Maker Breaker on Digraphs

Alan Frieze* and Wesley Pegden†
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh PA 15213

March 29, 2020

Abstract

We study two biased Maker-Breaker games played on the complete digraph $\vec{K}_n$. In the strong connectivity game, Maker wants to build a strongly connected subgraph. We determine the asymptotic optimal bias for this game viz. $\frac{n}{\log n}$. In the Hamiltonian game, Maker wants to build a Hamiltonian subgraph. We determine the asymptotic optimal bias for this game up to a constant factor.

1 Introduction

We consider some biased Maker-Breaker games played on the complete digraph $\vec{K}_n$ on $n$ vertices. This is in contrast to the large literature already existing on games played on the complete graph $K_n$. For a very nice summary of the main results in this area, we refer the reader to the monograph by Hefetz, Krivelevich, Stojaković and Szabo [4]. Our aim here is to analyse the directed versions of the connectivity game and the Hamiltonicity game. The connectivity game was solved in Chvátal and Erdős [2] and Gebauer and Szabó [3]. The Hamiltonicity game for graphs was solved by Krivelevich [5].

In the games analysed below, Maker goes first, claiming an edge of $\vec{K}_n$. Breaker then claims $b$ edges and so on, with Maker and Breaker taking one and $b$ edges respectively until there are no edges left to take. In addition, Maker and Breaker must claim disjoint sets of edges. Maker is aiming to construct a digraph with certain properties and Breaker is aiming to prevent this. The properties involved are monotone increasing and so there is a critical bias, $b_0$ say, such that if $b < b_0$ then Maker will win and if $b \geq b_0$ then Breaker will win. We will consider two properties here: strong connectivity and Hamiltonicity. We let $D_M, D_B$ denote the digraphs with vertex set $[n]$ and the edges taken by Maker, Breaker respectively. Maker wins the strong connectivity game if on termination $D_M$ is strongly connected. Maker wins the Hamiltonian game if on termination $D_M$ is Hamiltonian.

**Theorem 1.** Let $\varepsilon > 0$ be arbitrarily small and $n \geq n_\varepsilon$ sufficiently large. Then Breaker wins the strong connectivity game if $b \geq \frac{(1+\varepsilon)\log n}{n}$ and Maker wins if $b \leq \frac{(1-\varepsilon)\log n}{n}$.

*Research supported in part by NSF grant DMS1661063
†Research supported in part by NSF grant DMS1363136
Theorem 2. Let $\varepsilon > 0$ be arbitrarily small and $n \geq n_\varepsilon$ sufficiently large. Then Breaker wins the Hamiltonian game if $b \geq \frac{(1+\varepsilon) \log n}{n}$ and Maker wins if $b \leq \frac{\log n}{50n}$.

It is clear from this that there is room for improvement in the Hamiltonian game and we naturally conjecture that $1/50$ can be replaced by $1 - \varepsilon$.

2 Degree bound

Notation We let $d^+_M(v), d^-_M(v)$ denote the out-degree, in-degree of vertex $v$ in $D_M$ for $v \in [n]$. We define $d^+_B, d^-_B$ similarly. We let $E_M, E_B$ denote the edges claimed by Maker and Breaker respectively, on termination.

The following Theorem is a trivial generalisation of results from Chapter 5 of Hefetz, Krivelevich, Stojaković and Szabo [4]. Let $b = \beta n \log n$ be Breaker’s bias, where $\beta < 1$ is a constant.

Theorem 3. Let $G = (V, E)$ be an $n$-regular graph and that $\alpha \in (\beta, 1)$ and suppose that $2/\alpha \leq K \leq \theta \log n$ where $\theta < \frac{\alpha - \beta}{\beta}$ is a constant and $\theta \beta < \alpha$. Then the following holds: Maker has a randomised strategy that with positive probability can in at most $K|V|$ rounds ensure that Maker’s graph has minimum degree $K$ and Breaker’s graph has maximum degree at most $\alpha n$. Furthermore, Maker always randomly chooses edges from a set of size at least $(1 - \alpha)n$.

The proof of this involves a minor modification of the proof in [4]. We have for completeness provided a condensed proof in an appendix. Of course, having a randomized strategy in this context, also means having a deterministic strategy.

Now a digraph $D$ on vertex set $[n]$ can be associated with a bipartite graph $G$ on vertex set $A \cup B$ where $A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_n\}$ and where oriented edge $(i, j)$ is replaced by the edge $\{a_i, b_j\}$. In this way the out-degree of $k$ in $D$ is the degree of $a_k$ in $G$ and the in-degree of $k$ is the degree of $b_k$ in $G$. It follows from Theorem 3 that Maker can ensure that Maker’s graph has minimum in- and out-degree at least $K$ after at most $2Kn$ rounds. And that Breaker’s graph has maximum in- and out-degree at most $\alpha n$.

3 Strong Connectivity

3.1 Breaker win

We now consider the game to be played on the complete bipartite graph $K_{n,n}$ where the bipartition is $A \cup B$ with $|A| = |B| = n$. Breaker’s aim is to claim all the edges incident with some vertex $a \in A$. This is essentially the box game of Chvátal and Erdős [2]. We let $box_A = \{\{i, b\} : b \in B\}$ for $i \in A$. Breaker claims $b$ elements from the boxes and Maker claims one whole box in each turn. The claimed upper bound follows from Theorem 2.1 of [2].

Note that this also verifies the Breaker win in Theorem 2.
3.2 Maker win

Because Maker chooses neighbors randomly, small sets must have edges entering and leaving.

**Lemma 4.** Suppose that $K > 1/\alpha$. Then, w.h.p., $S \subseteq [n]$, $|S| \leq (1 - \alpha)^2 n$ implies that
\[
\{(i, j) \in E_M : i \in S, j \notin S\} \neq \emptyset \quad \text{and} \quad \{(i, j) \in E_M : i \notin S, j \in S\} \neq \emptyset.
\]

**Proof.** The probability that there exists a set violating the condition in the lemma is at most
\[
2^{(1 - \alpha)^2 n} \left(\frac{n}{s}\right)^K \left(\frac{s}{(1 - \alpha)n}\right)^K \leq 2 \sum_{s=K}^{(1 - \alpha)^2 n} \left(\frac{n}{s}\right)^K \left(\frac{s}{(1 - \alpha)n}\right)^K = o(1).
\]

Assume now that $\beta = 1 - \varepsilon$ is a close to one and that $\beta = (1 + \alpha)/2$. Now consider the DAG with one vertex for each strong component of $D_M$ in which there is an edge $(A, B)$ if there is an edge in $D_M$ directed from $A$ to $B$. We observe that w.h.p. each source and sink in $D_M$ must be associated with a subset of $[n]$ of size at least $(1 - \alpha)^2 n$. This follows directly from Lemma 4. A smaller sink would have an edge oriented from it to another strong component, contradiction.

It follows that w.h.p. after $2Kn$ rounds, Maker can make $D_M$ strongly connected in a further $(1 - \alpha)^{-4}$ rounds by adding an edge from each sink to each source. There will be by construction $\Omega(n^2)$ choices of edge available for each such pair and Breaker can only claim $o(n)$ edges in this number of rounds.

4 Hamiltonicity

We show that w.h.p. the digraph constructed by Maker is Hamiltonian. For each $v \in [n]$ there are sets $IN(v), OUT(v)$ of size $K$, where each of the $2nK$-sets have been chosen uniformly from sets $A(v), B(v)$ of size $(1 - \alpha)n$. The sets $A(v), B(v), v \in [n]$ are chosen adversarially.

Our analysis uses the following values for parameters:

\[
\alpha = 0.20, \quad \beta = 0.02, \quad \delta = 1/2 - \alpha, \quad \theta = 4 \quad \text{and} \quad K = \theta \log n,
\]

where

\[
P\left(Bin\left(\frac{(1.01)n\log n}{\delta - \alpha}, \frac{1}{(1 - \alpha)n}\right) \geq K\right) = o(n^{-1}). \tag{1}
\]

Using the bound, $P(Bin(n, p) \geq k) \leq \left(\frac{n}{k}\right)^k \leq \left(\frac{np}{\theta}\right)^k$, we see that (1) is implied by

\[
\theta \left(\log \left(\frac{(1.01)e}{(\delta - \alpha)(1 - \alpha)}\right) - \log \theta\right) < -1,
\]

which can be verified numerically. Note that we require $2\theta \beta \leq \alpha$ so that $2K\beta n/\log n \leq \alpha n$ in order not to violate Maker’s choices. We also need $\theta < \frac{\alpha - \beta}{\beta}$, which is required by Theorem \ref{thm:sensitivity}.

We will follow an approach similar to that of Angluin and Valiant \cite{AngluinValiant}. We choose an arbitrary vertex $x$ to start and at any point during the execution of the algorithm we have (i) a path $P$ from its start $s_P$ to its
finish $f_P$, (ii) a cycle disjoint from $P$ and (iii) a set $U = [n] \setminus (V(P) \cup V(C))$. We let $P[a,b]$ denote the sub-path of $P$ that goes from $a$ to $b$. At certain points $P,C$ may be empty and we denote this by $\Lambda$. We will assume that $IN(v), OUT(v)$ are ordered randomly and that there are pointers $in(v), out(v)$ to vertices in the lists. These are updated to the next vertex, after a selection is made. Initially, $in(v), out(v)$ point to the first vertex in each list. The choices of vertices are only exposed as necessary. This is usually referred to as deferred decisions. Let

$$P^* = \{v \in P : \exists w \in U \text{ s.t. } v \in IN(w)\} \quad \text{and} \quad C^* = \{v \in C : \exists w \in U \text{ s.t. } v \in IN(w)\}.$$ 

A general step of the process proceeds as follows: we begin with $P = (x), C = \Lambda$ and $U = [n] \setminus \{x\}$.

**Step 1** If $P = \Lambda$ then remove a random edge $e$ from $C$. Then $P \leftarrow C - e, C \leftarrow \Lambda$.

**Step 2** If $P \neq \Lambda$ let $y = out(f_P)$.

Case (a) (i) If $y \in U$ then $P \leftarrow P + (f_P, y)$.

(ii) If $y \notin U$, but $f_P \in P^*$ then $P \leftarrow P + (f_P, y)$ where $y \in U$ and $f_P = in(y)$.

Case (b) If $y \in C$ then $P \leftarrow P + C - e$ where $e = (z, y) \in C$. Also $C \leftarrow \Lambda$. Note that $f_P = z$ now.

Case (c) If $C = \Lambda$ and $y \in P$ and $y$ is distance at least $\delta n$ from $f_P$ along $P$, then $P \leftarrow P[s_P, z]$ where $(z, y) \in P$ and $C \leftarrow P[y, f_P] + (f_P, y)$.

If $z \in P^*$ then $P \leftarrow P + (z, u)$ where $u \in U$ and $z = in(u)$.

Case (d) If $C = \Lambda$ and $y = s_P$ then $C \leftarrow P + (y, s_P)$ and $P \leftarrow \Lambda$. Go to Step 1.

Case (e) If none of (a) – (e) are applicable, move $out(f_P)$ to the next vertex on its list.

It follows that $|C| = 0$ or $|C| \geq \delta n$ throughout. The pointers $in, out$ are updated if necessary to the next vertex on the list, if they are used in a step. Also, the above procedure fails if it reaches the end of a vertex list before creating a Hamilton cycle.

Next let $X_i$ be the number of edges examined in order to increase $|P| + |C|$ from $i$ to $i + 1$. Note that all random choices can be ascribed to a choice of either $out(f_P)$ or of $in(u), u \in U$. These choices can be thought of as being independent, assuming only that Maker makes her choices with replacement. In effect, this involves the rare possibility of her skipping a move.

(a) If $|U| \geq 2\alpha n$ then $X_i$ is dominated by the geometric random variable $Geo(p_1)$ where $p_1 = \frac{|U| - \alpha n}{n} \geq \alpha$.

This is because $f_P$ has at least $|U| - \alpha n$ choices available to it in $U$ for the next choice of vertex in $OUT(f_P)$.

(b) If $|U| < 2\alpha n$ then $|C^*| + |P^*| \geq (1 - 3\alpha)n$. It follows that $X_i$ is dominated by the geometric random variable $Geo(p_2)$ where $p_2 = \min \left\{ \frac{1 - 3\alpha - \delta}{n}, \frac{\delta - \alpha}{n} \right\}$.

(The minimands in the definition of $p_2$ are equal from the definition of $\delta$.)

Here we are bounding the probability of finding $f_P \in P^* \cup C^*$. The first term in $p_2$ corresponds to the case $C = \Lambda$ and $1 - 3\alpha - \delta$ lower bounds the probability of choosing a vertex in $P^* \cup C^*$ and we subtract $\delta$ to account for $y$ in Step 2(c) being close to $s_P$. The second term corresponds to the case $C \neq \Lambda$ and of choosing $z \in C^*$ in Case (b).
If we ignore the problem of the size of the sets $|N(v), OUT(v), v \in [n]$ then we can see from the Chebyshev inequality that w.h.p. we obtain $V \cup C = [n]$ in less than $(1.01)n \log n$ trials. Here a trial means exposure of $out(v)$ or $in(u)$. This follows from the fact that $\mathbb{E}(Geo(p)) = \frac{1}{p}$ and $\text{Var}(Geo(p)) = \frac{1-p}{p^2}$. So, if $T$ is the time to reach this stage then

$$\mathbb{E}(T) \leq n \log \frac{n}{\delta - \alpha} \text{ and } \text{Var}(T) \leq \frac{\pi^2 n^2}{6(\delta - \alpha)} = o(\mathbb{E}(T)^2).$$

Now a given vertex $v$ has probability at most $q = \frac{1}{1-(1-\alpha)n}$ of being selected as the next $y$ and this implies that the probability $K$ items on its list are examined is at most $\mathbb{P}(\text{Bin}(\frac{(1.01)n \log n}{\delta - \alpha}, q) \geq K) = o(n^{-1})$ by construction.

Once $V \cup C = [n]$, it takes $O(n)$ expected time to create a Hamilton cycle. Let us go through the possibilities.

(i) If $C = \Lambda$ and $s_P \in B(f_P)$ or $f_P \in A(s_P)$ then the process finishes in one more step with probability at least $1/n$.

(ii) If $C = \Lambda$ and (i) does not hold, then we update $out(f_P)$ and we are in (i).

(iii) If $C \neq \Lambda$ then there is a probability of at least $\delta - \alpha$ that $out(f_P) \in C$ and we are in (i).

5 Conclusion

We solved the strong connectivity game, but there is a big gap between the upper and lower bounds for Hamiltonicity. Closing this gap is an interesting open problem.

References


A Proof of Theorem 3

We let $G_M, G_B$ denote the subgraphs of $G$ with the edges taken by Maker, Breaker respectively. We let $d_M(v)$ denote the degree of vertex $v$ in $G_M$ for $v \in V$. We define $d_B$ similarly. Let $\text{dang}(v) = d_B(v) - 2bd_M(v)$ be the danger of vertex $v$ at any time.
**Maker’s Strategy:** In round $i$, choose a vertex $v_i$ of maximum danger and choose a random edge incident with $v_i$, not already taken. This is called *easing* $v_i$.

Let $M_i, B_i$ denote Maker and Breaker’s $i$th moves. Suppose that Breaker wins in round $g - 1$, so that after $B_{g-1}$ there is a vertex $v_g$ such that $d_B(v_g) > \alpha n$. Let $J_i = \{v_{i+1}, \ldots, v_g\}$. Next define

$$
\overline{\text{dang}}(M_i) = \frac{\sum_{v \in J_i} \text{dang}(v)}{|J_i - 1|} \quad \text{and} \quad \overline{\text{dang}}(B_i) = \frac{\sum_{v \in J_i} \text{dang}(v)}{|J_i|},
$$

computed before the $i$th moves of Maker, Breaker respectively.

Then $\overline{\text{dang}}(M_1) = 0$ and $\overline{\text{dang}}(M_g) = \text{dang}(v_g) = (\alpha - o(1))n$. Let $a(i)$ be the number of edges contained in $J_i$ that are claimed by Breaker in his first $i$ moves. We have

**Lemma 5.**

$$\overline{\text{dang}}(M_i) \geq \overline{\text{dang}}(B_i). \quad (2)$$

$$\overline{\text{dang}}(M_i) \geq \overline{\text{dang}}(B_i) + \frac{2b}{|J_i|}, \text{ if } J_i = J_{i-1}. \quad (3)$$

$$\overline{\text{dang}}(B_i) \geq \overline{\text{dang}}(M_{i+1}) - \frac{2b}{|J_i|} \quad (4)$$

$$\overline{\text{dang}}(B_i) \geq \overline{\text{dang}}(M_{i+1}) - \frac{b + a(i) - a(i - 1)}{|J_i|} - 1. \quad (5)$$

*Proof.* Equation (2) follows from the fact that a move by Maker does not increase danger. Equation (3) follows from the fact that if $v_i \in J_{i-1}$ then its danger, which is a maximum, drops by $2b$. Equation (4) follows from the fact that Breaker takes at most $b$ edges inside $J_i$. For equation (5), let $e_{\text{double}}$ be the number of edges that Breaker adds to $J_i$ in round $B_i$. Then

$$\overline{\text{dang}}(B_i) \geq \overline{\text{dang}}(M_{i+1}) - \frac{b + e_{\text{double}}}{|J_i|}$$

and

$$a(i) - e_{\text{double}} \geq a(i - 1) - |J_i|.$$

\[ \Box \]

It follows that

$$\overline{\text{dang}}(M_i) \geq \overline{\text{dang}}(M_{i+1}) \text{ if } J_i = J_{i-1}. \quad (6)$$

$$\overline{\text{dang}}(M_i) \geq \overline{\text{dang}}(M_{i+1}) - \min \left\{ \frac{2b}{|J_i|}, \frac{b + a(i) - a(i - 1)}{|J_i|} - 1 \right\}. \quad (7)$$

Next let $1 \leq i_1 \leq \cdots \leq i_r \leq g - 1$ be the indices where $J_i \neq J_{i-1}$. Then we have $|J_{i_r}| = |J_{g-1}| = 1$ and $|J_{i-1}| = |J_0| = r + 1$. Let $k = \frac{n}{\log n}$ and assume first that $r \geq k$ and then use the first minimand in (7) for
\[ i_1, \ldots, i_{r-k} \text{ and the second minimand otherwise.} \]

\[ 0 = \overline{\text{dang}}(M_1) \geq \overline{\text{dang}}(M_g) - \frac{b + a(i_r) - a(i_r - 1)}{|J_r|} - \ldots - \frac{b + a(i_{r-k+1}) - a(i_{r-k-1} + 1)}{|J_{r-k+1}|} - k - \frac{2b}{|J_{r-k}|} - \ldots - \frac{2b}{|J_1|} \tag{8} \]

\[ \geq \overline{\text{dang}}(M_g) - \frac{b}{1} - \ldots - \frac{b}{k} - \frac{a(i_r)}{1} - k - \frac{2b}{k + 1} - \ldots - \frac{2b}{r} \tag{9} \]

\[ \geq \alpha n - K b - b(1 + \log k) - k - 2b(\log n - \log k). \]

To go from (8) to (9) we use \( a(i_{r-j}) - 1 \geq a(i_{r-j-1}), j > 0 \) which follows from \( J_{i_r-j-1} = J_{i_r-j} - 1 \) and then the coefficient of \( a(i_{r-j-1}) \) is at least \( \frac{1}{j+1} - \frac{1}{j+2} \geq 0 \). Also, \( a(i_r) = 0 \) because \( J_{i_r} = J_{g-1} = \{v_g\} \).

It follows that

\[ b \geq \frac{\alpha n - k}{K + 1 + \log n + \log \log n + o(1)} = \frac{(\alpha - 1/\log n)n}{(1 + \theta + o(1)) \log n}, \]

contradicting our upper bound, \( \theta < \frac{\alpha - \beta}{\beta} \).

If \( r < k \) then we replace (9) by

\[ 0 = \overline{\text{dang}}(M_1) \geq \overline{\text{dang}}(M_g) - \frac{b}{1} - \ldots - \frac{b}{k} - \frac{a(i_r)}{1} - k \geq \alpha n - K b - b(1 + \log k) - k \]

and obtain the same contradiction.