ON RAINBOW HAMILTON CYCLES IN RANDOM HYPERGRAPHS

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ABSTRACT. Let $H_{n,p,\kappa}^{(k)}$ denote a *randomly colored random hypergraph*, constructed on the vertex set $[n]$ by taking each k-tuple independently with probability p, and then independently coloring it with a random color from the set $[\kappa]$. Let H be a k-uniform hypergraph of order n. An *ℓ-Hamilton cycle* is a spanning subhypergraph C of H with $n/(k-\ell)$ edges and such that for some cyclic ordering of the vertices each edge of C consists of k consecutive vertices and every pair of adjacent edges in C intersects in precisely ℓ vertices.

In this note we study the existence of *rainbow* ℓ-Hamilton cycles (that is every edge receives a different color) in $H_{n,p,\kappa}^{(k)}$. We mainly focus on the most restrictive case when $\kappa = n/(k-\ell)$. In particular, we show that for the so called tight Hamilton cycles ($\ell = k-1$) $p = e^2/n$ is the sharp threshold for the existence of a rainbow tight Hamilton cycle in $H_{n,p,n}^{(k)}$ for each $k \geq 4$.

1. INTRODUCTION

Suppose that $k > \ell \geq 1$. An *ℓ-Hamilton cycle* C in a k-uniform hypergraph $H = (V, \mathcal{E})$ on n vertices is a collection of $m_\ell = n/(k-\ell)$ edges of H such that for some cyclic order of $[n]$ every edge consists of k consecutive vertices and for every pair of consecutive edges E_{i-1}, E_i in C (in the natural ordering of the edges) we have $|E_{i-1} \cap E_i| = \ell$ (see Figure 1). Thus, in every ℓ-Hamilton cycle the sets $C_i = E_i \setminus E_{i-1}$, $i = 1, 2, \ldots, m_\ell$, are a partition of V into sets of size $k - \ell$. Hence, $m_\ell = n/(k-\ell)$. We thus always assume, when discussing ℓ-Hamilton cycles, that this necessary condition, $k - \ell$ divides n, is fulfilled. In the literature, when $\ell = k-1$ we have a *tight* Hamilton cycle and when $\ell = 1$ we have a *loose* Hamilton cycle.

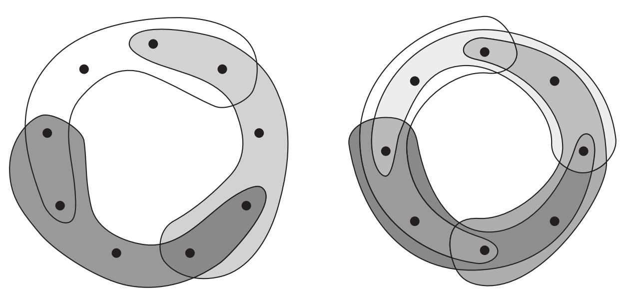

FIGURE 1. A 2-Hamilton and a 3-Hamilton 5-uniform cycles.

The first author was supported in part by a grant from the Simons Foundation (522400, AD).
The third author was supported in part by NSF grant DMS1661063.

Let $H_{n,p}^{(k)}$ denote a *random hypergraph*, constructed on the vertex set $[n]$ by taking each k-tuple from $\binom{[n]}{k}$ independently with probability p. When $k = 2$ we have the well-known Erdős-Rényi-Gilbert model $G_{n,p}$.

The threshold for the existence of Hamilton cycles in the random graph $G_{n,p}$ has been known for many years, see, e.g., [1], [4] and [16]. Recently these results were extended to hypergraphs, see, e.g., [2, 6, 7, 8, 9, 12, 13, 15, 17, 18]. Below we summarize some of them.

In the following and throughout the paper, $\omega = \omega(n)$ can be any function tending to infinity with n. All logarithms in this paper are natural (base e). Recall that an event $\mathcal{E}_n$ occurs *with high probability*, or **whp** for brevity, if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

Theorem 1.1 ([6])**.** *Let $\varepsilon > 0$ be fixed. Then:*

(i) For all integers $k > \ell \geq 2$, if $p \leq (1-\varepsilon)e^{k-\ell}/n^{k-\ell}$, then **whp** *$H_{n,p}^{(k)}$ is not ℓ-Hamiltonian.*
(ii) For all integers $k > \ell \geq 3$, there exists a constant $K = K(k)$ such that if $p \geq K/n^{k-\ell}$ and n is a multiple of $k-\ell$ then $H_{n,p}^{(k)}$ is ℓ-Hamiltonian **whp**.
(iii) If $k > \ell = 2$ and $p \geq \omega/n^{k-2}$ and n is a multiple of $k-2$, then $H_{n,p}^{(k)}$ is 2-Hamiltonian **whp**.
(iv) For all $k \geq 4$, if $p \geq (1+\varepsilon)e/n$, then **whp** *$H_{n,p}^{(k)}$ is $(k-1)$-Hamiltonian, i.e. it contains a* tight *Hamilton cycle.*

In particular, this theorem shows that e/n is the sharp threshold for the existence of a tight Hamilton cycle in a k-uniform hypergraph, when $k \geq 4$. As it was explained in [6], quite surprisingly, the proof of (ii)-(iv) in Theorem 1.1 is based on the second moment method.

Theorem 1.2 ([7, 8, 9, 12])**.** *Fix $k \geq 3$ and suppose that n is a multiple of $k-1$. Let $p \geq \omega(\log n)/n^{k-1}$. Then,* **whp** *$H_{n,p}^{(k)}$ contains a loose Hamilton cycle.*

Thus, $(\log n)/n^{k-1}$ is the asymptotic threshold for the existence of loose Hamilton cycles. This is because if $p \leq (1-\varepsilon)(k-1)!(\log n)/n^{k-1}$ and $\varepsilon > 0$ is constant, then **whp** $H_{n,p}^{(k)}$ contains isolated vertices.

In this note we study the existence of rainbow Hamilton cycles in $H_{n,p}^{(k)}$ with independently colored edges. Let $H_{n,p,\kappa}^{(k)}$ denote a *randomly colored random hypergraph*, constructed on the vertex set $[n]$ by taking each k-tuple independently with probability p, and then independently coloring it with a random color from the set $[\kappa]$. We also denote $H_{n,p,\kappa}^{(2)}$ by $G_{n,p,\kappa}$. Rainbow properties of $G_{n,p,\kappa}$ attracted a considerable amount of attention, see, e.g., [3, 5, 10, 14, 11].

Here we only focus on rainbow Hamilton cycles, which are Hamilton cycles where every edge of the cycle receives a different color. Improving the previous results of Cooper and Frieze [5] and Frieze and Loh [14], Ferber and Krivelevich [10] determined the very sharp threshold for the existence of the rainbow Hamilton cycle in $G_{n,p,\kappa}$ assuming nearly optimal number of colors.

Theorem 1.3 ([10])**.** *Let $\varepsilon > 0$, $\kappa = (1+\varepsilon)n$ and let $p = (\log n + \log\log n + \omega)/n$. Then,* **whp** *$G_{n,p,\kappa}$ contains a rainbow Hamilton cycle.*

For expressions such as $\kappa = (1+\varepsilon)n$ that clearly have to be an integer, we round up or down but do not specify which: the reader can choose either one, without affecting the argument.

Ferber and Krivelevich [10] were the first to study rainbow Hamilton cycles in $H^{(k)}_{n,p}$. They showed the following. (Recall that $m_\ell = n/(k-\ell)$ is the number of edges in an ℓ-Hamilton cycle.)

Theorem 1.4 ([10])**.** *Let $k > \ell \geq 1$ be integers. Suppose that n is a multiple of $k-\ell$. Let $p \in [0,1]$ be such that* **whp** *$H^{(k)}_{n,p}$ contains an ℓ-Hamilton cycle. Then, for every $\varepsilon = \varepsilon(n) \geq 0$, letting $\kappa = (1+\varepsilon)m_\ell$ and $q = \kappa p/(\varepsilon m_\ell + 1)$ we have that* **whp** *$H^{(k)}_{n,p,\kappa}$ contains a rainbow ℓ-Hamilton cycle.*

Observe that if ε is a constant, then by losing a multiplicative constant in the threshold, a rainbow ℓ-Hamilton **whp** exists. By combining this result with Theorems 1.1 and 1.2 one can obtain some explicit values of q. However, for small ε (including $\varepsilon = 0$) Theorem 1.4 does not provide optimal q. In our results we focus on the case when $\kappa = m_\ell$. (But we also allow more colors.) Here we state our first result.

Theorem 1.5. *Let $k > \ell \geq 2$ and $\varepsilon > 0$ be fixed. Let $c \geq 1/(k-\ell)$ and $\kappa = cn$. Then:*

(i) For all integers $k > \ell \geq 2$, if

$$p \leq \begin{cases} (1-\varepsilon)e^{k-\ell+1}/n^{k-\ell} & \text{if } c = 1/(k-\ell) \\ (1-\varepsilon)\left(\frac{c-1/(k-\ell)}{c}\right)^{(k-\ell)c-1} e^{k-\ell+1}/n^{k-\ell} & \text{if } c > 1/(k-\ell), \end{cases}$$

then **whp** *$H^{(k)}_{n,p,\kappa}$ is not rainbow ℓ-Hamiltonian.*

(ii) For all integers $k > \ell \geq 3$, there exists a constant $K = K(k)$ such that if $p \geq K/n^{k-\ell}$ and n is a multiple of $k-\ell$ then $H^{(k)}_{n,p,\kappa}$ is rainbow ℓ-Hamiltonian **whp**.

(iii) If $k > \ell = 2$ and $p \geq \omega/n^{k-2}$ and n is a multiple of $k-2$, then $H^{(k)}_{n,p,\kappa}$ is rainbow 2-Hamiltonian **whp**.

(iv) For all $k \geq 4$, if

$$p \geq \begin{cases} (1+\varepsilon)e^2/n & \text{if } c = 1 \\ (1+\varepsilon)\left(\frac{c-1}{c}\right)^{c-1} e^2/n & \text{if } c > 1, \end{cases}$$

then **whp** *$H^{(k)}_{n,p,\kappa}$ is rainbow $(k-1)$-Hamiltonian, i.e. it contains a rainbow* tight *Hamilton cycle.*

Consequently, if $k \geq 4$, then

$$p = \begin{cases} e^2/n & \text{if } c = 1 \\ \left(\frac{c-1}{c}\right)^{c-1} e^2/n & \text{if } c > 1 \end{cases}$$

is the sharp threshold for the existence of a rainbow tight Hamilton cycle. Furthermore, observe that $\lim_{c\to 1^+}\left(\frac{c-1}{c}\right)^{c-1} = 1$. Thus, in (iv) the case $c > 1$ approaches the case $c = 1$ in the continuous way. Finally, also observe that $\lim_{c\to\infty}\left(\frac{c-1}{c}\right)^{c-1} = 1/e$. Hence, when c tends to infinity (that means that each edge receives a different color) the threshold function is e/n, which is consistent with Theorem 1.1. The proof of Theorem 1.5 modifies the proof of Theorem 1.1.

We also establish a similar result for loose Hamilton cycles. Recall that a loose Hamilton cycle of order n has exactly $n/(k-1)$ edges. So for a rainbow loose Hamilton cycle we always need at least $n/(k-1)$ colors. Here we only consider this most restrictive case with $\kappa = n/(k-1)$.

Theorem 1.6. *Fix $k \geq 3$ and suppose that n is a multiple of $k-1$. Let $\kappa = n/(k-1)$ and $p \geq \omega(\log n)/n^{k-1}$. Then,* **whp** *$H^{(k)}_{n,p,\kappa}$ contains a rainbow loose Hamilton cycle.*

The proof is a modification of the proof of Theorem 1.2.

Some notation: For sequences $A_n, B_n, n \geq 1$ we write $A_n \approx B_n$ to mean that $A_n = (1+o(1))B_n$ as $n \to \infty$. Similarly, we write $A_n \lesssim B_n$ to mean that $A_n \leq (1+o(1))B_n$ as $n \to \infty$.

2. Proof of Theorem 1.5

The proof modifies the proof of Theorem 3 from [6].

Let $([n], \mathcal{E})$ be a k-uniform hypergraph. A permutation π of $[n]$ is *ℓ-Hamilton cycle inducing* if

$$E_\pi(i) = \{\pi((i-1)(k-\ell)+j) : \ j \in [k]\} \in \mathcal{E} \ for\ all\ i \in [n/(k-\ell)].$$

(We use the convention $\pi(n+r) = \pi(r)$ for $r > 0$.) Let the term *hamperm* refer to such a permutation.

Let Y be the random variable that counts the number of rainbow hamperms π for $H^{(k)}_{n,p,\kappa}$. Every ℓ-Hamilton cycle induces at least one hamperm and so we can concentrate on estimating $\mathbf{Pr}(Y > 0)$.

Observe that

$$\mathbf{E}(Y) = n! \cdot p^{n/(k-\ell)} \cdot \frac{(\kappa)_{n/(k-\ell)}}{\kappa^{n/(k-\ell)}},$$

where $(x)_t = x(x-1)\cdots(x-t+1)$ is the *falling factorial*. This is because π induces an ℓ-Hamilton cycle if and only if a certain $n/(k-\ell)$ edges are present and are colored rainbow.

Now let $c > 1/(k-\ell)$. Then, by Stirling's formula we get

$$\begin{aligned}
\mathbf{E}(Y) &= n! p^{n/(k-\ell)} \frac{\kappa!}{\kappa^{n/(k-\ell)}(\kappa - n/(k-\ell))!} \\
&\approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n p^{n/(k-\ell)} \frac{\sqrt{\frac{\kappa}{\kappa-n/(k-\ell)}}\left(\frac{\kappa}{e}\right)^\kappa}{\kappa^{n/(k-\ell)}\left(\frac{\kappa-n/(k-\ell)}{e}\right)^{\kappa-n/(k-\ell)}} \\
&= \sqrt{\frac{2\pi n\kappa}{\kappa - n/(k-\ell)}}\left(\frac{np^{1/(k-\ell)}}{e^{1+1/(k-\ell)}} \cdot \left(\frac{\kappa}{\kappa - n/(k-\ell)}\right)^{\kappa/n - 1/(k-\ell)}\right)^n \\
&= \sqrt{\frac{2\pi n\kappa}{\kappa - n/(k-\ell)}}\left(\frac{np^{1/(k-\ell)}}{e^{1+1/(k-\ell)}} \cdot \left(\frac{c}{c - 1/(k-\ell)}\right)^{c - 1/(k-\ell)}\right)^n.
\end{aligned}$$

Similarly for $c = 1/(k-\ell)$ we get

$$\mathbf{E}(Y) \approx 2\pi n\sqrt{\frac{1}{k-\ell}}\left(\frac{np^{1/(k-\ell)}}{e^{1+1/(k-\ell)}}\right)^n.$$

Thus, if

$$p \le \begin{cases} (1-\varepsilon)e^{k-\ell+1}/n^{k-\ell} & \text{if } c = 1/(k-\ell) \\ (1-\varepsilon)\left(\frac{c-1/(k-\ell)}{c}\right)^{(k-\ell)c-1} e^{k-\ell+1}/n^{k-\ell} & \text{if } c > 1/(k-\ell), \end{cases}$$

then $\mathbf{E}(Y) = o(1)$. This verifies part (i).

Now we prove parts (ii)-(iv) by the second moment method. First observe that if

$$p \ge \begin{cases} (1+\varepsilon)e^{k-\ell+1}/n^{k-\ell} & \text{if } c = 1/(k-\ell) \\ (1+\varepsilon)\left(\frac{c-1/(k-\ell)}{c}\right)^{(k-\ell)c-1} e^{k-\ell+1}/n^{k-\ell} & \text{if } c > 1/(k-\ell), \end{cases}$$

then $\mathbf{E}(Y) \to \infty$ together with n.

Fix a hamperm π. Let $H(\pi) = (E_\pi(1), E_\pi(2), \ldots, E_\pi(m_\ell))$ be the Hamilton cycle induced by π. Then let $N(b,a)$ be the number of permutations π' such that $|E(H(\pi)) \cap E(H(\pi'))| = b$ and $E(H(\pi)) \cap E(H(\pi'))$ consists of a edge disjoint paths. Here a path is a maximal sub-sequence $F_1, F_2, \ldots, F_q$ of the edges of $H(\pi)$ such that $F_i \cap F_{i+1} \neq \emptyset$ for $1 \le i < q$. The set $\bigcup_{j=1}^q F_j$ may contain other edges of $H(\pi)$. Observe that $N(b,a)$ does not depend on π.

Now,

$$\frac{\mathbf{E}(Y^2)}{\mathbf{E}(Y)^2} \le \frac{n!N(0,0)p^{2n/(k-\ell)}\left(\frac{(\kappa)_{n/(k-\ell)}}{\kappa^{n/(k-\ell)}}\right)^2}{\mathbf{E}(Y)^2} + \sum_{b=1}^{n/(k-\ell)} \sum_{a=1}^{b} \frac{n!N(b,a)p^{2n/(k-\ell)-b}}{\mathbf{E}(Y)^2} \cdot \frac{(\kappa)_{n/(k-\ell)}}{\kappa^{n/(k-\ell)}} \cdot \frac{(\kappa-b)_{n/(k-\ell)-b}}{\kappa^{n/(k-\ell)-b}}.$$

Since trivially $N(0,0) \le n!$, we get

$$\frac{\mathbf{E}(Y^2)}{\mathbf{E}(Y)^2} \le 1 + \sum_{b=1}^{n/(k-\ell)} \sum_{a=1}^{b} \frac{n!N(b,a)p^{2n/(k-\ell)-b}}{\mathbf{E}(Y)^2} \cdot \frac{(\kappa)_{n/(k-\ell)}}{\kappa^{n/(k-\ell)}} \cdot \frac{(\kappa-b)_{n/(k-\ell)-b}}{\kappa^{n/(k-\ell)-b}}.$$

Let X be the number of ℓ-hamperms in $H^{(k)}_{n,p}$. Then,

$$\mathbf{E}(X) = n!p^{n/(k-\ell)} \quad \text{and} \quad \mathbf{E}(Y) = \mathbf{E}(X) \cdot \frac{(\kappa)_{n/(k-\ell)}}{\kappa^{n/(k-\ell)}}.$$

Consequently,

$$\begin{aligned}
\frac{\mathbf{E}(Y^2)}{\mathbf{E}(Y)^2} &\le 1 + \sum_{b=1}^{n/(k-\ell)} \sum_{a=1}^{b} \frac{n!N(b,a)p^{2n/(k-\ell)-b}}{\mathbf{E}(X)^2} \cdot \frac{(\kappa)_{n/(k-\ell)}}{\kappa^{n/(k-\ell)}} \cdot \frac{(\kappa-b)_{n/(k-\ell)-b}}{\kappa^{n/(k-\ell)-b}} \cdot \left(\frac{\kappa^{n/(k-\ell)}}{(\kappa)_{n/(k-\ell)}}\right)^2 \\
&= 1 + \sum_{b=1}^{n/(k-\ell)} \sum_{a=1}^{b} \frac{N(b,a)p^{n/(k-\ell)-b}}{\mathbf{E}(X)} \cdot \kappa^b \cdot \frac{(\kappa-b)_{n/(k-\ell)-b}}{(\kappa)_{n/(k-\ell)}} \\
&= 1 + \sum_{b=1}^{n/(k-\ell)} \sum_{a=1}^{b} \frac{N(b,a)p^{n/(k-\ell)-b}}{\mathbf{E}(X)} \cdot \kappa^b \cdot \frac{(\kappa-b)!}{\kappa!} \\
&\le 1 + \sum_{b=1}^{n/(k-\ell)} \sum_{a=1}^{b} \frac{N(b,a)p^{n/(k-\ell)-b}}{\mathbf{E}(X)} \cdot e^b \left(\frac{\kappa-b}{\kappa}\right)^{\kappa-b}. \qquad (1)
\end{aligned}$$

Part (ii)**:** $\ell \ge 3$

We trivially bound $\left(\frac{\kappa-b}{\kappa}\right)^{\kappa-b} \le 1$. It was shown in [6] (equation (10)) that

$$\frac{N(b,a)p^{n/(k-\ell)-b}}{\mathbf{E}(X)} \lesssim \left(\frac{2k!ke^k}{n^{k-\ell}p}\right)^b \frac{1}{n^{a(\ell-2)}}.$$

Thus,

$$\begin{aligned}
\frac{\mathbf{E}(Y^2)}{\mathbf{E}(Y)^2} &\le 1 + \sum_{b=1}^{n/(k-\ell)} \sum_{a=1}^{b} \frac{N(b,a)p^{n/(k-\ell)-b}}{\mathbf{E}(X)} \cdot e^b \\
&\lesssim 1 + \sum_{b=1}^{n/(k-\ell)} \sum_{a=1}^{b} \left(\frac{2k!ke^k}{n^{k-\ell}p}\right)^b \frac{1}{n^{a(\ell-2)}} \cdot e^b \\
&\le 1 + \sum_{b=1}^{n/(k-\ell)} \sum_{a=1}^{b} \left(\frac{2k!ke^{k+1}}{n^{k-\ell}p}\right)^b \frac{1}{n^a}.
\end{aligned}$$

Set $K = 4k!ke^{k+1}$ and $p = K/n^{k-\ell}$. Thus,

$$\frac{\mathbf{E}(Y^2)}{\mathbf{E}(Y)^2} \le 1 + \sum_{b=1}^{n/(k-\ell)} \sum_{a=1}^{b} \frac{1}{2^b} \cdot \frac{1}{n^a} \le 1 + \left(\sum_{b=1}^{n} \frac{1}{2^b}\right)\left(\sum_{a=1}^{n} \frac{1}{n^a}\right) \approx 1.$$

Part (iii)**:** $\ell = 2$

Let $p \geq \omega/n^{k-2}$. Similarly as in the previous case

$$\begin{aligned}\frac{\mathbf{E}(Y^2)}{\mathbf{E}(Y)^2} &\leq 1 + \sum_{b=1}^{n/(k-2)} \sum_{a=1}^{b} \left(\frac{2k!ke^k}{n^{k-2}p} \right)^b \cdot e^b \\ &\leq 1 + \sum_{b=1}^{n/(k-2)} \sum_{a=1}^{b} \left(\frac{2k!ke^{k+1}}{\omega} \right)^b \\ &\leq 1 + \sum_{b=1}^{n/(k-2)} b \left(\frac{2k!ke^{k+1}}{\omega} \right)^b \approx 1.\end{aligned}$$

Part (iv)**:** $\ell = k-1$ (tight cycles)

If $c = 1$ (that means $\kappa = n$), then we trivially bound $\left(\frac{\kappa-b}{\kappa}\right)^{\kappa-b} \leq 1$. Otherwise, we use a simple fact.

Claim 2.1. *Let $\kappa = cn$, where $c > 1$. Then,*

$$\max_{0<b\leq n} \left(\frac{\kappa-b}{\kappa} \right)^{\frac{\kappa-b}{b}} = \left(\frac{\kappa-n}{\kappa} \right)^{\frac{\kappa-n}{n}} = \left(\frac{c-1}{c} \right)^{c-1}.$$

Proof of the claim. Let $x = b/n$. Note that since $1 \leq b \leq n$, $x \in (0,1]$ and $c > x$. Then

$$\left(\frac{\kappa-b}{\kappa} \right)^{\frac{\kappa-b}{b}} = \left(\frac{c-b/n}{c} \right)^{\frac{c-b/n}{b/n}} = \left(\frac{c-x}{c} \right)^{\frac{c-x}{x}}.$$

Taking the derivative gives us

$$\frac{d}{dx} \left(\frac{c-x}{c} \right)^{\frac{c-x}{x}} = -\frac{c}{x^2} \left(\frac{c-x}{c} \right)^{\frac{c-x}{x}} \left(\log\left(\frac{c-x}{c} \right) + \frac{x}{c} \right).$$

Since $\frac{c}{x^2}\left(\frac{c-x}{c}\right)^{\frac{c-x}{x}} > 0$, we have

$$\operatorname{sgn}\left(\frac{d}{dx} \left(\frac{c-x}{c} \right)^{\frac{c-x}{x}} \right) = -\operatorname{sgn}\left(\log\left(\frac{c-x}{c} \right) + \frac{x}{c} \right) = -\operatorname{sgn}\left(\log\left(1 - \frac{x}{c}\right) + \frac{x}{c} \right)$$

and since $\log\left(1-\frac{x}{c}\right) < \log e^{-\frac{x}{c}} = -\frac{x}{c}$ we get $\log\left(1-\frac{x}{c}\right) + \frac{x}{c} < 0$. Thus

$$\frac{d}{dx} \left(\frac{c-x}{c} \right)^{\frac{c-x}{x}} > 0$$

for $0 < x \leq 1$ and $c > x$. Thus $\left(\frac{c-x}{c}\right)^{\frac{c-x}{x}}$ is maximized at $x = 1$ in our domain, which corresponds to $b = n$, proving the claim. $\square$

Due to (1) and the above claim we obtain

$$\frac{\mathbf{E}(Y^2)}{\mathbf{E}(Y)^2} \leq \begin{cases} 1 + \sum_{b=1}^{n} \sum_{a=1}^{b} \frac{N(b,a)p^{n-b}}{\mathbf{E}(X)} \cdot e^b, & \text{if } c = 1 \\ 1 + \sum_{b=1}^{n} \sum_{a=1}^{b} \frac{N(b,a)p^{n-b}}{\mathbf{E}(X)} \cdot \left(e \left(\frac{c-1}{c}\right)^{c-1} \right)^b, & \text{if } c > 1. \end{cases}$$

Moreover, it was shown in [6] (equation (13)) that for $k \geq 4$,

$$\sum_{b=1}^{n}\sum_{a=1}^{b}\frac{N(b,a)p^{n-b}}{\mathbf{E}(X)} \leq \frac{2c_k k! e^{k-1}}{n^{k-3}}\exp\left\{\frac{2k!e^{k-1}}{n^{k-4}}\right\}\sum_{b=1}^{n}\left(\frac{e}{np}\right)^b$$

for some positive constant c_k that depends on k only. Thus,

$$\frac{\mathbf{E}(Y^2)}{\mathbf{E}(Y)^2} \leq \begin{cases} 1+\frac{2c_k k! e^{k-1}}{n^{k-3}}\exp\left\{\frac{2k!e^{k-1}}{n^{k-4}}\right\}\sum_{b=1}^{n}\left(\frac{e}{np}\right)^b \cdot e^b, & \text{if } c=1\\ 1+\frac{2c_k k! e^{k-1}}{n^{k-3}}\exp\left\{\frac{2k!e^{k-1}}{n^{k-4}}\right\}\sum_{b=1}^{n}\left(\frac{e}{np}\right)^b \cdot \left(e\left(\frac{c-1}{c}\right)^{c-1}\right)^b, & \text{if } c>1. \end{cases}$$

Hence, both for $c=1$, $p \geq \frac{(1+\varepsilon)e^2}{n}$ and for $c>1$, $p \geq (1+\varepsilon)\left(\frac{c-1}{c}\right)^{c-1}\frac{e^2}{n}$, we get that

$$\frac{\mathbf{E}(Y^2)}{\mathbf{E}(Y)^2} \leq 1+\frac{2c_k k! e^{k-1}}{n^{k-3}}\exp\left\{\frac{2k!e^{k-1}}{n^{k-4}}\right\}\sum_{b=1}^{n}\frac{1}{(1+\varepsilon)^b} \approx 1.$$

In all three cases we showed that $\frac{\mathbf{E}(Y^2)}{\mathbf{E}(Y)^2} \lesssim 1$. Thus, the Chebyshev inequality completes the proof of Theorem 1.5.

3. Proof of Theorem 1.6

Let $n=(k-1)m$ and assume that m is even. Clearly, $m=m_1=\kappa$. In this case the proof is a straightforward modification of the proof of Theorem 2 from [7].

Let $X=[m]$ and $Y=[m+1,n]$ and $Z=[n+1,n+m]$. Given $H=H^{(k)}_{n,p,m}$ we define the $(k+1)$-uniform hypergraph Γ with vertex set $[n]$ and an edge $\phi(e)$ for each edge $e=\{x_1,x_2,y_1,\ldots,y_{k-2}\}$ of H that satisfies $|e\cap X|=2$. Here $x_1,x_2\in X$ and $y_i\in Y, 1\leq i\leq k-2$. We then let $\phi(e)=\{x_1,x_2,y_1,\ldots,y_{k-2},c(e)+n\}$, where $c(e)$ is the color of e and $c(e)+n\in Z$. The proof in [7] can be adapted (and therefore we need to assume that m is even) to show that **whp** Γ contains a loose Hamilton cycle where consecutive edges intersect in vertices of X. We will give sufficient detail in Appendix to justify this claim. A loose Hamilton cycle in Γ corresponds to a rainbow loose Hamilton cycle of H, where we re-interpret the vertex z of an edge as the color $z-n$.

We can easily remove the requirement that m be even by using an idea of Ferber [9]. In particular, one can follow his proof of Theorem 1.2 to show that Γ contains a loose Hamilton cycle in this case.

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Appendix A. Modifying the proof in [7]

Suppose that $p = \omega(\log n)/n^{k-1}$, where $\omega = o(\log n)$ and $\omega \to \infty$. Let $M = \binom{n}{k}p$ and consider a random $(k+1)$-uniform hypergraph K with approximately $M' \approx M$ edges. Then

$$\begin{aligned}\mathbf{Pr}(\exists\, e_1, e_2 \in E(K) : |e_1 \cap e_2| = k) &\leq \binom{n}{k+1}\binom{k+1}{k} n \frac{\binom{\binom{n}{k+1}-2}{M'-2}}{\binom{\binom{n}{k+1}}{M'}} \\ &\leq n^{k+2}\left(\frac{M'}{\binom{n}{k+1}}\right)^2 \\ &\leq n^{k+1}\left(\frac{2(k+1)!\omega n \log n}{n^{k+1}}\right)^2 \\ &= o(1).\end{aligned}$$

In this way we can justify viewing $H_{n,p,m}^{(k)}$ as a random $(k+1)$-uniform hypergraph. This would be a problem if the latter model gave an edge more than one color.

Let $m = 2m_1$. Then m_1 will replace m in the proof in [7]. The proof in [7] involves proving that **whp** $H_{n,p}^{(k)}$ contains a loose Hamilton cycle that respects a certain vertex partition. Such a Hamilton cycle will consist of $2m_1$ edges of the form $\{x_i, x_{i+1}, y_{i,1}, \ldots, y_{i,\kappa}\}$, where $\kappa = k-2$, $1 \leq i \leq 2m_1$, $x_{2m_1+1} = x_1$, $\{x_1, \ldots, x_{2m_1}\} = X$ and $\{y_{1,1}, \ldots, y_{2m_1,\kappa}\} = Y$.

This is done as follows: we choose a large positive integer d. Let $\mathcal{X}$ be a set of size $2dm_1$ representing d copies of each $x \in X$. Denote the jth copy of $x \in X$ by $x^{(j)} \in \mathcal{X}$ and let $\mathcal{X}_x = \{x^{(j)}, j = 1, 2, \ldots, d\}$. Then let $X_1, X_2, \ldots, X_d$ be a uniform random partition of $\mathcal{X}$ into d sets of size $2m_1$. Define $\psi_1 : \mathcal{X} \to X$ by $\psi_1(x^{(j)}) = x$ for all j and $x \in X$. Similarly, we let $\mathcal{Y}$ be a set of size $d\kappa m_1$ representing $d/2$ copies of each $y \in Y$. Denote the jth copy of $y \in Y$ by $y^{(j)} \in \mathcal{Y}$ and let $\mathcal{Y}_y = \{y^{(j)}, j = 1, 2, \ldots, d/2\}$. Then let $Y_1, Y_2, \ldots, Y_d$ be a uniform random partition of $\mathcal{Y}$ into d sets of size κm_1. Define $\psi_2 : \mathcal{Y} \to Y$ by $\psi_2(y^{(j)}) = y$ for all $y \in Y$. Finally, let $\psi : \binom{\mathcal{X}}{2} \times \binom{\mathcal{Y}}{\kappa} \to X^2 \times Y^\kappa$ be such that $\psi(\nu_1, \nu_2, \xi_1, \xi_2, \ldots, \xi_\kappa) = (\psi_1(\nu_1), \psi_1(\nu_2), \psi_2(\xi_1), \psi_2(\xi_2), \ldots, \psi_2(\xi_\kappa))$.

All we need do is add a set $\mathcal{Z}$ of size dm_1 representing $d/2$ copies of each $z \in Z$. We denote the jth copy of $z \in Z$ by $z^{(j)} \in \mathcal{Z}$ and let $\mathcal{Z}_z = \{z^{(j)}, j = 1, 2, \ldots, d/2\}$. Then let $Z_1, Z_2, \ldots, Z_d$ be a uniform random partition of $\mathcal{Z}$ into d sets of size m_1. Define $\psi_3 : \mathcal{Z} \to Z$ by $\psi_3(z^{(j)}) = z$ for all $z \in Z$. We then modify ψ so that $\psi : \binom{\mathcal{X}}{2} \times \binom{\mathcal{Y}}{\kappa} \times \mathcal{Z} \to X^2 \times Y^\kappa \times Z$ be such that $\psi(\nu_1, \nu_2, \xi_1, \xi_2, \ldots, \xi_\kappa, \zeta) = (\psi_1(\nu_1), \psi_1(\nu_2), \psi_2(\xi_1), \psi_2(\xi_2), \ldots, \psi_2(\xi_\kappa), \psi_3(\zeta))$. After this the proof in [7] can be carried out with straightforward modifications involving adding a component for members of Z.

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