Abstract

We study two biassed Maker-Breaker games played on the complete digraph \( \vec{\mathcal{K}}_n \). In the strong connectivity game, Maker wants to build a strongly connected subgraph. We determine the asymptotic optimal bias for this game viz. \( \frac{n}{\log n} \). In the Hamiltonian game, Maker wants to build a Hamiltonian subgraph. We determine the asymptotic optimal bias for this game up to a constant factor.

1 Introduction

We consider some biassed Maker-Breaker games played on the complete digraph \( \vec{\mathcal{K}}_n \) on \( n \) vertices. This is in contrast to the large literature already existing on games played on the complete graph \( \mathcal{K}_n \). For a very nice summary of the main results in this area, we refer the reader to the monograph by Hefetz, Krivelevich, Stojaković and Szabo [4]. Our aim here is to analyse the directed versions of the connectivity game and the Hamiltonicity game. The connectivity game was solved in Chvátal and Erdős [2] and Gebauer and Szabó [3].

In the games analysed below, Maker goes first, claiming an edge of \( \vec{\mathcal{K}}_n \). Breaker then claims \( b \) edges and so on, with Maker and Breaker taking one and \( b \) edges respectively until there are no edges left to take. In addition, Maker and Breaker must claim disjoint sets of edges. Maker is aiming to construct a digraph with certain properties and Breaker is aiming to prevent this. The properties involved are monotone increasing and so there is a critical bias, \( b_0 \) say, such that if \( b < b_0 \) then Maker will win and if \( b \geq b_0 \) then Breaker will win. We will consider two properties here: strong connectivity and Hamiltonicity. We let \( D_M, D_B \) denote the digraphs with vertex set \([n]\) and the edges taken by Maker, Breaker respectively. Maker wins the strong connectivity game if on termination \( D_M \) is strongly connected. Maker wins the Hamiltonian game if on termination \( D_M \) is Hamiltonian.

**Theorem 1.** Let \( \varepsilon > 0 \) be arbitrarily small and \( n \geq n_\varepsilon \) sufficiently large. Then Breaker wins the strong connectivity game if \( b \geq \frac{(1 + \varepsilon) \log n}{n} \) and Maker wins if \( b \leq \frac{(1 - \varepsilon) \log n}{n} \).
Theorem 2. Let \( \varepsilon > 0 \) be arbitrarily small and \( n \geq n_\varepsilon \) sufficiently large. Then Breaker wins the Hamiltonian game if \( b \geq (1 + \varepsilon) \log n \). Furthermore, there exists an absolute constant \( \beta > 0 \) such that Maker wins if \( b \leq \frac{\beta \log n}{n} \).

2 Degree bound

Notation We let \( d^+_M(v), d^-_M(v) \) denote the out-degree, in-degree of vertex \( v \) in \( D_M \) for \( v \in [n] \). We define \( d^+_B, d^-_B \) similarly. We let \( E_M, E_B \) denote the edges claimed by Maker and Breaker respectively, on termination.

The following Theorem is a trivial generalisation of results from Chapter 5 of Hefetz, Krivelevich, Stojaković and Szabo \[4\]. Let \( b = \frac{\beta n}{\log n} \) be Breaker’s bias, where \( \beta < 1 \) is a constant.

Theorem 3. Let \( G = (V, E) \) be an \( n \)-regular graph and that \( \alpha \in (\beta, 1) \) and suppose that \( \frac{2}{\alpha} \leq K \leq \theta \log n \) where \( \theta < \frac{\alpha - \beta}{\beta} \) is a constant and \( \theta \beta < \alpha \). Then the following holds: Maker has a randomised strategy that with positive probability can in at most \( K |V| \) rounds ensure that Maker’s graph has minimum degree \( K \) and Breaker’s graph has maximum degree at most \( \alpha n \). Furthermore, Maker always randomly chooses edges from a set of size at least \( (1 - \alpha)n \).

The proof of this involves a minor modification of the proof in \[4\]. We have for completeness provided a condensed proof in an appendix. Of course, having a randomized strategy in this context, also means having a deterministic strategy.

Now a digraph \( D \) on vertex set \([n]\) can be associated with a bipartite graph \( G \) on vertex set \( A \cup B \) where \( A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_n\} \) and where oriented edge \((i, j)\) is replaced by the edge \(\{a_i, b_j\}\). In this way the out-degree of \( k \) in \( D \) is the degree of \( a_k \) in \( G \) and the in-degree of \( k \) is the degree of \( b_k \) in \( G \). It follows from Theorem 3 that Maker can ensure that Makers graph has minimum in- and out-degree at least \( K \) after at most \( 2Kn \) rounds. And that Breaker’s graph has maximum in- and out-degree at most \( \alpha n \).

3 Strong Connectivity

3.1 Breaker win

We now consider the game to be played on the complete bipartite graph \( K_{n,n} \) where the bipartition is \( A \cup B \) with \( |A| = |B| = n \). Breaker’s aim is to claim all the edges incident with some vertex \( a \in A \). This is essentially the box game of Chvátal and Erdős \[2\]. We let box \( A_i = \{i, b\} : b \in B \) for \( i \in A \). Breaker claims \( b \) elements from the boxes and Maker claims one whole box in each turn. The claimed upper bound follows from Theorem 2.1 of \[2\].

Note that this also verifies the Breaker win in Theorem 2.

3.2 Maker win

Because Maker chooses neighbors randomly, small sets must have edges entering and leaving.
Lemma 4. Suppose that $K > 1/\alpha$. Then, w.h.p., $S \subseteq [n]$, $|S| \leq (1-\alpha)^2 n$ implies that

$$\{(i,j) \in E_M : i \in S, j \notin S\} \neq \emptyset \text{ and } \{(i,j) \in E_M : i \notin S, j \in S\} \neq \emptyset.$$  

Proof. The probability that there exists a set violating the condition in the lemma is at most

$$2 \sum_{s=K}^{(1-\alpha)^2n} \binom{n}{s} \left(\frac{s}{(1-\alpha)n}\right)^{Kn} \leq 2 \sum_{s=K}^{(1-\alpha)^2n} \left(\frac{n/2}{s/2}\right)^{K/2} = o(1).$$


Assume now that $\beta = 1 - \varepsilon$ is a close to one and that $\beta = (1 + \alpha)/2$. Now consider the DAG with one vertex for each strong component of $D_M$ in which there is an edge $(A, B)$ if there is an edge in $D_M$ directed from $A$ to $B$. We observe that w.h.p. each source and sink in $D_M$ must be associated with a subset of $[n]$ of size at least $(1-\alpha)^2 n$. This follows directly from Lemma 4. A smaller sink would have an edge oriented from it to another strong component, contradiction.

It follows that w.h.p. after $2Kn$ rounds, Maker can make $D_M$ strongly connected in a further $(1-\alpha)^{-4}$ rounds by adding an edge from each sink to each source. There will be by construction $\Omega(n^2)$ choices of edge available for each such pair and Breaker can only claim $o(n)$ edges in this number of rounds.

4 Hamiltonicity

We show that w.h.p. the digraph constructed by Maker is Hamiltonian. For each $v \in [n]$ there are sets $IN(v), OUT(v)$ of size $K = \theta \log n$, where each of the $2n$ $K$-sets have been chosen uniformly from sets $A(v), B(v)$ of size $(1-\alpha)n$. The sets $A(v), B(v), v \in [n]$ are chosen adversarially.

Our analysis assumes that $\alpha$ is sufficiently small and $\theta$ is sufficiently large, given $\alpha, \theta > 10/\alpha^2$ will suffice. Note that we require $2\theta \beta \leq \alpha$ so that $2K \beta n/\log n \leq \alpha n$ in order not to violate Maker’s choices. We also need $\theta < 2\alpha/\beta$, which is required by Theorem 3. This makes $\beta$ small compared to $\alpha$.

We will follow an approach similar to that of Angluin and Valiant. We choose an arbitrary vertex denoted $s_P$ to start and at any point during the execution of the algorithm we have (i) a path $P$ that begins at $s_P$ and ends at some vertex $f_P$, (ii) a cycle $C$ disjoint from $P$ and (iii) a set $U = [n] \setminus (V(P) \cup V(C))$. We let $P[a,b]$ denote the sub-path of $P$ that goes from $a$ to $b$. At certain points $P,C$ may be empty and we denote this by $\Lambda$. We will assume that $OUT(v)$ is kept as a randomly list and that there is a pointer $out(v)$ to vertices in the lists. These are updated to the next vertex, after a selection is made from $OUT(v)$. Initially, $out(v)$ point to the first vertex in each list. The choices of vertices in the OUT sets are only exposed as necessary. This is usually referred to as deferred decisions. Imagine then that the $OUT(v)$ are ordered lists of boxes, each containing a random integer (from some large set).

Next let

$$\bar{U}^* = \{v \notin U : \exists u \in U \text{ s.t. } v \in IN(u)\}.$$  

A general step of the process proceeds as follows: we begin with $P = (s_P = f_P), C = \Lambda$ and $U = [n] \setminus \{s_P\}$ for an arbitrary choice of vertex $s_P$.

For $x \in P \cup C$ we let $\pi(x)$ denote the unique vertex $z$ such that $(z,x)$ is an edge of $P \cup C$.
While \(|U| \geq 2\alpha n\), we simply try to grow \(P\) by attaching an edge \((f_P, u)\) where \(u = \text{out}(f_P) \in U\). If \(|U| < 2\alpha n\) and \(C = \Lambda\) then we try to create a large cycle \(C\) by adding an edge from \(f_P\) to \(y \neq s_P, y \in P\). We then delete the previous edge \((x, y)\) of \(P\) that points to \(y\). In this situation we wait until we add another edge \((f_P, z)\) where \(z \in C\) and then make one long path. The size of \(U\) is decreased when \(C = \Lambda\) and the vertex \(x \in \bar{U}\). In this case we can extend \(P\) by adding an edge \((x, u), u \in U\) where \(x \in IN(u)\). The reader will notice that we avoid adding the edge \((f_P, s_P)\) if \(s_P = \text{out}(f_P)\). This is just a matter of convenience. It saves adding a case. The details are as follows: Recall that in what follows all random choices can be ascribed to a choice of either \(in, out\) or \(in(u), u \in U\). We now discuss the distribution of the \(X_i\).

(a) If \(|U| \geq 2\alpha n\) then \(X_i\) is dominated by the geometric random variable \(Geo(p_1)\) where \(p_1 = \frac{|U| - \alpha n}{n} \geq \alpha\).

This is because \(f_P\) has at least \(|U| - \alpha n\) choices available to it in \(U\) for the next choice of vertex in \(OUT(f_P)\). A step here means opening an OUT box and looking inside.

(b) Now consider the case where \(C = \Lambda\) and \(1 \leq |U| < 2\alpha n\). At this point we need a lower bound on the size of \(|\bar{U}\)|. We will prove the following bound that holds w.h.p. throughout the process:

\[
|\bar{U}| \geq \begin{cases} 
\frac{n}{20} & |U| \geq \frac{n}{\theta \log n}, \\
\theta^{1/2}|U| \log n & |U| < \frac{n}{\theta \log n}.
\end{cases}
\]  

(1)

We will now use the above to estimate how long it takes to finish the process. We will justify it at the end of this section. Let us first ignore the sizes of the sets \(OUT(v), v \in [n]\) and deal with this issue later. Each \(X_i\) can be coupled with and bounded by an independent geometric random variable with probability of success, \(p_i\) say. We will see that w.h.p. we obtain \(V \cup C = [n]\) in less than \(O(n)\) trials. Here a trial means exposure of \(out(v)\) or \(in(u)\). Our high probability bound on the number of trials will follow from the Chebyshev inequality and from the fact that \(\mathbb{E}(Geo(p)) = \frac{1}{p}\) and \(\text{Var}(Geo(p)) = \frac{1-p^2}{p^2}\).

When \(|U| \geq 2\alpha n\) we take \(p_i \geq \alpha\). When \(|U| < 2\alpha n\) we take \(p_i = \Omega(\alpha(1 - \theta(U)/2n))\). We can argue this as follows. With probability at least \(\alpha\) we find that \(C = \Lambda\) after one step. If at least half of \(\bar{U}\) is further than
2\alpha n from \( f_P \) then we are in good shape. In this case there is a probability of at least \((1 - \alpha)/2n\) times the RHS of (1) of reducing |\( U \)| by one. Otherwise, with probability at least \((1 - \alpha)/2\), we create a cycle of size at least \(n/2\) and then with probability at least \(1/3\) we will be in good shape. In this case we reduce |\( U \)| by one in at most four steps. Thus if \( T \) denotes the total number of trials then we have,

\[
\mathbb{E}(T) \leq \frac{n}{\alpha} + 4 \sum_{u=1}^{n/\theta \log n} \frac{6}{\alpha(1 - \alpha)(1/40 - \alpha)} + 4 \sum_{u=n/\theta \log n}^{2\alpha n} \frac{n}{6\alpha(1 - \alpha)u\theta^{1/2}\log n} = \Theta(n).
\]

\[
\text{Var}(T) \leq \frac{n}{\alpha} + 16 \sum_{u=1}^{n/\theta \log n} \frac{36}{\alpha^2(1 - \alpha)^2(1/40 - \alpha)^2} + 16 \sum_{u=n/\theta \log n}^{2\alpha n} \frac{n^2}{36\alpha^2(1 - \alpha)^2u^2\theta \log^2 n} = o(n^2).
\]

The Chebyshev inequality now implies that w.h.p. the number of trials needed is at most \( Cn \) for some constant \( C = C(\alpha) > 0 \).

We now deal with the sizes of the sets \( OUT(v), v \in [n] \). We need to show that we do not come to the end of a list. A given vertex \( v \) has probability at most \( q = \frac{1}{(1 - \alpha)^n} \) of being selected as the next \( y \) and this implies that the probability \( K = \theta \log n \) items on its OUT list are examined is at most \( \mathbb{P}(\text{Bin}(Cn, q) \geq K) = o(n^{-1}) \).

Once \( P \cup C = [n] \), it takes \( O(n) \) expected time to create a Hamilton cycle. Let us go through the possibilities.

(i) If \( C = \Lambda \) and \( s_P \in B(f_P) \) or \( f_P \in A(s_P) \) then the process finishes in one more step with probability at least \( 1/n \).

(ii) If \( C = \Lambda \) and (i) does not hold, then we update \( out(f_P) \). Note that there is a probability of at least \( 1 - 2\alpha \) that we will now be in case (i).

(iii) If \( C \neq \Lambda \) then there is a probability of at least \( \alpha \) that \( out(f_P) \in C \) and we are in (i).

\[\text{Explanation for } (2): \text{ we choose a set of size } k \text{ for } U \text{ and then a set of size } \gamma k \text{ for } U^*. \text{ Then we estimate the probability that each choice in } IN(U) \text{ is in } U \text{ or the set of size } \gamma k \text{ that we have chosen.} \]

When \( k = |U| \geq n/\theta \log n \) we take \( \gamma + 1 = n/10 \leq \theta \log n/10 \) and we note that \( \gamma k \geq n/20 \). We then have

\[
\sum_{k=n/\theta \log n}^{2\alpha n} \left( \frac{(\gamma + 1)k}{(1 - \alpha)n} \cdot e^{2\gamma} \right)^k \leq \sum_{k=n/\theta \log n}^{2\alpha n} \left( \frac{1}{10(1 - \alpha)} \cdot e^{\theta \log n/10} \right)^k = o(1).
\]

4.1 Proof of (1)

We estimate the probability that there is a subset \( U, |U| \leq 2\alpha n \) for which \( |\bar{U}^*| \leq \gamma |U| \) for \( \gamma \) inferred by (1). We bound this probability by

\[
\sum_{k=1}^{2\alpha n} \binom{n}{k} \left( \frac{n - k}{\gamma k} \right) k^{\theta \log n} \leq \sum_{k=1}^{2\alpha n} \left( \frac{ne}{k} \right)^k \left( \frac{ne}{\gamma k} \right)^{\gamma k} \left( \frac{(\gamma + 1)k}{(1 - \alpha)n} \right)^{k^{\theta \log n}}
\]

\[
= \sum_{k=1}^{2\alpha n} \left( \frac{(\gamma + 1)k}{(1 - \alpha)n} \right)^{\theta \log n - \gamma - 1} \cdot \frac{(1 - \alpha)e}{\gamma + 1} \cdot \left( \frac{(\gamma + 1)e}{(\gamma(1 - \alpha))} \right)^k
\]

\[
\leq \sum_{k=1}^{2\alpha n} \left( \frac{(\gamma + 1)k}{(1 - \alpha)n} \right)^{\theta \log n - \gamma - 1} \cdot e^{2\gamma}
\]
For $1 \leq k \leq n/\theta \log n$ we take $\gamma = \theta^{1/2} \log n$. And then,

$$\
sum_{k=1}^{n/\theta \log n} \left( \left( \frac{(\gamma + 1)k}{(1 - \alpha)n} \right)^{\theta \log n - \gamma - 1} \cdot e^{2\gamma} \right)^k \leq \sum_{k=1}^{n/\theta \log n} \left( \frac{2}{\theta^{1/2}} \right)^{9\theta \log n/10} \cdot e^{2\theta^{1/2} \log n} \right)^k = o(1).
$$

5 Conclusion

We solved the strong connectivity game, but there is a big gap between the upper and lower bounds for Hamiltonicity. Closing this gap is an interesting open problem.

References


A Proof of Theorem 3

We let $G_M, G_B$ denote the subgraphs of $G$ with the edges taken by Maker, Breaker respectively. We let $d_M(v)$ denote the degree of vertex $v$ in $G_M$ for $v \in V$. We define $d_B$ similarly. Let $\text{dang}(v) = d_B(v) - 2bd_M(v)$ be the danger of vertex $v$ at any time.

Maker’s Strategy: In round $i$, choose a vertex $v_i$ of maximum danger and choose a random edge incident with $v_i$, not already taken. This is called easing $v$.

Let $M_i, B_i$ denote Maker and Breaker’s $i$th moves. Suppose that Breaker wins in round $g - 1$, so that after $B_{g-1}$ there is a vertex $v_g$ such that $d_B(v_g) > \alpha n$. Let $J_i = \{v_{i+1}, \ldots, v_g\}$. Next define

$$\overline{\text{dang}}(M_i) = \frac{\sum_{v \in J_{i-1}} \text{dang}(v)}{|J_{i-1}|}$$

and $$\overline{\text{dang}}(B_i) = \frac{\sum_{v \in J_i} \text{dang}(v)}{|J_i|},$$

computed before the $i$th moves of Maker, Breaker respectively.

Then $\overline{\text{dang}}(M_1) = 0$ and $\overline{\text{dang}}(M_g) = \text{dang}(v_g) = (\alpha - o(1))n$. Let $a(i)$ be the number of edges contained in $J_i$ that are claimed by Breaker in his first $i$ moves.
Lemma 5.
\[ \overline{\text{dang}}(M_i) \geq \overline{\text{dang}}(B_i). \]  \hspace{1cm} (4)
\[ \overline{\text{dang}}(M_i) \geq \overline{\text{dang}}(B_i) + \frac{2b}{|J_i|}, \text{ if } J_i = J_{i-1}. \]  \hspace{1cm} (5)
\[ \overline{\text{dang}}(B_i) \geq \overline{\text{dang}}(M_{i+1}) - \frac{2b}{|J_i|}. \]  \hspace{1cm} (6)
\[ \overline{\text{dang}}(B_i) \geq \overline{\text{dang}}(M_{i+1}) - \frac{b + a(i) - a(i - 1)}{|J_i|} - 1. \]  \hspace{1cm} (7)

Proof. Equation (4) follows from the fact that a move by Maker does not increase danger. Equation (5) follows from the fact that if \( v_i \in J_{i-1} \) then its danger, which is a maximum, drops by 2b. Equation (6) follows from the fact that Breaker takes at most b edges inside \( J_i \). For equation (7) let \( e_{\text{double}} \) be the number of edges that Breaker adds to \( J_i \) in round \( B_i \). Then
\[ \overline{\text{dang}}(B_i) \geq \overline{\text{dang}}(M_{i+1}) - \frac{b + e_{\text{double}}}{|J_i|} \]
and
\[ a(i) - e_{\text{double}} \geq a(i - 1) - |J_i|. \]

It follows that
\[ \overline{\text{dang}}(M_i) \geq \overline{\text{dang}}(M_{i+1}) \text{ if } J_i = J_{i-1}. \]  \hspace{1cm} (8)
\[ \overline{\text{dang}}(M_i) \geq \overline{\text{dang}}(M_{i+1}) - \min \left\{ \frac{2b}{|J_i|}, \frac{b + a(i) - a(i - 1)}{|J_i|} - 1 \right\}. \]  \hspace{1cm} (9)

Next let \( 1 \leq i_1 \leq \cdots \leq i_r \leq g - 1 \) be the indices where \( J_i \neq J_{i-1} \). Then we have \( |J_{i_r}| = |J_{g-1}| = 1 \) and \( |J_{i_1 - 1}| = |J_0| = r + 1 \). Let \( k = \frac{n}{\log n} \) and assume first that \( r \geq k \) and then use the first minimand in (9) for \( i_1, \ldots, i_{r-k} \) and the second minimand otherwise.
\[ 0 = \overline{\text{dang}}(M_1) \]
\[ \geq \overline{\text{dang}}(M_g) - \frac{b + a(i_r) - a(i_r - 1)}{|J_r|} - \cdots - \frac{b + a(i_{r-k+1}) - a(i_{r-k+1} - 1)}{|J_{r-k+1}|} \\
- k - \frac{2b}{|J_{r-k}|} - \cdots - \frac{2b}{|J_1|} \]  \hspace{1cm} (10)
\[ \geq \overline{\text{dang}}(M_g) - \frac{b}{1} - \cdots - \frac{b}{k} - \frac{a(i_r)}{1} - k - \frac{2b}{k+1} - \cdots - \frac{2b}{r} \]  \hspace{1cm} (11)
\[ \geq \alpha n - Kb - b(1 + \log k) - k - 2b(\log n - \log k). \]

To go from (10) to (11) we use \( a(i_{r-j}) - 1 \geq a(i_{r-j-1}), j > 0 \) which follows from \( J_{i_{r-j-1}} = J_{i_{r-j}} - 1 \) and then the coefficient of \( a(i_{r-j-1}) \) is at least \( \frac{1}{j+1} - \frac{1}{j+2} \geq 0 \). Also, \( a(i_r) = 0 \) because \( J_{i_r} = J_{g-1} = \{v_g\} \).

It follows that
\[ b \geq \frac{\alpha n - k}{K + 1 + \log n + \log \log n + o(1)} = \frac{(\alpha - 1/\log n)n}{(1 + \theta + o(1))\log n}, \]
contradicting our upper bound, \( \theta < \frac{\alpha - \beta}{\beta} \).
If $r < k$ then we replace (11) by

\[ 0 = \text{dang}(M_1) \geq \text{dang}(M_g) - \frac{b}{1} - \cdots - \frac{b}{k} - \frac{a(i_r)}{1} - k \geq \alpha n - K b - b(1 + \log k) - k \]

and obtain the same contradiction.