suppose that the vertices of $G$ are denoted $A = \{a_i, i \in [n]\}$ and $B = \{b_j, j \in [n]\}$. We will need to assume that

$$a_1, a_2, \ldots, a_n$$

constitutes a random ordering of the vertices in $A$.

We will use the notation $(a, b)$ for edges of $G$, where $a \in A$ and $b \in B$. We will let $w(a, b)$ denote the weight of $(a, b)$. Let $C(n, r)$ denote the weight of the minimum weight matching

$$M_r = \{(a_i, \phi_r(a_i)) : i = 1, 2, \ldots, r\} \text{ of } A_r \text{ into } B.$$
We will prove that
\[ \mathbb{E}(C(n, r) - C(n, r - 1)) \approx \frac{1}{\alpha} \sum_{i=1}^{r} \frac{1}{r(n - i + 1)}. \] (17)
for \( r = 1, 2, \ldots, n - o(n) \).

Using this and a simple argument for \( r \geq n - o(n) \) we argue that
\[ \mathbb{E}(C_n) = \mathbb{E}(C(n, n)) \approx \frac{1}{\alpha} \sum_{i=1}^{n} \sum_{j=1}^{r} \frac{1}{r(n - i + 1)} \approx \frac{1}{\alpha} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6\alpha}. \] (18)

### 4.1 Proof details

We use the ideas of [42], [43] and some ideas from [18]. We add a special vertex \( b_{n+1} \) to \( B \), with edges to all \( n \) vertices of \( A \). Each edge adjacent to \( b_{n+1} \) is assigned an \( E(\lambda) \) weight independently, \( \lambda > 0 \). Here \( E(\lambda) \) is an exponential random variable of rate \( \lambda \) i.e. \( \mathbb{P}(E(\lambda) \geq x) = e^{-\lambda x} \). We now consider \( M_r \) to be the minimum weight matching of \( A_r \) into \( B^* = B \cup \{b_{n+1}\} \). We denote this matching by \( M_r^* \) and we let \( B_r^* \) denote the corresponding set of vertices of \( B^* \) that are covered by \( M_r^* \).

Define \( P(n, r) \) as the normalized probability that \( b_{n+1} \) participates in \( M_r^* \), i.e.
\[ P(n, r) = \lim_{\lambda \to 0} \frac{\mathbb{P}(b_{n+1} \in B_r^*)}{\lambda}. \] (19)

Its importance lies in the following lemma:

**Lemma 5.**
\[ \mathbb{E}(C(n, r) - C(n, r - 1)) = \frac{P(n, r)}{r}. \] (20)

**Proof.** Choose \( i \) randomly from \([r]\) and let \( \widehat{B}_i \subseteq B_r \) be the \( B \)-vertices in the minimum weight matching of \((A_r \setminus \{a_i\})\) into \( B^* \). Let \( X = C(n, r) \) and let \( Y = C(n, r - 1) \). Let \( w_i \) be the weight of the edge \((a_i, b_{n+1})\), and let \( I_i \) denote the indicator variable for the event that the minimum weight of an \( A_r \) matching that contains this edge is smaller than the minimum weight of an \( A_r \) matching that does not use \( b_{n+1} \). In other words, \( I_i \) is the indicator variable for the event \( \{Y_i + w_i < X\} \), where \( Y_i \) is the minimum weight of a matching from \( A_r \setminus \{a_i\} \) to \( B \). This uses the fact that \( \widehat{B}_i \subseteq B_r \) by assuming that after deleting the edge \((a_i, b_{n+1})\) from \( M_r \) we have a matching from \( A_r \setminus \{a_i\} \) to \( \widehat{B}_i \) of weight \( Y_i \). Note that \( Y \) and \( Y_i \) have the same distribution. They are both equal to the minimum weight of a matching of a random \((r - 1)\)-set of \( A \) into \( B \).

If \((a_i, b_{n+1}) \in M_r^* \) then \( w_i < X - Y_i \). Conversely, if \( w_i < X - Y_i \) and no other edge from \( b_{n+1} \) has weight smaller than \( X - Y_i \), then \((a_i, b_{n+1}) \in M_r^* \), and when \( \lambda \to 0 \), the probability that there are two distinct edges from \( b_{n+1} \) of weight smaller than \( X - Y_i \) is of order \( O(\lambda^2) \). Indeed, let \( F \) denote the existence of two distinct edges from \( b_{n+1} \) of weight smaller than \( X \) and let \( F_{i,j} \) denote the event that \((a_i, b_{n+1}) \) and \((a_j, b_{n+1}) \) both have weight smaller than \( X \). Then,
\[ \mathbb{P}(F) \leq n^2 \mathbb{E}_X(\max_{i,j} \mathbb{P}(F_{i,j} \mid X)) = n^2 \mathbb{E}((1 - e^{-\lambda X})^2) \leq n^2 \lambda^2 \mathbb{E}(X^2), \] (21)
and since \( \mathbb{E}(X^2) \) is finite and independent of \( \lambda \), this is \( O(\lambda^2) \).
Since \( w_i \) is \( E(\lambda) \) distributed, as \( \lambda \to 0 \) we have from (21) that
\[
P(n, r) = \lim_{\lambda \to 0} \left( \frac{1}{\lambda} \sum_{i \in A_r} \mathbb{P}(w_i < X - Y_i) + O(\lambda) \right) = \lim_{\lambda \to 0} \mathbb{E} \left( \frac{1}{\lambda} \sum_{i \in A_r} (1 - e^{-\lambda(X - Y_i)}) \right)
= \sum_{i \in A_r} \mathbb{E}(X - Y_i) = r \mathbb{E}(X - Y).
\]
\[\square\]

We now proceed to estimate \( P(n, r) \). Fix \( r \) and assume that \( b_{n+1} \notin B_{r-1}^* \). Suppose that \( M_r^* \) is obtained from \( M_{r-1}^* \) by finding an augmenting path \( P = (a_r, \ldots, a_\sigma, b_r) \) from \( a_r \) to \( B \setminus B_{r-1} \) of minimum additional weight. We condition on (i) \( \sigma \), (ii) the lengths of all edges other than \( (a_\sigma, b_j), b_j \in B \setminus B_{r-1} \) and (iii) \( \min \{ w(a_\sigma, b_j) : b_j \in B \setminus B_{r-1} \} \). With this conditioning \( M_{r-1} = M_{r-1}^* \) will be fixed and so will \( P' = (a_r, \ldots, a_\sigma) \). We can now use the following fact: Let \( X_1, X_2, \ldots, X_M \) be independent exponential random variables of rates \( \lambda_1, \lambda_2, \ldots, \lambda_M \). Then the probability that \( X_i \) is the smallest of them is \( \lambda_i/(\lambda_1 + \lambda_2 + \cdots + \lambda_M) \). Furthermore, the probability stays the same if we condition on the value of \( \min \{ X_1, X_2, \ldots, X_M \} \). Thus
\[
\mathbb{P}(b_{n+1} \in B_r^* \mid b_{n+1} \notin B_{r-1}^*) = \mathbb{E} \left( \frac{\lambda}{\delta_r + \lambda} \right)
\]
where \( \delta_r = d_{r-1}(a_\sigma) \) is the number of neighbors of \( a_\sigma \) in \( B \setminus B_{r-1} \).

Lemma 6.
\[
P(n, r) = \mathbb{E} \left( \frac{1}{\delta_1} + \frac{1}{\delta_2} + \cdots + \frac{1}{\delta_r} \right). \tag{22}
\]

Proof.
\[
\lim_{\lambda \to 0} \lambda^{-1} \mathbb{P}(b_{n+1} \in B_r^*) = \lim_{\lambda \to 0} \lambda^{-1} \mathbb{E} \left( 1 - \frac{\delta_1}{\delta_1 + \lambda} \cdot \frac{\delta_2}{\delta_2 + \lambda} \cdots \frac{\delta_r}{\delta_r + \lambda} \right)
= \lim_{\lambda \to 0} \lambda^{-1} \mathbb{E} \left( 1 - \left( 1 + \frac{\lambda}{\delta_1} \right)^{-1} \cdots \left( 1 + \frac{\lambda}{\delta_r} \right)^{-1} \right)
= \lim_{\lambda \to 0} \lambda^{-1} \mathbb{E} \left( \left( \frac{1}{\delta_1} + \frac{1}{\delta_2} + \cdots + \frac{1}{\delta_r} \right) \lambda + O(\lambda^2) \right)
= \mathbb{E} \left( \frac{1}{\delta_1} + \frac{1}{\delta_2} + \cdots + \frac{1}{\delta_r} \right). \tag{23}
\]
\[\square\]

We now state (part of) Theorem 2 of Thomason [38] in terms of our notation. Assume that \( G(m) \) is as in Theorem 3.

Theorem 7. If \( X \subseteq A, Y \subseteq B \) and \( \alpha |X| > 1 \) and \( x = |X|, y = |Y| \), then
\[
|e(X,Y) - \alpha xy| \leq (xy(\alpha n + \mu x))^{1/2}.
\]
where \( e(X,Y) \) is the number of edges with one end in \( X \) and the other in \( Y \).
4.1.1 Upper bound

We begin with an upper bound estimate for (23). This means finding lower bounds for the $δ_i$. Let

$$r_0 = \frac{n}{\log^2 n}, \quad \omega = \log^5 n, \quad θ = \frac{1}{\omega^2}, \quad ε = \frac{1}{\omega}, \quad k = \omega^3.$$  \hspace{1cm} (24)

We have the trivial bound $δ_i ≥ αn - r$ which implies that

$$\sum_{r=1}^{r_0} \frac{1}{r} \sum_{i=1}^{r} \frac{1}{δ_i} ≤ \sum_{r=1}^{r_0} \frac{1}{αn - r_0} = o(1).$$  \hspace{1cm} (25)

Now suppose that $r ≥ r_0$ and let

$$E_r = \{ \exists S \subseteq [r, r + θr] : |S| = k, δ_i ≤ α(1 - ε)(n - r - i) \text{ for } i \in S\}.$$  \hspace{1cm} (26)

We claim that Theorem 7 implies that $E_r$ cannot occur for $r ≤ n - r_0$. Indeed, suppose that $E_r$ occurs. Then

$$e(S, B_{r+θr}) ≥ α \sum_{i \in S} (r + i + ε(n - r - i)) ≥ αk(r + ε(n - r)).$$  \hspace{1cm} (27)

Plugging in the values from (24) into (26) and (27) we see that after subtracting $αkr$ the RHS of (26) is $Ω\left(\frac{kn}{\omega \log^2 n}\right)$ and the RHS of (27) is $O\left(\frac{kn}{\omega^{3/2}}\right)$, contradiction.

We will prove below that if $r ≤ n_1 = n - n^{3/4}$ then w.h.p. and in expectation

$$δ_r ≥ ν_1 = ω^7$$  \hspace{1cm} (28)

for some constant $c_1 > 0$.

Let $ξ(r)$ be the indicator for the exceptions in (28) and let $ζ_a$ denote the number of times that vertex $a$ takes the role of $a_α$. We will show that w.h.p.

$$ζ_a ≤ ν_2 = c_1 \log n,$$  \hspace{1cm} (29)

for all $a ∈ A$,

Let $I_1, I_2, \ldots, I_s = \left\lceil \frac{n - n_1}{θn} \right\rceil$ be an equitable partition of $[r_0, n - n_1]$ into consecutive intervals of length $≈ θn$. By equitable we mean that $|I_k - I_l| ≤ 1$ for all $k ≠ l$. Given that $E_r$ doesn’t occur and (28) we see that w.h.p.

$$\sum_{j \in I_ℓ} \frac{1}{δ_j} ≤ \frac{1}{α} \sum_{j \in I_ℓ} \frac{1}{(1 - ε)(n - j)} + \frac{kν_2}{ν_1} + ξ(I_ℓ).$$  \hspace{1cm} (30)

Consequently, if $γ_ℓ = |I_1 + \cdots + |I_ℓ|,$

$$E(C(n, n - n_1)) ≤ o(1) + \frac{1 + o(1)}{α} \sum_{ℓ=1}^{s} \sum_{r=r_0+γ_ℓ-1+1}^{r_0+γ_ℓ} \frac{1}{r} \sum_{j=1}^{r} \frac{1}{n-j+1} + \frac{kν_2}{ν_1} + \frac{ν_2ω^9}{r_0},$$

$$= o(1) + \frac{1 + o(1)}{α} \sum_{r=r_0}^{n-n_1} \frac{1}{r} \sum_{j=1}^{r} \frac{1}{n-j+1} ≈ \frac{π^2}{6α}. \hspace{1cm} (31)$$

The first $o(1)$ term in (31) comes from (25).

We show later that

$$E(C(n, n) - C(n, n - n_1)) = o(1) \hspace{1cm} (32)$$

and this proves the upper bound on $E(C_n)$. 

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Let $\ell = n^{1/3}$ and $s = \left\lceil \frac{n-n_1}{\ell} \right\rceil \approx n^{2/3}$ and equitably partition $[n-n_1]$ into intervals $I_j$, $j = 1, 2, \ldots, \ell$ of length $\approx s$ and define $\gamma_t$ as before. Fix $j$ and let $I = I_j$. Next let $S_i, i \leq \nu_2$ denote the set of elements of $A$ that appear $i$ times as $a_\alpha$ in $I$ and let $s_i = |S_i|$. Let $T_i$ denote the subset of $I_j$ corresponding to $S_i$. Partition $T_i = U_1 \cup \cdots U_i$ into $i$ copies of $S_i$ in a natural way. Then it follows from Theorem 7 that if $s_i \alpha > 1$ then for $1 \leq k \leq i$,

$$\sum_{j \in U_k} \delta_j - s_i \alpha(n - (j - 1)s) \leq (s_i(n - (j - 1)s)(\alpha n + \mu s_i))^{1/2}.$$ 

Therefore,

$$\sum_{i:s_i \alpha \geq 1} \left( \sum_{j \in T_i} \delta_j - is_i \alpha(n - (j - 1)s) \right) \leq \sum_{i:is_i \alpha > 1/\alpha} i(s_i(n - (j - 1)s)(\alpha n + \mu s_i))^{1/2} \leq n^{1/2}(n - (j - 1)s)^1/2 \sum_{i:is_i \alpha > 1/\alpha} \frac{s_i^{1/2}}{\alpha} \leq \nu_2^2 s^{1/2} n^{1/2}(n - (j - 1)s).$$

It follows then that

$$\sum_{k \in I_j} \delta_k \leq \alpha s(n - (j - 1)s) + \nu_2^2 s^{1/2} n^{1/2}(n - (j - 1)s)^{1/2} + \alpha^{-1} \nu_2(n - (j - 1)s). \tag{33}$$

We have from (33) and the fact that the harmonic mean is at most the arithmetic mean that

$$\sum_{i \in I_j} \frac{1}{\delta_i} \geq \frac{s^2}{\sum_{i \in I_j} \delta_i} \geq \frac{s}{\alpha(n - (j - 1)s) \left( 1 + \frac{\nu_2^2}{s(n - (j - 1)s)} \right)^{1/2} + \frac{s_1}{\alpha s}} = \frac{s}{\alpha(n - (j - 1)s)} \left( 1 + O \left( \frac{\nu_2^2}{s(1 - (j - 1)s/n)} \right) \right) = \frac{1}{\alpha} \sum_{i \in I_j} \frac{1}{n - i + 1} \left( 1 + O \left( \frac{\nu_2^2}{s(1 - (j - 1)s/n)} \right) \right) + O \left( \frac{s}{n - (j - 1)s} \right) \approx \frac{1}{\alpha} \sum_{i \in I_j} \frac{1}{n - i + 1}. \tag{34}$$

Therefore,

$$E(C(n, n - n_1)) = E \left( \sum_{r=1}^{n-n_1} \sum_{i=1}^{r} \frac{1}{\delta_i} \right) = E \left( \sum_{j=1}^{\ell} \sum_{r \in I_j} \sum_{i=1}^{r} \frac{1}{\delta_i} \right) \geq \frac{1}{\alpha} \sum_{j=1}^{\ell} \sum_{r \in I_j} \sum_{i=1}^{r} \frac{1}{n - i + 1} \geq \frac{1}{\alpha} \sum_{r=1}^{n-n_1} \sum_{i=1}^{r} \frac{1}{n - i + 1} - \frac{s}{n - s} - \sum_{j=2}^{\ell} \frac{s}{(j-1)s(n - js)} = \frac{1}{\alpha} \sum_{r=1}^{n-n_1} \sum_{i=1}^{r} \frac{1}{n - i + 1} - o(1) \approx \frac{\pi^2}{6} \alpha. \tag{34}$$

For a proof of the final estimate in (34) we refer the reader to the proof of Equation (2) of [18]. This gives the correct lower bound for Theorem 3.

We now have to verify (28), (32). These claims rest on a bound on the maximum weight of an edge in the minimum weight perfect matching. Our proof is similar to that in [18].
4.1.3 No long edges

Let \( V_r = A_{r+1} \cup B \) and let \( G_r \) be the subgraph of \( G(m) \) induced by \( V_r \). For \( v \in V_r \), define the \( k \)-neighborhood \( N_k(v) \) to be the \( k \) other endpoints of the \( k \) shortest edges incident to \( v \).

Let the \( k \)-neighborhood of a set be the union of the \( k \)-neighborhoods of its vertices. In particular, for \( S \subseteq A_{r+1} \), \( T \subseteq B \),

\[
N_k(S) = \{ b \in B : \exists a \in S : y \in N_k(a) \}, \quad \text{(35)}
\]

\[
N_k(T) = \{ a \in A_{r+1} : \exists b \in T : a \in N_k(b) \}. \quad \text{(36)}
\]

Given a function \( \phi_r \) defining a matching \( M \) of \( A_r \) into \( B \), we define the following digraph: let \( \Gamma_r = (V_r, \vec{X}) \) where \( \vec{X} \) is an orientation of \( X \cup Y \) where

\[
X = \{ \{ a, b \} \in G : (a \in A_{r+1}, b \in N_{40}(a)) \text{ or } (b \in B, a \in N_{40}(b)) \} \cup \{ (\phi_r(a_i), a_i) : i = 1, 2, \ldots, r. \}
\]

\[
Y = \left \{ e = \{ a, b \} \in R_p : a \in A_{r+1}, X(e) \leq \frac{\log n}{n} \right \}.
\]

An edge \( e \in M \) is oriented from \( B \) to \( A \) and has weight \(-X(e)\). The remaining edges are oriented from \( A \) to \( B \) and have weight equal to their weight in \( G \).

The arcs of directed paths in \( \Gamma_r \) are alternately forwards \( A \to B \) and backwards \( B \to A \) and so they correspond to alternating paths with respect to the matching \( M \). It helps to know (Lemma 8, next) that given \( a \in A_{r+1}, b \in B \) we can find an alternating path from \( a \) to \( b \) with \( O(\log n) \) edges. The \( ab \)-diameter will be the maximum over \( a \in A_{r+1}, b \in B \) of the length of a shortest alternating path from \( a \) to \( b \).

**Lemma 8.** W.h.p., for every \( \phi_r \), the (unweighted) \( ab \)-diameter of \( \Gamma_r \) is at most \( k_0 = \lceil 3 \log_4 n \rceil \).

**Proof.** For \( S \subseteq A_{r+1}, T \subseteq B \), let

\[
\Lambda(S) = \{ b \in B : \exists a \in S \text{ such that } (a, b) \in \vec{X} \},
\]

\[
\Lambda(T) = \{ a \in A_{r+1} : \exists b \in T \text{ such that } (a, b) \in \vec{X} \}.
\]

We first prove an expansion property: that w.h.p., for all \( S \subseteq A_{r+1} \) with \( |S| \leq n_0 = \lceil n/\log n \rceil \), \( |\Lambda(S)| \geq 4|S| \).

(Note that \( \Lambda(S), \Lambda(T) \) are defined independently of \( \phi_r \).)

\[
\mathbb{P}(\exists S : |S| \leq n_0, |\Lambda(S)| < 4|S|) \leq \sum_{s=1}^{n_0} \binom{r+1}{s} \left( \frac{n}{4s} \right) \left( \frac{4s}{40} \right)^s \leq \sum_{s=1}^{n_0} \left( \frac{n}{s} \right)^s \left( \frac{4s}{40} \right)^s \left( \frac{4s}{\alpha n} \right)^{40s}
\]

\[
= \sum_{s=1}^{n_0} \left( \frac{e^{436}s^{35}}{\alpha^{40n^{35}}} \right)^s = o(1). \quad \text{(37)}
\]

Similarly, w.h.p., for all \( T \subseteq B \) with \( |T| \leq n_0, |\Lambda(T)| \geq 4|T| \). Now, choose an arbitrary \( a \in A_{r+1} \), and define \( S_0, S_1, S_2, \ldots \) as the endpoints of all alternating paths starting from \( a \) and of lengths \( 0, 2, 4, \ldots \). That is,

\[
S_0 = \{ a \} \text{ and } S_i = \phi_r^{-1}(\Lambda(S_{i-1})).
\]

We can assume that \( |S_i| \geq 4|S_{i-1}| \) provided \( |S_{i-1}| \leq n_0 \), and so there exists a smallest index \( i_0 \) such that \( |S_{i_0}| \geq n_0 \) and \( i_0 \leq \log_4(n_0) \leq \log_4 n \). Similarly, for an arbitrary \( b \in B \), define \( T_0, T_1, \ldots \), by

\[
T_0 = \{ b \} \text{ and } T_i = \phi_r(\Lambda(T_{i-1})).
\]

\[
T_0 = \{ b \} \text{ and } T_i = \phi_r(\Lambda(T_{i-1})).
\]
Again, we will find an index \( j_0 \leq \log_4 n \) where \( |T_{j_0}| \geq n_0 \).

If \( \phi_r(S_{i_0}) \cap T_{j_0} \neq \emptyset \) then this establishes the existence of an alternating walk and hence (removing any cycles) an alternating path of length at most \( 2(i_0 + j_0 + 1) \leq 2 \log_4 n \) from \( a \) to \( b \) in \( \Gamma_r \). Otherwise we use

\[
P(\bar{e} = (a, b) \in R_p : a \in S_{i_0}, b \in T_{j_0}) \leq \left( 1 - \frac{\omega \log^3 n}{n^2} \right)^{n_0^2} = o(n^{-3}), \tag{38}\]

where \( \omega \to \infty \).

\[ \square \]

We will need the following lemma from [18]:

**Lemma 9.** Suppose that \( k_1 + k_2 + \cdots + k_M \leq a \log N \), and \( Z_1, Z_2, \ldots, Z_M \) are independent random variables with \( Z_i \) distributed as the \( k_i \)th minimum of \( N \) independent exponential rate one random variables. If \( \mu > 1 \) then

\[
P\left( Z_1 + \cdots + Z_M \geq \frac{\mu a \log N}{N - a \log N} \right) \leq N^{a(1 + \log \mu - \mu)}.\]

Given this we can bound the weighted diameter of \( \Gamma_r \).

**Lemma 10.** W.h.p., for all \( \phi_r \), the weighted ab-diameter of \( \Gamma_r \) is at most \( c_2 \frac{\log n}{an} \) for some absolute constant \( c_2 > 0 \).

**Proof.** Let

\[
Z_1 = \max \left\{ \sum_{i=0}^{k} X(x_i, y_i) - \sum_{i=0}^{k-1} X(y_i, x_{i+1}) \right\}, \tag{39}\]

where the maximum is over sequences \( x_0, y_0, x_1, \ldots, x_k, y_k \) where \( (x_i, y_i) \) is one of the 40 shortest arcs of \( G \) leaving \( x_i \) for \( i = 0, 1, \ldots, k \leq k_0 = \lceil 3 \log_4 n \rceil \), and \( (y_i, x_{i+1}) \) is a backwards matching edge.

We compute an upper bound on the probability that \( Z_1 \) is large. For any \( \eta > 0 \) we have

\[
P\left( Z_1 \geq \eta \frac{\log n}{an} \right) \leq \sum_{k=0}^{k_0} \left( (r + 1)n \right)^{k+1} \left( \frac{1 + o(1)}{an} \right)^{k+1} \times \int_{y=0}^{\infty} \frac{1}{(k-1)!} \left( \frac{y \log n}{an} \right)^{k-1} \sum_{\rho_0 + \rho_1 + \cdots + \rho_k \leq 40(k+1)} q(\rho_0, \rho_1, \ldots, \rho_k; \eta + y) \, dy
\]

where

\[
q(\rho_0, \rho_1, \ldots, \rho_k; \eta) = P\left( Z_0 + Z_1 + \cdots + Z_k \geq \eta \frac{\log n}{an} \right),
\]

\( Z_0, Z_1, \ldots, Z_k \) are independent and \( Z_j \) is distributed as the \( \rho_j \)th minimum of \( r \) independent exponential random variables. (When \( k = 0 \) there is no term \( \frac{1}{(k-1)!} \left( \frac{y \log n}{an} \right)^{k-1} \).)

**Explanation:** We have at most \( (r + 1)n)^{k+1} \) choices for the sequence \( x_0, y_0, x_1, \ldots, x_k, y_k \). The term \( \frac{1}{(k-1)!} \left( \frac{y \log n}{an} \right)^{k-1} \) \( dy \) bounds the probability that the sum of \( k \) independent exponentials, \( X(y_0, x_1) + \cdots + X(y_{k-1}, x_k) \), is in \( \log n \) \( [y, y + dy] \). (The density function for the sum of \( k \) independent exponentials is \( \frac{y^{k-1} e^{-y}}{(k-1)!} \).)

We integrate over \( y \). \( \frac{1 + o(1)}{an} \) is the probability that \((x_i, y_i)\) is the \( \rho_i \)th shortest edge of \( G \) leaving \( x_i \), and these
events are independent for $0 \leq i \leq k$. The final summation bounds the probability that the associated edge lengths sum to at least $\frac{(\eta+y) \log n}{an}$.

It follows from Lemma 9 with $a \leq 3, N = (1 + o(1)) \alpha n, \mu = (\eta + y)/a$ that if $\eta$ is sufficiently large then, for all $y \geq 0$,

$$q(\rho_1, \ldots, \rho_k; \eta + y) \leq (\alpha n)^{-(\eta+y)\log n/(2\log n)} = n^{-(\eta+y)/2}.$$ 

Since the number of choices for $\rho_0, \rho_1, \ldots, \rho_k$ is at most $\binom{41k + 40}{k+1}$ (the number of positive integral solutions to $a_0 + a_1 + \ldots + a_{k+1} \leq 40(k+1)$) we have

$$\mathbb{P}\left(Z_1 \geq \eta \frac{\log n}{\alpha n}\right) \leq 2n^{-\eta/2} \sum_{k=0}^{k_0} \frac{(\log n)^{k-1}}{(k-1)!} \binom{41k + 40}{k+1} \int_{y=0}^{\infty} y^{k-1} n^{-y/2} dy$$

$$\leq 2n^{-\eta/2} \sum_{k=0}^{k_0} \frac{(\log n)^{k-1}}{(k-1)!} 2^{41k+40} \left(\frac{2}{\log n}\right)^k \int_{z=0}^{\infty} z^{k-1} e^{-z} dz$$

$$= 2^{39} n^{-\eta/2} \sum_{k=0}^{k_0} 2^{42} = o(n^{-4}),$$

for $\eta$ sufficiently large.

Now as we have seen in Lemma 8 we might need to use one edge of $R_\rho$ of weight at most $\frac{\log n}{n}$ to find a path from $a$ to $b$. There will be one w.h.p., see (38).

**Remark 1.** Lemma 10 shows that with probability $1 - o(n^{-4})$ we never need to use an edge of weight more than $\frac{c_2 \log n}{\alpha n}$ in $M_n$. Otherwise, we could use an alternating path to reduce the weight of the matching.

This proves that with $r_0$ as in (32)

$$\mathbb{E}(C(n,n) - C(n,n - r_0)) = O\left( r_0 \frac{\log n}{n} \right) = O\left( \frac{n}{\log^2 n} \frac{\log n}{n} \right) = o(1).$$

This verifies (32).

To prove (28) we argue

$$\mathbb{P}\left( \exists a \in A : \left\{ e : v \in e, X_e \leq \frac{c_2 \log n}{\alpha n} \right\} \right) \geq 10c_2 \log n \leq \mathbb{P}\left( \text{Bin}(\alpha n, \frac{c_2 \log n}{\alpha n}) \geq 10c_2 \log n \right)$$

$$\leq \left( \frac{\alpha n}{10c_2 \log n} \right)^{10c_2 \log n} \leq \left( \frac{e}{10} \right)^{10c_2 \log n} = O(n^{-2}).$$

This verifies (29) with $c_1 = 10c_2$.

We finally consider (28). Consider how a vertex $a \in A$ loses neighbors in $B \setminus B_r$. It can lose up to $\nu_2$ for the times when $a = a_\sigma$. Otherwise, it loses a neighbor when $a_\sigma \neq a$ chooses a common neighbor with $a$. The important point here is that this choice depends on the structure of $G$, but not on the weights of edges incident with $a$. It follows that the cheapest neighbors at any time are randomly distributed among the current set of available neighbors. To get to the point where $d_\nu(a) \leq \nu_1$, we must have at least one of the $\nu_2$ original cheapest neighbors occuring in a random $\nu_1$ subset of a set of size $\approx \alpha n$. This has probability $O(\nu_1 \nu_2/n) = o(\omega^8/n)$ and (28) follows from the Markov inequality.
A perfect matching in the bipartite graph \(G\) corresponds to a set of vertex disjoint directed cycles that cover all of the vertices of \(D\). It will be important to bound the number of cycles \(\nu_C\) in the minimum weight matching of Section 4.

**Lemma 11.** W.h.p. \(\nu_C = \Theta(n^{3/4} \log n)\).

**Proof.** A matching \(M_{r-1}\) induces a collection \(C_{r-1}\) of vertex disjoint paths and cycles in \(D\). An augmenting path \(P\) with respect to \(M_{r-1}\) changes this collection in the following way. We first add an edge \(a_r, b_k\) for some index \(k\). If \(b_k\) is isolated in \(M_r\) then \(C_r = C_{r-1}\) plus the edge \((r, k)\). Otherwise, suppose that \((a_s, b_k) \in M_r\). This means that adding the edge \((r, k)\) to \(C_r\) increases the in-degree of \(k\) to two. So, we delete the edge \((s, k)\) and then continue along \(P\) to examine the other edge \((a_s, b_\ell)\) incident with \(a_s\) in the path. This continues until we reach \(a_\sigma\). We then add the edge \((a_\sigma, b_r)\). In the digraph \(D\) this either means that the added edge closes a cycle or connects two paths into one. Because \(b_r\) is randomly chosen, we see that the probability it closes a cycle is exactly \(1/\delta_r\). It follows that

\[
\mathbb{E}(\nu_C) \leq \mathbb{E}\left(\sum_{r=1}^{n_1} \frac{1}{\delta_r}\right) + (n - n_1).
\]  

Equations (28) and (30) imply that the expected value of the sum in (41) is \(\log O(1) n\) and the lemma then follows from the Markov inequality.

Now we know that w.h.p. the collection of cycles \(C_n = C_1, C_2, \ldots, C_{\nu_C}\) does not use any edge of weight greater than \(w_0 = \frac{c_2 \log n}{\alpha n}\). The edges \(E^+\) of higher weight have distribution \(w_0 + E(1)\). We condition on the edges of weight at most \(w_0\). Suppose now that \(|C_1| \geq n/\nu_C\) and \(C\) is any other cycle. We *patch* \(C\) into \(C_1\) by choosing edges \((u, v)\) of \(C_1\) and \((x, y)\) of \(C\), deleting them and replacing them by \(e = (x, v)\) and \(f = (u, y)\). If \(X_e, X_f \leq 2w_0\) then we have successfully merged \(C, C_1\) into a larger cycle at cost \(2w_0\). The probability we cannot use \(R_p\) to do this is at most \(e^{-n_1^2/\nu_C} = o(n^{-1})\). Note we assume here that either \(e, f \in E^+\) and are independently distributed as \(w_0 + E(1)\) or they are of weight at most \(w_0\). It follows that w.h.p. we can patch all cycles into a tour at extra cost \(\frac{2c_2 \nu_C \log n}{\alpha n} = o(1)\). In this process we can always patch the next cycle into the largest current cycle using edges that have not been exposed so far. So w.h.p. the weight of this tour is asymptotically equal to the weight of the minimum matching. This completes the proof of Theorem 4.

**6 Final remarks**

We have shown that adding sufficient random edges is enough to give “smooth out” the optimal value in certain optimization problems. There are several questions that remain. The first is to remove the pseudo-random requirement from Theorem 3. The problem is to control the sizes of the \(\delta_r\). Another possibility is to consider matchings and 2-factors in arbitrary regular graphs, not just bipartite ones. Then one can consider the Symmetric Travelling Salesperson problem. We could also consider relaxing \(\alpha\) to be \(o(1)\) and we can consider more general distributions than \(E(1)\).
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