

Pattern Colored Hamilton Cycles in Random Graphs

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Abstract

We consider the existence of patterned Hamilton cycles in randomly colored random graphs. Given a string Π over a set of colors $\{1, 2, \dots, r\}$, we say that a Hamilton cycle is Π -colored if the pattern repeats at intervals of length $|\Pi|$ as we go around the cycle. We prove a hitting time result for the existence of such a cycle. We also prove a hitting time result for the related notion of Π -connected.

1 Introduction

In recent years there has been a growing interest in the properties of randomly colored random graphs. The edge-colored random graph process can be described as follows: let G_0, G_1, \dots, G_N , $N = \binom{n}{2}$, be the random graph process. That is we start with the empty graph G_0 on vertex set $[n]$. At Step $i \geq 1$ we choose an edge e_i uniformly at random from the set of $N - i + 1$ edges not in $E(G_{i-1})$ and then let $E(G_i) = E(G_{i-1}) \cup \{e_i\}$. For $i \in [N]$, at step i , a random color c_i is chosen independently and uniformly at random from $[r]$ and is assigned to e_i . We denote this randomly $[r]$ -colored version of the random graph process by $G_0^r, G_1^r, \dots, G_N^r$.

Much of the interest in this model has been focused on *rainbow* colorings. A set S of edges is said to be rainbow colored if every edge of S has a different color. One of the earliest papers on this subject is due to Frieze and McKay [11]. In this paper, $r = cn$ where $c \geq 1$ is a constant. Let $\tau_0 = \min \{i : G_i \text{ is connected}\}$ be the hitting time for connectivity and let $\tau_1 = \min \{i : \text{at least } n - 1 \text{ distinct colors have been used}\}$. Then [11] shows that the hitting time τ for the existence of a rainbow spanning tree satisfies $\tau = \max \{\tau_0, \tau_1\}$ w.h.p. After this, the attention has been focussed on the existence of a rainbow Hamilton cycle. Cooper and Frieze [4] showed that $O(n \log n)$ random edges and $O(n)$ colors are sufficient. This was improved to $(\frac{1}{2} + o(1))n \log n$ random edges and $(1 + o(1))n$ colors by Frieze and Loh [10]. This was sharpened still further by Ferber and Krivelevich [8] who showed that the number of edges can be reduced to the exact threshold for Hamiltonicity. Bal and Frieze [2]

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considered the case where exactly n colors are available and showed that $O(n \log n)$ random edges are sufficient to obtain a rainbow Hamilton cycle.

The next phase of this study concerns Hamilton cycles in random k -uniform hypergraphs. There are various notions of Hamilton cycle in this context, and Ferber and Krivelevich [8] proved that if m_H edges are needed for a given type of Hamilton cycle to exist w.h.p. then $O(m_H)$ random edges and $(1 + \epsilon)m_H$ colors are sufficient for a rainbow Hamilton cycle. The hypergraph results in [8] were sharpened by Dudek, English and Frieze [6].

Cooper and Frieze [5] considered the related question of finding the threshold for every k -bounded coloring of the edges of $G_{n,m}$ to contain at least one rainbow Hamilton cycle. Here k -bounded means that no color can be used more than k times.

Rainbow is one pattern of coloring and Espig, Frieze and Krivelevich [7] considered other types of colorings. Suppose that r is constant. They considered the existence of Hamilton cycles where the edges of the cycle are colored in sequence $1, 2, \dots, r, 1, 2, \dots, r, \dots$. When $r = 2$ they called such colorings *Zebraic*. They gave tight results in terms of the number of random edges needed for such cycles. Our first result generalizes this and considers arbitrary patterns of coloring.

An $[r]$ -*pattern* Π is a finite sequence with elements in $[r]$. For a given $[r]$ -pattern Π let $\ell = |\Pi|$ be its length and for $1 \leq j \leq \ell$ let Π_j be its j 'th element. We say a path/cycle f_1, f_2, \dots, f_k is Π -colored if there exists an integer $0 \leq l \leq \ell - 1$ such that $\forall j \in [k]$, f_j has color Π_{j+l} , where $j + l$ is taken modulo ℓ . So for example if $[r] = 3$, $\Pi = 1, 2, 2, 3$ and $P = f_1, f_2, \dots, f_6$ is a Π -colored path then f_1, f_2, \dots, f_6 may have colors $1, 2, 2, 3, 1, 2$ or $2, 2, 3, 1, 2, 2$ or $2, 3, 1, 2, 2, 3$ or $3, 1, 2, 2, 3, 1$ respectively. Here and elsewhere for a given pattern Π and $s > |\Pi|$ we let $\Pi_s = \Pi_j$ where $j = s \bmod \ell$. In this paper we are interested in the following question. Given $r = O(1)$ and an r -pattern Π , when does the first Π -colored Hamilton cycle appear in the process $G_0^r, G_1^r, \dots, G_N^r$?

We will assume without loss of generality that $[r] = \{\Pi_1, \Pi_2, \dots, \Pi_\ell\}$ i.e. every color in $[r]$ appears at least once in Π . We will also assume that ℓ divides n and that $r \geq 2$, since the case $r = 1$ is essentially the uncolored case.

Let $r \in \mathbb{N}$ and let Π be an $[r]$ -pattern. Then let

$$\tau_\Pi := \min\{i : G_i^r \text{ contains a } \Pi\text{-colored Hamilton cycle}\}$$

For $v \in V$ we say that v *fits* the pattern Π in G_i^r if there exist two distinct edges $e_1, e_2 \in E(G)$ incident to v and $1 \leq j \leq \ell$ such that e_1 has color Π_j and e_2 has color Π_{j+1} . So in our example where $[r] = 3$, $\Pi = 1, 2, 2, 3$ v fits Π if it is incident to some e_1, e_2 with colors c_1, c_2 that satisfy $\{c_1, c_2\} \in \{\{1, 2\}, \{2, 2\}, \{2, 3\}, \{3, 1\}\}$. Furthermore we define the hitting time

$$\tau_{fit-\Pi} := \min\{i : \text{every } v \in V \text{ fits } \Pi \text{ in } G_i^r\}.$$

Observe that before $\tau_{fit-\Pi}$ occurs there is at least one vertex v that “does not fit the pattern”. That is there do not exist two colors that appear in adjacent places in the pattern and in the neighborhood of v . Thus clearly for any pattern Π we have $\tau_{fit-\Pi} \leq \tau_\Pi$.

Theorem 1. *Let $2 \leq r = O(1)$ and Π be an $[r]$ -pattern. Then, w.h.p. $\tau_{fit-\Pi} = \tau_\Pi$.*

Our second result is related to the notion of *rainbow connection*. Given a connected graph G , the rainbow connection number $rc(G)$ is defined as the smallest r such that there

exists an r -coloring of the edges of G so that there is a rainbow path between every pair of vertices of G . This is a well studied concept, see Li, Shi and Sun [18] for a survey. The rainbow connection number of random graphs has also been studied. Heckel and Riordan [16] and He and Liang [15] studied the rainbow connection number of dense random graphs. Frieze and Tsourakakis [13] studied the rainbow connection number of a random graph at the connectivity threshold. Dudek, Frieze and Tsourakakis [14] and Kamceř, Krivelevich and Sudakov [17] and Molloy [19] studied the rainbow connection number of random regular graphs. Suffice it to say that in general the rainbow connection number is close to the diameter in all cases. Espig, Frieze and Krivelevich [7] introduced the notion of *zebraic connection*. Given a 2-coloring of the edges of a connected graph G we say that a path is *zebraic* if the colors of edges alternate along the path. A colored graph is *zebraically connected* if there is a zebraic path joining every pair of vertices. In the paper [7], they proved a hitting time result for the zebraic connectivity of a random 2-coloring of the edges of a random graph. In this paper we generalize this notion to Π -connectivity. We say that G is Π -connected if every pair of vertices are joined by a Π -colored path. We give a hitting time result for when G_i^r becomes Π -connected. In order to state the result about Π -connectivity we define the following hitting times:

$$\tau_1 := \min\{i : G_i \text{ has minimum degree at least } 1\},$$

$$\tau_{\Pi\text{-connected}} := \min\{i : G_i^r \text{ is } \Pi\text{-connected}\}.$$

Theorem 2. *Let $r = O(1)$ and Π be an $[r]$ -pattern. Then w.h.p.*

$$\tau_{\Pi\text{-connected}} = \tau_1.$$

The following corollary is then immediate from our knowledge of τ_1 .

Corollary 3. *Let $m = \frac{1}{2}n(\log n + c_n)$. Then,*

$$\lim_{\substack{n \rightarrow \infty \\ \ell | n}} \Pr(G_i^r \text{ is } \Pi\text{-connected}) = \begin{cases} 0 & c_n \rightarrow -\infty. \\ e^{-e^{-c}} & c_n \rightarrow c. \\ 1 & c_n \rightarrow +\infty. \end{cases}$$

1.1 Directed versions

There are some natural directed versions of the results that we just have just stated. For that we consider the directed random graph process $D_0, D_1, \dots, D_{N'}$, $N' = n(n-1)$. That is we start with the empty digraph D_0 on vertex set $[n]$. At Step $i \geq 1$ we choose an edge e_i uniformly at random from the set of $N' - i + 1$ edges not in $E(D_i)$ and then let $E(D_i) = E(D_{i-1}) \cup \{e_i\}$. For $i \in [N']$, at step i , a random color c_i is chosen independently and uniformly at random from $[r]$ and is assigned to e_i . We denote this randomly colored version of the directed random graph process by $D_0^r, D_1^r, \dots, D_{N'}^r$.

The notion of Π paths/cycle/connectivity can be extended in a straightforward manner to the directed setting by substituting directed path/ directed cycle in the place of cycle/paths.

For $v \in V$ we say that v fits the pattern Π if there exist arcs $e_1, e_2 \in E(D)$ and $1 \leq j < \ell$ such that e_1 has color Π_j , e_2 has color Π_{j+1} and such that the head of e_1 and the tail of e_2 are both v . We replace connectivity in the undirected setting by strong connectivity in the directed case. Finally let $\vec{\tau}_\Pi$, $\vec{\tau}_{fit-\Pi}$, $\vec{\tau}_{\Pi\text{-connected}}$ and $\vec{\tau}_1$ be the directed analogs of τ_Π , $\tau_{fit-\Pi}$, $\tau_{\Pi\text{-connected}}$ and τ_1 .

Theorem 4. *Let $r = O(1)$ and Π be an $[r]$ -pattern. Then, w.h.p.*

$$\vec{\tau}_{fit-\Pi} = \vec{\tau}_\Pi.$$

Theorem 5. *Let $r = O(1)$ and Π be an $[r]$ -pattern. Then w.h.p.*

$$\vec{\tau}_{\Pi\text{-connected}} = \vec{\tau}_1.$$

Corollary 6. *Let $m = n(\log n + c)$. Then,*

$$\lim_{\substack{n \rightarrow \infty \\ \ell \mid n}} \Pr(G_i^r \text{ is } \Pi\text{-connected}) = \begin{cases} 0 & c_n \rightarrow -\infty. \\ e^{-2e^{-c}} & c_n \rightarrow c. \\ 1 & c_n \rightarrow +\infty. \end{cases}$$

1.2 Notation-Preliminaries

For $i \in [N]$ and $c \in [r]$ let $G_i^r(c)$ denote the subgraph of G_i^r induced by the edges of color c . Furthermore denote the degree of v in $G_i^r(c)$ by $\deg_i(v, c)$. By extension, for $C \subset [r]$ denote by $G_i^r(C)$ the subgraph of G_i^r induced by the edges with color in C and set $\deg_i(v, C)$ to be the degree of v in $G_i^r(C)$. For $e \in E(G_N)$ let $c(e)$ be the color that is assigned to e by the end of the process. Throughout the paper we will use the following estimate.

Lemma 7. *Let $a, b, c, d, t \in \mathbb{Z}_{\geq 0}$ and $i, Q \in \mathbb{Z}_{> 0}$ be such that $b, c, i, t = o(Q)$, $d, i = o(t)$ and $i = o(b)$ then,*

$$\frac{\binom{a}{i} \binom{Q-c-b}{t-d-i}}{\binom{Q-c}{t-d}} \leq \left(\frac{3at}{iQ} \right)^i \exp \left\{ -\frac{(1+o(1))bt}{Q} \right\}.$$

Proof.

$$\begin{aligned} \frac{\binom{a}{i} \binom{Q-c-b}{t-d-i}}{\binom{Q-c}{t-d}} &= \frac{\binom{a}{i} \binom{Q-c-b}{t-d-i} \binom{t-d}{i}}{\binom{Q-c}{i} \binom{Q-c-i}{t-d-i}} = \binom{t-d}{i} \prod_{j=0}^{i-1} \frac{a-j}{Q-c-j} \prod_{h=0}^{t-d-i-1} \frac{Q-c-b-h}{Q-c-i-h} \\ &\leq \left(\frac{et}{i} \right)^i \cdot \prod_{j=0}^{i-1} \frac{a}{Q-c-j} \prod_{h=0}^{t-d-i-1} \left(1 - \frac{b-i}{Q-c-i-h} \right) \\ &\leq \left(\frac{et}{i} \right)^i \left(\frac{(1+o(1))a}{Q} \right)^i \left(1 - \frac{(1+o(1))b}{Q} \right)^{(1+o(1))t} \\ &\leq \left(\frac{(1+o(1))eat}{iQ} \right)^i \exp \left\{ -\frac{(1+o(1))bt}{Q} \right\}. \end{aligned}$$

□

We will also use the following elementary result.

Lemma 8. *Let $\nu = \nu(n)$ be a positive integer and let $0 < p < 1$ be such that $\nu p \rightarrow \infty$ and let X be a Binomial(ν, p) random variable. Then w.h.p. $X = (1 + o(1))\nu p$.*

Remark 9. *Let $C \subset [r]$ and $t \geq n$. Then $G_t^r(C)$ is distributed as $G(n, m)$ i.e. a random graph with m edges chosen at random from all N edges. Here m is the number of edges in $G_t^r(C)$ colored by a color in C . Since each edge is colored independently and at random, m is distributed as a Binomial($t, |C|/r$) random variable. Therefore w.h.p. $m = (1 + o(1))|C|t/r$.*

The paper contains various constants that are used throughout. We collect them here for ease of reference:

- r equals the number of colors available.
- ℓ equals the length of pattern Π .
- $\epsilon = 10^{-9}\ell^{-1}$.
- $t_i = i\mu$ where $\mu = \epsilon n \log n$ for $1 \leq i \leq 2\ell$.
- $X_i = \{e_j : j \in [(i-1)\mu + 1, i\mu] \text{ and } c(e_j) = \Pi_i\}$ for $i \in [\ell]$.
- $Y_i = \{e_j : j \in [(\ell+i-1)\mu + 1, (\ell+i)\mu] \text{ and } c(e_j) = \Pi_i\}$ for $i \in [\ell]$.
- $V_i = \left\{ \frac{(i-1)n}{\ell} + 1, \frac{(i-1)n}{\ell} + 2, \dots, \frac{in}{\ell} \right\}$ for $1 \leq i \leq \ell$, a partition of the vertex set into ℓ equal size subsets.
- $n_\ell = \frac{n}{\ell}$ equals the size of the V_i 's. We use this notation to stop formulae looking too ugly.
- $\beta = 10^{-3}\ell^{-1}r^{-1}\epsilon$.
- BAD is a set of low degree vertices with size bounded by $2\ell n_b$ where $n_b = n^{1-10\beta}$.
- $n_r = 2\ell|BAD|$.

2 Demand of a pattern

For a given $[r]$ -pattern Π the hitting time $\tau_{fit-\Pi}$ clearly depends on both the number of colors r and the pattern itself. In order to determine the the influence of the latter we define the *demand* of a pattern.

Definition 10. *Let*

$$\mathcal{D}(\ell) := \{S \subset [\ell] : \{i, i+1\} \cap S \neq \emptyset \text{ for all } i \in [\ell]\}. \quad (\ell + 1 = 1 \text{ here})$$

Definition 11. Let $r \in \mathbb{N}$ and let Π be an $[r]$ -pattern. The “demand” of Π is

$$d(\Pi) := \min \{ |\Pi_S| : S \in \mathcal{D}(\ell) \}$$

where

$$\Pi_S = \{ \Pi_i : i \in S \}.$$

The motive for giving the above definition is the following. For a given $[r]$ -pattern Π , if there exists a set $S \in \mathcal{D}(\ell)$ and a vertex $v \in V$ such that v is not incident with any edge colored with one of the at least $d(\Pi)$ colors in Π_S then v does not fit Π . Conversely if v does not fit Π then for $i \in [\ell]$, v is incident to at most 1 edge of color Π_i or to at most 1 edge of color Π_{i+1} . Here we say at most one instead of none since we have to consider the case $\Pi_i = \Pi_{i+1}$. Therefore $S = [\ell] \setminus \{i : v \text{ is incident with two edges of color } i\}$ satisfies $S \in \mathcal{D}(\ell)$ and $|\Pi_S| \geq d(\Pi)$ and certifies that v does not fit Π .

In the following lemma we use the following well known result (see [12]). Let $\epsilon > 0$ then, w.h.p. $G(n, \frac{(1-\epsilon)n \log n}{2})$ contains an isolated vertex. On the other hand $G(n, \frac{(1+\epsilon)n \log n}{2})$ does not contain a vertex of degree at most $r+1$.

Lemma 12. Let $r = O(1)$, Π be an $[r]$ -pattern and $\epsilon > 0$. Then, w.h.p.

$$\frac{r}{d(\Pi)} \cdot \frac{(1-\epsilon)n \log n}{2} \leq \tau_{fit-\Pi} \leq \frac{r}{d(\Pi)} \cdot \frac{(1+\epsilon)n \log n}{2}. \quad (1)$$

Proof. Let $t = \frac{r}{d(\Pi)} \cdot \frac{(1-\epsilon)n \log n}{2}$. Let $S \in \mathcal{D}(\ell)$ be such that $|\Pi_S| = d(\Pi)$. Then $G_t^r(\Pi_S)$ is distributed as $G(n, m)$ where w.h.p. $m \leq \frac{(1-\epsilon/2)n \log n}{2}$ (see Remark 9). Hence w.h.p. $G_t^r(\Pi_S)$ has an isolated vertex. This vertex is not incident to any edge with a color in Π_S and it does not fit Π . Consequently, w.h.p. $\frac{r}{d(\Pi)} \cdot \frac{(1-\epsilon)n \log n}{2} \leq \tau_{fit-\Pi}$.

Now let $t = \frac{r}{d(\Pi)} \cdot \frac{(1+\epsilon)n \log n}{2}$. In the event that $\tau_{fit-\Pi} \geq t$ we have that there is a vertex $v \in V$ and a set $S \in \mathcal{D}(\ell)$ such that S certifies that v does not fit Π i.e. for every $i \in \Pi_S$, v is incident to at most one edge with color i . Hence v has degree at most r in $G_t^r(\Pi_S)$.

Fix $S \in \mathcal{D}(\ell)$. Then $|\Pi_S| \geq d(\Pi)$. Furthermore $G_t^r(\Pi_S)$, is distributed as $G(n, m')$ where w.h.p. $m' \geq \frac{(1+\epsilon/2)n \log n}{2}$ (see Remark 9). Hence w.h.p. $G_t^r(\Pi_S)$ has no vertex of degree at most r . Then by taking a union bound over $S \in \mathcal{D}(\ell)$ we get that w.h.p. $\tau_{fit-\Pi} \leq \frac{r}{d(\Pi)} \cdot \frac{(1+\epsilon)n \log n}{2}$. \square

Corollary 13. Let $r = O(1)$, Π be an $[r]$ -pattern. Then w.h.p. $0.4n \log n \leq \tau_{fit-\Pi} \leq rn \log n$.

Proof. Follows from Lemma 12 and the fact that $1 \leq d(\Pi) \leq r$. \square

The above corollary can be tightened. In actual fact, Theorem 1 implies the following:

Corollary 14. Let $d = d(\Pi)$ and let

$$m = \frac{rn}{2d}(\log n + c_n).$$

Then,

$$\lim_{\substack{n \rightarrow \infty \\ \ell | n}} \Pr(G_m^r \text{ contains a } \Pi\text{-colored Hamilton cycle}) = \begin{cases} 0 & c_n \rightarrow -\infty. \\ e^{-\lambda} & c_n \rightarrow c. \\ 1 & c_n \rightarrow +\infty. \end{cases}$$

Here λ depends on the pattern Π and it is equal to the expected number of vertices that do not fit Π .

The justification for Corollary 14 comes from Theorem 1 and the fact that in the case $c_n \rightarrow c$, the number of vertices that do not fit Π is asymptotically Poisson with mean λ . The proof of the corollary follows a standard “method of moments” proof and is omitted.

3 Outline proof of Theorem 1

We have defined a partition of V into sets V_1, \dots, V_ℓ of equal size (see Section 1.2). We begin by identifying sets of *bad* vertices that have low degree and then show that (i) there are few of them and (ii) they are spread out in $G_{\tau_{fit-\Pi}}$. This is the content of Lemma 17. Next, we cover the vertices in *BAD* by a set of Π -colored paths \mathcal{P}_{bad} . Moreover every path in \mathcal{P}_{bad} is a $\Pi_{1,\ell}$ -colored path i.e. its length equals $-1 \pmod{\ell}$, its first edge is colored Π_1 its second edge is colored Π_2 e.t.c. (its last edge is colored $\Pi_{\ell-1}$). To generate \mathcal{P}_{bad} we use Algorithm *CoverBAD* and prove that it is successful in Lemma 20.

If \mathcal{A} is the set of vertices not covered by \mathcal{P}_{bad} then the sets $V_i \cap \mathcal{A}$ may be unbalanced. We move a small set of vertices around so that \mathcal{A} is now partitioned into equal sized sets V_i'' . Then for each $i \in [\ell - 1]$ we find a perfect matching of color Π_i from V_i'' to V_{i+1}'' . These matchings together form a collection of Π -colored paths \mathcal{P}_{good} that cover the vertices in \mathcal{A} , each with an endpoint in V_1, V_ℓ . The edges used in the construction of these matchings are all in $\bigcup_{i=1}^{\ell-1} X_i \cup \bigcup_{i=1}^{\ell-1} Y_i$ (the sets X_i, Y_i are defined in Section 1.2). Hence every path in \mathcal{P}_{good} is also a $\Pi_{1,\ell}$ -colored path. Let $\mathcal{P} = \mathcal{P}_{bad} \cup \mathcal{P}_{good} = \{P_1, P_2, \dots, P_{n_\ell}\}$. Let the endpoints of P_i be $v_i^- \in V_1$ and $v_i^+ \in V_\ell$ for $1 \leq i \leq n_\ell$.

After this, we find a perfect matching $M = \{(v_j^+, v_{\pi(j)}^-) : j \in [n_\ell]\}$ of color Π_ℓ from $\{v_j^- : P_j \in \mathcal{P}\}$ to $\{v_j^+ : P_j \in \mathcal{P}\}$, using a subset of the edges $X_\ell \cup Y_\ell$. Here π is a permutation of $[n_\ell]$ and so the digraph $\Gamma = ([n_\ell], \{(i, \pi(i))\})$ is a collection of vertex disjoint cycles. We argue by symmetry that π is a random permutation so that w.h.p. it has at most $2 \log n$ cycles. A cycle $j, \pi(j), \pi^2(j), \dots, j$ can be expanded into a Π -colored cycle $v_j^-, P_j, v_j^+, v_{\pi(j)}^-, P_{\pi(j)}, v_{\pi(j)}^+, v_{\pi^2(j)}^-, \dots, v_j^-$. And in this way we cover the vertex set $[n]$ by $O(\log n)$ Π -colored cycles.

After this we focus on converting this set of cycles into a single Π -colored Hamilton cycle using Π_ℓ edges of $E(G_{\tau_{fit-\Pi}}) \setminus \bigcup_{i=1}^{\ell} (X_i \cup Y_i)$. This turns out to be essentially equal to the task successfully faced in the construction of a directed Hamilton cycle in [9], and which is laid out more explicitly in [1]. The reduction to [9], [1] is laid out in Lemma 18.

4 Structural results

For every $i \in [N]$ an ordering of the endpoints of e_i is chosen independently and uniformly at random. Hence we may consider that e_i is given to us in the form of an ordered pair $\vec{e}_i = (v_i, w_i)$. Note that in the proof of Theorem 1, we know that we will not be presented with both of (v, w) and (w, v) .

Definition 15. For $j \in [\ell]$ define the sets

$$\begin{aligned} BAD_j &= \{v : |\{i : v_i = v, w_i \in V_j, \{v_i, w_i\} \in X_i, t_{j-1} < i \leq t_j, c_i = \Pi_j\}| \leq \beta \log n\}. \\ BAD_{j+\ell} &= \{v : |\{i : w_i = v, v_i \in V_j, \{v_i, w_i\} \in Y_i, t_{j+\ell-1} < i \leq t_{j+\ell}, c_i = \Pi_j\}| \leq \beta \log n\}. \end{aligned}$$

Set $BAD = \bigcup_{j \in [2\ell]} BAD_j$ and call every vertex in BAD bad. Also set $GOOD = V \setminus BAD$ and call every vertex in $GOOD$ good.

Definition 16. We also define the vertex set

$$\begin{aligned} TBAD &= \left\{ v \in V : \exists C \subset [r] \text{ such that } |C| = d(\Pi) \text{ and } \deg_{\tau_{fit-\Pi}}(v, c) \leq \log \log n \text{ for all } c \in C \right\} \\ &\subseteq BAD. \end{aligned}$$

We say that a vertex in $TBAD$ is *tbad* (terribly bad).

Lemma 17. The following hold w.h.p.

- (a) $|BAD| \leq 2\ell n^{1-10\beta}$.
- (b) Every vertex has at most $6\epsilon^{-1}r\ell^2$ bad vertices within distance 2ℓ of it in $G_{\tau_{fit-\Pi}}$.
- (c) $\nexists v, w \in V$ s.t. $v \in TBAD$, $w \in BAD$ and their distance in $G_{\tau_{fit-\Pi}}$ is less than 2ℓ .
- (d) The maximum degree in $G_{\tau_{fit-\Pi}}$ is less than $10r \log n$.

Proof. (a) Let $n_b = n^{1-10\beta}$. Due to symmetry, for $j \in [2\ell]$ the sizes of BAD_j follow the same distribution. Therefore it suffices to show that with probability $1 - o(1)$ we have $|BAD_1| \leq n_b$. In the case that $|BAD_1| > n_b$ there is a set $A \subset V$ of size n_b such that

$$|\{\vec{e}_i = (v_i, w_i) : v_i \in A, w_i \in V_1 \setminus A, c_i = \Pi_1 \text{ and } i \leq t_1\}| \leq \beta n_b \log n. \quad (2)$$

$G_{t_1}^r(\Pi_1)$ is distributed as a $G(n, t'_1)$ where w.h.p. $t'_1 = (1 + o(1))r^{-1}t_1$ (see Remark 9). We can choose A in $\binom{n}{n_b}$ ways. Then there are at least $n_b(n_\ell - n_b)$ and at most $n_b n_\ell$ edges with one endpoint in each of $A, V_1 \setminus A$. From these, if (2) occurs, then $k \leq \beta n_b \log n$ many appear in $G_{t_1}^r(\Pi_1)$. Therefore,

$$\begin{aligned} \Pr((2) \mid t'_1) &\leq \binom{n}{n_b} \sum_{k=0}^{\beta n_b \log n} \frac{\binom{n_b n_\ell}{k} \binom{N - n_b(n_\ell - n_b)}{t'_1 - k}}{\binom{N}{t'_1}} \\ &\leq \left(\frac{en}{n_b}\right)^{n_b} \sum_{k=0}^{\beta n_b \log n} \left(\frac{3t'_1 n_b n_\ell}{kN}\right)^k \exp\left\{-\frac{(1 + o(1))n_b n_\ell t'_1}{N}\right\} \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{en}{n_b}\right)^{n_b} \sum_{k=0}^{\beta n_b \log n} \left(\frac{6e^{o(1)}en_b \log n}{k\ell r}\right)^k \exp\left\{-\frac{2e^{o(1)}en_b \log n}{r\ell}\right\} \\
&\leq e^{(1+10\beta \log n)n_b} \cdot \beta n_b \log n \cdot (6e^{o(1)}10^3 \cdot e^{-2e^{o(1)}\cdot 10^3})^{\beta n_b \log n} \\
&= o(1).
\end{aligned}$$

Here and elsewhere we follow the convention that $(x/0)^0 = 1$ for all x . For the second inequality we use Lemma 7 (with $a = n_b n_\ell, b = n_b(n_\ell - n_b), c = d = 0, i = k, t = t'_1, Q = N$). For the third one we use that $t'_1 = (1 + o(1))\epsilon r^{-1}n \log n, n_\ell = n/\ell$ and $N = \binom{n}{2}$. For the last one we use that $\left(\frac{6e^{o(1)}en_b \log n}{k\ell r}\right)^k$ has a unique maximum obtained when $\frac{6e^{o(1)}en_b \log n}{k\ell r} = e$. Thus for $k \in [0, \beta n_b \log n]$, this is maximized when $k = \beta n_b \log n$.

(b) We will show that with probability at least $1 - o(1)$ every vertex has at most $s = 3\epsilon^{-1}r\ell$ vertices within distance 2ℓ of it that belong to BAD_1 in $G_{rn \log n}$, hence in $G_{\tau_{fit-\Pi}}$. The result follows by symmetry and the union bound. In the case that there exists a vertex v with at least s vertices in BAD_1 within distance 2ℓ of v we can find sets A, B such that the following is satisfied: (i) $|A| = s$, (ii) $|B| \leq 2s(\ell - 1) + 1$ (iii) $A \cup B$ spans a tree in $G_{rn \log n}$, (iv) there are at most $\beta s \log n$ edges from A to $V_1 \setminus (A \cup B)$ in $G_{t'_1}^r(\Pi_1)$ (here A consists of vertices in BAD_1 and B consists of a vertex v and at most $(2\ell - 1)s$ vertices that are spanned by paths of length at most 2ℓ from v to vertices in A).

Fix sets A, B satisfying (i), (ii) and a tree T that is spanned by $A \cup B$. Let $|B| = b$.

$$\begin{aligned}
\Pr(E(T) \subset E(G_{rn \log n}^r)) &= \frac{\binom{N}{rn \log n - (s+b-1)}}{\binom{N}{rn \log n}} = \frac{\binom{rn \log n}{s+b-1}}{\binom{N - rn \log n + (s+b-1)}{s+b-1}} \\
&\leq \left(\frac{rn \log n}{N - rn \log n + (s+b-1)}\right)^{s+b-1} \\
&\leq \left(\frac{3r \log n}{n}\right)^{s+b-1}. \tag{3}
\end{aligned}$$

$G_{t'_1}^r(\Pi_1)$ is distributed as a $G(n, t'_1)$ with $t'_1 = (1 + o(1))t_1/r$ (see Remark 9). Conditioned on t'_1 and $E(T) \subset E(G_{rn \log n}^r)$ and on $z \leq s + b - 1$ edges of T appearing in $G_{t'_1}^r(\Pi_1)$ we have that the probability of condition (iv) being satisfied is bounded above by

$$\begin{aligned}
\sum_{k=0}^{\beta s \log n} \frac{\binom{sn_\ell}{k} \binom{N - s(n_\ell - s - b) - z}{t'_1 - k - z}}{\binom{N - (s+b-1)}{t'_1 - z}} &\leq \sum_{k=0}^{\beta s \log n} \left(\frac{6e^{o(1)}snt'_1}{\ell kn^2}\right)^k \exp\left\{-\frac{(1 + o(1))snt'_1}{\ell N}\right\} \\
&= \sum_{k=0}^{\beta s \log n} \left(\frac{6e^{o(1)}s\epsilon \log n}{\ell kr}\right)^k \exp\left\{-\frac{2e^{o(1)}s\epsilon \log n}{\ell r}\right\} \\
&\leq \beta s \log n (6e^{o(1)} \cdot 10^3 \cdot e^{-2e^{o(1)}\cdot 10^3})^{\beta s \log n} \\
&\leq \frac{1}{n^2}.
\end{aligned}$$

To get the first expression observe that $G_{t'_1}^r(\Pi_1)$ consists of t'_1 edges. z of those have already being chosen from $E(T)$. Thereafter k of those are chosen so that they have an endpoint

in each $A, V_1 \setminus (A \cup B)$ and the rest are chosen from those not in $E(T)$ or those having both an endpoint in each $A, V_1 \setminus (A \cup B)$. We then apply Lemma 7 with $a = sn_\ell, b = s(n_\ell - s - b), c = d = z, i = k, t = t'_1, Q = N$. For the last equality we have used that $\left(\frac{6e^{o(1)}s\epsilon \log n}{k}\right)^k$ is maximized when $k = \frac{6e^{o(1)}s\epsilon \log n}{e}$. Thus for $k \in [0, \beta s \log n]$ this is maximized when $k = \beta s \log n$.

Summarizing, there are $\binom{n}{s}$ ways to choose A and thereafter $\binom{n-s}{b}$ ways to choose a set B of size $b \leq (2\ell - 1)s + 1$. Given A, B there are $(s + b)^{s+b-2}$ trees that are spanned by $A \cup B$. Each such tree appears with probability at most $\left(\frac{3r \log n}{n}\right)^{s+b-1}$. Finally given the appearance of any such tree there are at most $\beta s \log n$ edges of color Π_1 with an endpoint in each of $A, V_1 \setminus (A \cup B)$ in G_{t_1} with probability at most $1/n^2$. Therefore the probability that conditions (i), (ii), (iii) and (iv) are satisfied is bounded by

$$\sum_{b=0}^{(2\ell-1)s+1} \binom{n}{s} \binom{n-s}{b} (s+b)^{s+b-2} \left(\frac{3r \log n}{n}\right)^{s+b-1} \frac{1}{n^2} = o(1).$$

(c) It is enough to show the above statement for $v \in TBAD, w \in BAD_1$. In the case that the statement is false $\exists v, w \in V, C \subset [r]$ with $|C| = d$ and $S \subset V$ with $|S| = s \leq 2\ell - 1$ such that the following hold: (i) $\{v, w\} \cup S$ spans a path P in $G_{rn \log n}$, (ii) there are at most $d \log \log n$ edges adjacent to v in $G_{\tau_{fit-\Pi}}^r(C)$, (iii) there are at most $\beta \log n$ edges from w to $V_1 \setminus S \cup \{w\}$ in $G_{t_1}^r(\Pi_1)$.

Fix such v, w, P, S, C . As shown in (b), (see (3)), (i) is satisfied with probability at most $\left(\frac{3r \log n}{n}\right)^{s+1}$. $G_{\tau_{fit-\Pi}}^r(C)$ is distributed as a $G(n, m_d)$ where w.h.p. $\frac{1}{2}(1 - \alpha)n \log n \leq m_d \leq rn \log n$ for arbitrarily small $\alpha > 0$, (see Corollary 13 and Remark 9). Therefore conditional on (i) and on $u \leq s + 1$ edges of P appearing in $G_{\tau_{fit-\Pi}}^r(C)$ out of which at most one is adjacent to v we have

$$\begin{aligned} \Pr(\deg_{\tau_{fit-\Pi}}(v, C) \leq d \log \log n) &\leq \sum_{k=0}^{d \log \log n - 1} \frac{\binom{n-2}{k} \binom{N-(s+1)-(n-2)}{m_d-u-k}}{\binom{N-(s+1)}{m_d-u}} \\ &\leq \sum_{k=0}^{d \log \log n} \left(\frac{6e^{o(1)}nm_d}{kn^2}\right)^k \exp\left\{-\frac{(1+o(1))nm_d}{N}\right\} \\ &\leq \sum_{k=0}^{d \log \log n} \left(\frac{6e^{o(1)}r \log n}{k}\right)^k e^{-(1-2\alpha) \log n} \\ &\leq n^{3\alpha-1}. \end{aligned}$$

For the first inequality we used Lemma 7 with $a = b = n - 2, c = s + 1, d = u, i = k, t = m_d, Q = N$.

$G_{t_1}^r(\Pi_1)$ is distributed as a $G(n, t'_1)$ where w.h.p. $t_1 = (1 + o(1))t_1/r$ (see Remark 9). Conditional on (i), (ii) occurring, if (iii) also occurs, then in $E(G_{t_1}(\Pi_1))$ there are $k \leq \beta \log n$ edges from w to $V_1 \setminus \{v, w\}$ and $h \leq s + 1 + d \log \log n$ edges that either belong to $E(P)$ or are adjacent to v and lie in $E(G_{\tau_{fit-\Pi}}^r(C \cap \{\Pi_1\})) \cap E(G_{t'_1}^r(\Pi_1))$. The remainder of the $t_1 - k - h$ edges are chosen from those not in $E(P)$ and not in $\{w\} \times V_1 \setminus \{v, w\}$. If $\Pi_1 \in C$

then these edges are also chosen from those not incident to v while if $\Pi_1 \notin C$ then these edges are also chosen from the $d_{\tau_{fit-\Pi}}(v, C)$ edges not incident to v in $E(G_{\tau_{fit-\Pi}}^r(C))$. Let $j = n - 2$ if $\Pi_1 \in C$ and $j = d_{\tau_{fit-\Pi}}(v, C)$ otherwise. Then,

$$\begin{aligned} \Pr((iii)|(i), (ii), j) &\leq \sum_{k=0}^{\beta \log n} \frac{\binom{n_\ell}{k} \binom{N-(s+1)-(n_\ell-2)-(j-1)}{t'_1-h-k}}{\binom{N-(s+1)-j}{t'_1-h}} \\ &\leq \sum_{k=0}^{\beta \log n} \left(\frac{6e^{o(1)}t'_1n}{k\ell n^2} \right)^k \exp \left\{ -\frac{(1+o(1))t'_1n}{\ell N} \right\} \\ &\leq \sum_{k=0}^{\beta \log n} \left(6e^{o(1)} \cdot 10^3 \right)^{\beta \log n} \exp \left\{ -\frac{3\epsilon \log n}{2r\ell} \right\} \\ &\leq n^{-\epsilon/(r\ell)}. \end{aligned}$$

For the first inequality we used Lemma 7 with $a = n_\ell, b = n_\ell - 2, c = j + s + 1, d = h, i = k, t = t'_1, Q = N$. For the second one we used the fact that $\left(\frac{6e^{o(1)}t'_1n}{k\ell n^2}\right)^k$ has a unique maximum that occurs when $k = \frac{6e^{o(1)}t'_1n}{\ell n^2}$. Thus for $k \in [0, \beta \log n]$ this is maximized when $k = \beta \log n$. Furthermore we used that w.h.p. $\frac{3t_1}{4r} \leq t'_1 \leq \frac{7t_1}{6r}$ (see Remark 9).

Taking the union bound over v, w, P, S, C we have that the probability the event described by (c) does not hold can be bounded by

$$n(n-1) \sum_{s=0}^{2\ell-1} \binom{n-2}{s} s! \binom{r}{d} \left(\frac{3r \log n}{n} \right)^{s+1} n^{3\alpha-1} \cdot n^{-\epsilon/(r\ell)} = o(1),$$

for α sufficiently small.

(d) In the event that (d) fails there exist a vertex with degree at least $10r \log n$ in $G_{rn \log n}$. Thus

$$\Pr(\neg(d)) \leq n \sum_{k=10r \log n}^n \frac{\binom{n-1}{k} \binom{N-n-1}{rn \log n - k}}{\binom{N}{rn \log n}} \leq n \sum_{k=10r \log n}^{n^2} \left(\frac{6e^{o(1)}n \cdot rn \log n}{kn^2} \right)^k = o(1).$$

For the first inequality we used Lemma 7 with $a = n - 1, c = d = 0, e = n + 1, i = k, t = rn \log n, Q = N$. \square

5 Proof of Theorem 1

We say a pattern is trivial if $\ell=1$. In the case of a trivial pattern, Theorems 1-5 reduce to well known results about hitting times and the random graph process (see [3], [12]). Hence we may assume that $\ell > 1$.

We prove Theorem 1 using the 3-phase approach for finding a Hamilton cycle in the directed random graph process used in [9]. In the first phase a 1-factor is created consisting of $O(\log n)$ cycles. Then in the second phase we sequentially merge pairs of cycles by performing two arc exchanges until no such arc exchange is available. W.h.p. at the end of Phase 2 the

largest cycle is of order $n - o(n)$. Finally as a last phase we merge one by one the smaller cycles with the largest one. In each merging we start by merging the two cycles into a path. Then we use double rotations, a technique that generalizes Posa's rotations to the directed setting, in order to turn the initial path into a cycle. Instead of going through all 3 phases of the proof we give a reduction to the following Lemma. For its proof see Lemma 1.2 of [1] and Sections 10 and 11 of that paper.

Lemma 18. *Let F, H, G_1 be digraphs such that: (i) F is a 1-factor consisting of $O(\log n)$ directed cycles, (ii) H has maximum degree $O(\log n)$, and (iii) G_1 is a random graph where every edge appears independently with probability $p_1 = \Omega(\frac{\log n}{n})$. Then w.h.p. $E(F) \cup (E(G_1) \setminus E(H))$ span a Hamilton cycle.*

In the random digraph setting Lemma 18 states the following. Assume that we are able to perform Phase 1 and find a directed 1-factor F consisting of $O(\log n)$ cycles without exposing too many arcs. Even if we forbid re-using any the arcs that we have exposed but not used in the construction of F (each vertex is w.h.p. incident to $O(\log n)$ of them), then we have enough randomness left so that w.h.p. we are able to perform Phases 2, 3 and construct the Hamilton cycle while avoiding the edges that we have exposed in Phase 1.

5.1 Construction of the 2-factor

We now reveal the following. For $i \in [N]$ we reveal the color c_i . For every $i \leq t_\ell$ we reveal only v_i from the pair $\vec{e}_i = (v_i, w_i)$. Furthermore for $t_\ell < i \leq t_{2\ell}$ we reveal only w_i from the pair $\vec{e}_i = (v_i, w_i)$. Given the information that we have just revealed we can determine BAD . Now given the set BAD we reveal every edge with an endpoint in BAD . Observe that $TBAD \subset BAD$ hence $TBAD$ is now determined. Furthermore $\tau_{fit-\Pi}$ is determined (we use this information in the algorithm given below). Indeed let v^* be a vertex that does not fit Π at time $\tau_{fit-\Pi} - 1$. Every vertex in $GOOD$ fits Π by time $t_{2\ell}$. Corollary 13 implies that w.h.p. $t_{2\ell} = 2 \cdot 10^{-9} n \log n < \frac{(1-\epsilon)n \log n}{2} \leq \tau_{fit-\Pi}$, hence w.h.p. $v^* \in BAD$. Therefore w.h.p. $\tau_{fit-\Pi} = \max \{ \min \{ t : v \text{ fits } \Pi \text{ in } G_t \} : v \in BAD \}$ and the quantity in the left can be determined by looking at edges incident with BAD , information that we have revealed.

We now implement the algorithm *CoverBAD*, given below, in order to cover every vertex in BAD by a unique path **1 - ℓ , Π -colored path**. By $\{v, u\}$ we denote undirected edge between v and u while by (u, v) we denote the directed edge from u to v .

Given $v \in BAD$, *CoverBAD* grows a Π -colored path $P(v)$ with v in the interior. Suppose that v is incident with edges $\{v_1, v_0 = v\}, \{w_0 = v, w_1\}$ of color Π_i, Π_{i+1} respectively, see Step 2. If $i \neq \ell$ then in Step 3 we create a path $(v = w_0, w_1, w_2, \dots, w_s, s = \ell - i - 1)$ where edge (w_j, w_{j+1}) has color Π_{i+j} . In Step 4 we extend this path via $(v = v_0, v_1, \dots, v_t, t = i)$ where edge (v_j, v_{j+1}) has color Π_{i-j} . Thus in this case $P(v)$ has length $\ell - 1$. When $i = \ell$ we grow a path of length $2\ell - 1$ in the same manner. **In both cases $P(v)$ is a 1 - ℓ , Π colored path.**

We say that *CoverBAD* breaks if there is a step where no vertices satisfying the given conditions can be found.

Algorithm 1 COVERBAD

Initialize: $\mathcal{A} := GOOD, \mathcal{P}_{bad} := \emptyset$.

For $v \in BAD$ do begin:

1. Set $v_0 = w_0 = v, s = 1, t = 1$.
2. Find $v_1, w_1 \in \mathcal{A}, i \in [\ell]$ such that $\{v_0, v_1\}, \{w_0, w_1\} \in G_{\tau_{fit-\Pi}}, c(\{v_0, v_1\}) = \Pi_i$ and $c(\{w_0, w_1\}) = \Pi_{i+1}$. Update $\mathcal{A} = \mathcal{A} \setminus \{v_1, w_1\}$. Direct the edge $\{v_0, v_1\}$ from v_1 to v_0 and the edge $\{w_0, w_1\}$ from w_0 to w_1 .
3. While $s \neq \ell - 1 + i \cdot \mathbb{I}(i \neq \ell)$: expose all ordered edges $\vec{e}_i = (w_s, x)$ and find $x \in \mathcal{A}$ such that $c((w_s, x)) = \Pi_{i+s+1}$. Update $\mathcal{A} = \mathcal{A} \setminus \{x\}, s = s + 1, w_s = x$.
4. While $t \neq i$: expose all ordered edges $\vec{e}_i = (y, v_t)$ and find $y \in \mathcal{A}$ such that $c((y, v_t)) = \Pi_{i-(t-1)}$. Update $\mathcal{A} = \mathcal{A} \setminus \{y\}, t = t + 1, v_t = y$.
5. Set $P(v) = v_t, v_{t-1}, \dots, v_1, v, w_1, \dots, w_s$. Update $\mathcal{P}_{bad} = \mathcal{P}_{bad} \cup \{P(v)\}$.

End for.

Remark 19. If COVERBAD succeeds then every vertex $v \in BAD$ lies in the interior of some path in \mathcal{P}_{bad} and it is at distance at most ℓ from each of its endpoints.

Lemma 20. *W.h.p.* CoverBAD does not break.

Proof. We consider 3 cases.

Case 1: CoverBAD breaks at Step 2 for some $v \in TBAD$. Due the definition of the stopping time $\tau_{fit-\Pi}$, there exist $v_1, w_1 \in V$, and $i \in [\ell]$ such that $\{v_1, v\}, \{v, w_1\} \in G_{\tau_{fit-\Pi}}, c(\{v_0, v\}) = \Pi_i, c(\{v, w_1\}) = \Pi_{i+1}$. It is enough to show that at the beginning of the iteration in which we construct $P(v)$, we can find $v_1, w_1 \in \mathcal{A} \subseteq GOOD$. Lemma 17 (iii) implies that $v_1, w_1 \notin BAD$. If $v_1 \notin \mathcal{A}$ then there exists $v' \in BAD$ such that $v_1 \in P(v')$. Then Remark 19 implies that v, v' are within distance 2ℓ contradicting Lemma 17 (iii). Hence w.h.p. $v_1 \in \mathcal{A}$. Similarly $w_1 \in \mathcal{A}$ w.h.p.

Case 2: CoverBAD breaks at Step 2 for some $v \in BAD \setminus TBAD$. Observe that if for every $i \in [\ell]$ we have either $\deg_{\tau_{fit-\Pi}}(v, \Pi_i) \leq \log \log n$ or $\deg_{\tau_{fit-\Pi}}(v, \Pi_{i+1}) \leq \log \log n$ then $\left| \left\{ c \in C : \deg_{\tau_{fit-\Pi}}(v, c) \leq \log \log n \right\} \right| \geq d$. On the other hand since $v \notin TBAD$ we have that $\left| \left\{ c \in C : \deg_{\tau_{fit-\Pi}}(v, c) \leq \log \log n \right\} \right| \leq d - 1$. Hence there exists $j \in [\ell]$ such that $\deg_{\tau_{fit-\Pi}}(v, \Pi_j) \geq \log \log n$ and $\deg_{\tau_{fit-\Pi}}(v, \Pi_{j+1}) \geq \log \log n$. Let

$$\mathcal{C}_j(v) = \{w \in V : \{v, w\} \in G_{\tau_{fit-\Pi}} \text{ and } c(\{v, w\}) = \Pi_j\}.$$

Similarly define $\mathcal{C}_{j+1}(v)$. Then $|\mathcal{C}_j(v)|, |\mathcal{C}_{j+1}(v)| \geq \log \log n$. Since the algorithm breaks we have $\mathcal{C}_j(v) \cap \mathcal{A} = \emptyset$ or $\mathcal{C}_{j+1}(v) \cap \mathcal{A} = \emptyset$. From Remark 19 we have that if a vertex is removed from \mathcal{A} then it is within distance ℓ from some vertex in BAD . From Lemma 17 (ii) w.h.p. there are at most $6\epsilon^{-1}r\ell^2$ vertices within distance 2ℓ from v and for each such vertex at most

2ℓ vertices are removed from the \mathcal{A} . Hence w.h.p. $|\mathcal{C}_j(v) \cap \mathcal{A}| \geq |\mathcal{C}_j(v)| - 12\epsilon^{-1}r\ell^3 \geq 1$. Thus $\mathcal{C}_j(v) \cap \mathcal{A} \neq \emptyset$. Similarly $\mathcal{C}_{j+1}(v) \cap \mathcal{A} \neq \emptyset$.

Case 3: *CoverBAD* breaks at Step 3 or Step 4 for some $v \in \text{BAD}$. Assume that it breaks at Step 3 for $v \in V$ (the case that it breaks at Step 4 can be dealt with in the same way). Then there exists $j \in [\ell]$ and $t \leq \ell - 1$ such that no vertex v_{t+1} can be found such that $c((v_t, v_{t+1}))$ is of color Π_j . Let $\mathcal{C}_j(v) = \{v \in V : (v_t, v_{t+1}) \in G_{\tau_{fit-\Pi}} \text{ and } c((v_t, v_{t+1})) = \Pi_j\}$. By construction $v_s \in \text{GOOD}$ and hence $|\mathcal{C}_j(v)| = \Omega(\log n)$. The rest of the argument is identical to the one given for Case 2. \square

After the termination of *CoverBAD*, \mathcal{A} consists of all the vertices not spanned by some path in \mathcal{P}_{bad} . Our next step is to cover the vertices in \mathcal{A} by Π -colored paths. In order to do so we use the partition V_1, \dots, V_ℓ . For $i \in [\ell]$ let $V'_i = V_i \cap \mathcal{A}$. From each set V_i a set of size at most $2\ell|\text{BAD}|$ may have been used in the construction of paths in \mathcal{P}_{bad} . Thus for $i \in [\ell]$ we have $|V_i \cap \mathcal{A}| \geq |V_i| - 2\ell|\text{BAD}|$. Let $n_r = 2\ell|\text{BAD}|$. Then from Lemma 17 we have that $n_r = O(n^{1-10\beta})$. To equalize the sizes of V'_i 's, for $i \in [\ell]$ we now remove from V'_i a random set of size n_r . We redistribute the vertices that we have just removed to the sets V'_i 's in order to form sets of the same size. For $i \in [\ell]$ we let V''_i be the set obtained from V'_i .

We now define the following edge sets. For $v \in V$ and $i \in [\ell]$ let $E_i^+(v)$ be the first 6 edges (v, w) of color Π_i with $w \in V'_i \cap V''_i$ that appear after $e_{t_{i-1}}$. Similarly let $E_i^-(v)$ be the first 6 edges (w, v) of color Π_{i-1} with $w \in V'_i \cap V''_i$ that appear after $e_{t_{\ell+i-1}}$. We stress that $V'_i \cap V''_i$ equals the set of the vertices in V_i that are not covered by some path in \mathcal{P}_{bad} and have not been moved to some other set V'_j .

Lemma 21. *W.h.p. for every $i \in [\ell]$ and every $v \in V''_i$ we have $E_i^+(v) \subset E(G_{t_i}) \setminus E(G_{t_{i-1}})$ and $E_i^-(v) \subset E(G_{t_{\ell+i}}) \setminus E(G_{t_{\ell+i-1}})$.*

Proof. Because of symmetry it is enough to show that for a fixed $v \in V''_1$ we have

$$\Pr(|\{w \in V'_2 \cap V''_2 : \vec{e}_i = (v, w) \text{ with } t_1 < i \leq t_2\}| < 6) = o(n^{-1}).$$

Let $N_2(v) = \{w \in V'_2 : \vec{e}_i = (v, w) \text{ with } t_1 < i \leq t_2\}$. Since $v \in \mathcal{A}$ we have that $v \notin \text{BAD}_2$. Furthermore there are at most $O(1)$ vertices in BAD within distance 2ℓ of v hence at most $O(1)$ of its neighbors do not lie in \mathcal{A} . Thus $n_2(v) = |N_2(v)| = \Theta(\log n)$. Therefore

$$\begin{aligned} & \Pr(|\{w \in V'_2 \cap V''_2 : \vec{e}_i = (v, w) \text{ with } t_1 < i \leq t_2\}| \leq 5) \\ & \leq \sum_{k=0}^5 \frac{\binom{n_2(v)}{n_2(v)-k} \binom{|V'_2| - n_2(v)}{n_r - n_2(v) + k}}{\binom{|V'_2|}{n_r}} \\ & \leq \sum_{k=0}^5 n_2^k(v) \prod_{i=0}^{n_2(v)-k-1} \frac{n_r - i}{|V'_2| - i} \prod_{i=0}^{n_r - n_2(v) + k - 1} \frac{n_r - n_2(v) + k - i}{|V'_2| - n_2(v) + k - i} \cdot \frac{|V'_2| + n_2(v) - i}{n_r - n_2(v) + k - i} \\ & \leq \sum_{k=0}^5 n_2^k(v) \cdot \left(\frac{n_r}{|V'_2|}\right)^{n_2(v)-k} \\ & = o(n^{-1}). \end{aligned} \tag{4}$$

To see inequality (4) observe that on the event $\{|N_2(v) \cap V_2''| = k \leq 5\}$ at least $n_2(v) - k$ of the vertices in $N_2(v)$ were chosen and redistributed. The last inequality follows from the fact that $n_r = O(n^{1-10\beta})$, $n_2(v) = \Theta(\log n)$ and $|V_2'| = (1 + o(1))n_\ell$. \square

For $i \in [\ell]$ set

$$E_i^+ = \bigcup_{v \in V_i''} E_{i+1}^+(v) \quad \text{and} \quad E_i^- = \bigcup_{v \in V_i''} E_{i-1}^-(v).$$

Thus E_i^+ (E_i^- respectively) is a set of $6|V_i''|$ edges with an endpoint in each V_i'' , $V_{i+1}'' \cap V_{i+1}''$ (V_i'' , $V_{i-1}'' \cap V_{i-1}''$ resp.) such that each vertex in V_i'' is incident to 6 edges in it.

Lemma 22. *W.h.p. for $i \in [\ell - 1]$, $E_i^+ \cup E_{i+1}^-$ spans a complete matching M_i from V_i'' to V_{i+1}'' .*

Proof. Assume that no such matching exists. Then Hall's theorem implies that either (i) $\exists A \subseteq V_i''$, $B \subseteq V_{i+1}''$, with $|A| = s$, $|B| = s - 1$, $6 \leq s \leq \frac{|V_i''|}{2}$, and no edge from A to $V_{i+1}'' \setminus B$ lies in E_i^+ or (ii) $\exists A \subseteq V_{i+1}''$, $B \subseteq V_i''$, with $|A| = s$, $|B| = s - 1$, $6 \leq s \leq \frac{|V_{i+1}''|}{2}$, and no edge from A to $V_i'' \setminus B$ lies in E_{i+1}^- . Here (i) takes care of witnesses of size at most $|V_i''|/2$ and (ii) takes care of the larger ones (consider the sets $A' = V_i'' \setminus B$, $B' = V_{i+1}'' \setminus A$). In this context, given that we have shown that (i) is unlikely, we have to be sure that the edge choices involved in (ii) are independent of those considered in (i). This is achieved by the fact that the edges in E_i^+ , E_i^- are sampled from distinct sets. This does not mean complete independence because we cannot choose the same edge twice and this accounts for the $O(\log n)$ term in (5).

Therefore the probability that $\exists i \in [\ell - 1]$ such that $E_i^+ \cup E_{i+1}^-$ does not span a matching M_i between V_i'' and V_{i+1}'' is bounded by

$$2(\ell - 1) \sum_{s=6}^{|V_1''|/2} \sum_{A \in \binom{V_1''}{s}} \sum_{B \in \binom{V_2''}{s+1}} \prod_{v \in B} \frac{\binom{|A \cap V_1' \cap V_1''|}{6}}{\binom{|V_1' \cap V_1''| - O(\log n)}{6}} \quad (5)$$

$$\leq 2(\ell - 1) \sum_{s=6}^{|V_1''|/2} \binom{|V_1''|}{s} \binom{|V_1''|}{s+1} \left(\frac{e^{o(1)} \binom{s}{6}}{\binom{|V_1''|}{6}} \right)^s \quad (6)$$

$$\leq 2(\ell - 1) \sum_{s=6}^{|V_1''|/2} \left(\frac{e|V_1''|}{s} \right)^s \left(\frac{e|V_1''|}{s} \right)^{s+1} \left(\frac{e^{o(1)} s}{|V_1''|} \right)^{6s}$$

$$\leq n^2 \sum_{s=6}^{|V_1''|/2} \left(\frac{e^{2+o(1)} s^4}{|V_1''|^4} \right)^s$$

$$= o(1).$$

For (6) we used that $|V_1''| = |V_2''| = (1 - o(1))|V_1' \cap V_1''|$ and at the last equality that $|V_i''| = (1 + o(1))n_\ell$. \square

The edges in $\cup_{i \in [\ell-1]} M_i$ span a set of $1 - \ell$, **II-colored paths** with an endpoint in each of V_1'' , V_ℓ'' that covers \mathcal{A} . Let \mathcal{P}_M be this set of paths. Set $\mathcal{P} = \mathcal{P}_M \cup \mathcal{P}_{bad} = \{P_1, P_2, \dots, P_{n_h}\}$.

For $P_j \in \mathcal{P}$, we denote its endpoint in V_1 by v_j^- and its endpoint in V_ℓ by v_j^+ . In addition set $Q^+ = \{v_j^+ : P_j \in \mathcal{P}\}$ and $Q^- = \{v_j^- : P_j \in \mathcal{P}\}$. Let $\mathcal{P}'_{good} \subset \mathcal{P}$ be the set of paths with an endpoint in each of $V'_1 \cap V''_1, V'_\ell \cap V''_\ell$. We define the following edge sets. For $v \in Q^+$ let $E^+(v)$ be the first 6 edges (v, w) with $w \in \{v_j^- : P_j \in \mathcal{P}'_{good}\}$ that appear before e_{t_1} . Similarly, for $v \in Q^-$ let $E^-(v)$ be the first 6 edges (w, v) with $w \in \{v_j^+ : P_j \in \mathcal{P}'_{good}\}$ that appear after $e_{t_{2\ell-1}}$. Note that because $Q^+, Q^- \subseteq \mathcal{A}$, these edges are not conditioned by the edges of the matchings in Lemma 22. Finally set

$$E^+ = \bigcup_{v \in Q^+} E^+(v) \quad \text{and} \quad E^- = \bigcup_{v \in Q^-} E^-(v).$$

We have the following two Lemmas. Their proofs are identical to the proofs of Lemmas 21 and 22 respectively and hence are omitted.

Lemma 23. *W.h.p. we have $E^+ \subset E(G_{t_1})$ and $E^- \subset E(G_{t_{2\ell}}) \setminus E(G_{t_{2\ell-1}})$.*

Lemma 24. *W.h.p. $E^+ \cup E^-$ spans a complete matching M^* from Q^+ to Q^- .*

We now use M^* to join the paths in \mathcal{P} and create a 2-factor F' . Thus $E(F') = M^* \cup (\cup_{i \in [\ell-1]} M_i)$. We finish this subsection with the following Lemma.

Lemma 25. *W.h.p. F' consists of $O(\log n)$ cycles.*

Proof. Let $M_{bad}^+ \subset M^*$ be the edges in M^* incident to $\{v_j^+ : P_j \in \mathcal{P}_{bad}\}$. Then M_{bad}^+ is spanned by $\{v_j^+ : P_j \in \mathcal{P}_{bad}\} \times \{v_j^- : P_j \in \mathcal{P}'_{good}\}$. The key observation is that given $M_i, i \in [\ell-1]$ and M_{bad}^+ , there is a one to one correspondence between realizations of F' and permutations $\pi_{F'}$ on $\{v_j^+ : P_j \in \mathcal{P}'_{good}\}$. Here we are using the fact that our pattern is not trivial i.e. $\ell > 1$. In addition the number of cycles of F' equals the number of cycles of the permutation $\pi_{F'}$. Furthermore, due to symmetry every matching M^* that may occur, occurs equally likely and thus each possible permutation on $\{v_j^+ : P_j \in \mathcal{P}'_{good}\}$ is equally likely to occur. It is well known (see e.g. [12]) that w.h.p. a uniformly random permutation on M elements consists w.h.p. of at most $2 \log M$ cycles and the lemma follows. \square

5.2 Reduction to Lemma 18

We now generate graphs H, F, G_1 as follows that satisfy the conditions of Lemma 18. These graphs will have vertex set V' distinct from $[n]$. They are designed so that a Hamilton cycle in $E(F) \cup (E(G_i) \setminus E(H))$ can be used to construct a Hamilton cycle spanning V . There is a vertex $v(P_i) \in V'$ for each $P_i \in \mathcal{P}$. We let

$$F = \{(v(P_i), v(P_j)) : P_i, P_j \in \mathcal{P} \text{ and } (v_i^+, v_j^-) \in M^*\}.$$

Then w.h.p. F defines a collection of at most $2 \log |V'|$ cycles that span V' . We let $H' = \{(v_i^+, v_j^-) \in E(G_{t_{2\ell}}^r)\}$ and H be the graph consisting of the arcs in

$$H := \{(v(P_i), v(P_j)) : P_i, P_j \in \mathcal{P} \text{ and } (v_i^+, v_j^-) \in H'\}.$$

It follows from Lemma 17(d) that H has maximum degree $O(\log |V'|)$.

Let $E^* = \{(v_i^+, v_j^-) : P_i, P_j \in \mathcal{P}\}$ and $B = \{f_1, f_2, \dots, f_b\}$ be the edges of color Π_ℓ in $(E^* \cap E(G_{\hat{t}})) \setminus E(G_{t_{2\ell}}) = (E^* \cap E(G_{\hat{t}})) \setminus H'$, where \hat{t} is the LHS of (1). Let $p_1 = \frac{\log n}{100rn}$. In order to generate G_1 we generate $E'_1 \subseteq E^*$ by adding to E'_1 every $(v_i^+, v_j^-) \in E^*$ independently with probability $p_1 = \frac{\log n}{100rn}$. Equivalently, tentatively accept an ordered pair with probability $\frac{\log n}{100rn}$ and then randomly color it and finally accept it if the color is color ℓ . Then for every $(v_i^+, v_j^-) \in E'_1$ we include in $E(G_1)$ the arc $(v(P_i), v(P_j))$. G_1 is a random digraph with vertex set V' where every arc appears independently with probability $p_1 = \Omega(\frac{\log n}{n})$.

F, H and G_1 satisfy the conditions of Lemma 18 therefore $E(F) \cup (E(G_1) \setminus E(H))$ w.h.p. spans a Hamilton cycle on V' . Any such cycle Q corresponds to a cycle Q' on V spanned by $\{E(P_j) : P_j \in \mathcal{P}\} \cup E(F) \cup (E'_1 \setminus H')$ that traces all the paths in \mathcal{P} . Observe that if $E'_1 \setminus H'$ has the same distribution as a random subset of B then $\{E(P_j) : P_j \in \mathcal{P}\} \cup E(F) \cup B$ spans a Hamilton cycle on V . Any such Hamilton cycle is Π -colored. Indeed every path in \mathcal{P} is a **1 - ℓ , Π -colored path** and Q traces it in the right direction. Thereafter every edge in $E(F) \cup B$ corresponds to a Π_ℓ -colored edge in $G_{\tau_{fit-\Pi}}^r$. To show that $G_{\tau_{fit-\Pi}}^r$ has a Π -colored Hamilton cycle it suffices to show now that $E'_1 \setminus H'$ has the same distribution as a random subset of B . This will follow if we can show (i) B is distributed uniformly among all the edge sets in $E^* \setminus H'$ of size b and (ii) $|B| \geq |E'_1|$.

Let $|N_1| = |E'_1|$ and $|N_2| = |E'_1 \setminus H'|$. Recall (i) every edge in E^* has both of its endpoints in $GOOD$, (ii) we have not revealed any edge with both of its endpoints in $GOOD$ that appears after time $t_{2\ell} = 2 \cdot 10^{-9}n \log n$ (hence any edge not in H') and (iii) all the edges are randomly colored. It follows that given b , B is distributed uniformly among all the edge sets in $E^* \setminus H'$ of size b . Thus in the event that $b \geq N_1$, $E'_1 \setminus H'$ has the same distribution as a subset of B of size $N_2 \leq N_1$.

Lemma 26. *W.h.p. $|B| \geq N_1$.*

Proof. We will show that w.h.p. $N_1 < \frac{n \log n}{10r\ell^2} < |B|$. Let X be the number of Π_ℓ colored edges in $E(G_{\hat{t}}) \setminus (E(G_{t_{2\ell}}))$. Each edge is Π_ℓ colored independently with probability $1/r$. Therefore X is distributed as a *Binomial* $(\hat{t} - t_{2\ell}, 1/r)$ random variable. Lemma 8 and Lemma 12 imply that w.h.p. $X \geq \frac{3n \log n}{10r}$. The edges in X are chosen uniformly at random from those not in $G_{t_{2\ell}}$. Since $|\mathcal{P}| = n_\ell - |O(BAD)|$, Lemma 17(d) implies that w.h.p. at most $10rn_\ell \log n$ out of the $(1 + o(1))\frac{n^2}{\ell^2}$ edges in $\{(v_i^+, v_j^-) : P_i, P_j \in \mathcal{P}\}$ lie in $G_{t_{2\ell}}$. Therefore

$$\begin{aligned}
\Pr\left(|B| \leq \frac{n \log n}{10r\ell^2}\right) &\leq \sum_{k=0}^{\frac{n \log n}{10r\ell^2}} \frac{\binom{n^2/\ell^2}{k} \binom{N-t_{2\ell}-(1+o(1))n^2/\ell^2}{\frac{3n \log n}{10r}-k}}{\binom{N-t_{2\ell}}{\frac{3n \log n}{10r}}} \\
&\leq \sum_{k=0}^{\frac{n \log n}{10r\ell^2}} \left(\frac{9n^3 \log n}{10r\ell^2 k N}\right)^k \exp\left\{-\frac{(1+o(1))3n^3 \log n}{10r\ell^2 N}\right\} \\
&\leq \sum_{k=0}^{\frac{n \log n}{10r\ell^2}} \left(\frac{9n^3 \log n}{10r\ell^2 k N}\right)^k \exp\left\{-\frac{n \log n}{2r\ell^2}\right\} \\
&\leq \frac{n \log n}{10r\ell^2} \cdot 20 \frac{n \log n}{10r\ell^2} \exp\left\{-\frac{n \log n}{2r\ell^2}\right\} = o(1).
\end{aligned}$$

For the second inequality we use Lemma 7 with $a = \frac{n^2}{\ell^2}$, $b = (1 + o(1))\frac{n^2}{\ell^2}$, $c = t_{2\ell}$, $d = 0$, $i = k$, $t = \frac{3n \log n}{10r}$, $Q = N$. For the last inequality we used that $(\frac{9n^3 \log n}{10r\ell^2 k N})^k$ in the sum is maximized when $k = \frac{n \log n}{10r\ell^2}$.

On the other hand from Lemma 8, it follows that

$$\Pr\left(N_1 \geq \frac{n \log n}{10r\ell^2}\right) \leq \Pr\left(\text{Binomial}\left(\frac{n^2}{\ell^2}, \frac{\log n}{100rn}\right) > \frac{n \log n}{10r\ell^2}\right) = o(1).$$

□

6 Proof of Theorem 2

6.1 Outline of proof

We first construct a large Π -colored cycle C containing most of the good vertices. The construction follows the argument from the proof of Theorem 1 and is omitted. The cycle will be such that the vertices not in C can be paired up as v, v_1 where (v, v_1) has color Π_k , say. In addition there will be vertices $v_2, v_3 \in C$ such that there is an edge (v_1, v_2) of color Π_{k-1} and an edge (v_1, v_3) of color Π_{k+1} . Then to find a Π -colored path from a v to w we use $(v, v_1, v_2, Q, w_3, w_1, w)$ where Q is the cycle path starting at v_2 in which the indices of the colors “decrease”.

6.2 The proof itself

In this section we will use the following result. The first inequality is well-known, see e.g. [12].

Lemma 27. *Let $\epsilon > 0$. Then w.h.p. $\frac{(1-\epsilon)n \log n}{2} \leq \tau_1 \leq \tau_{\Pi\text{-connected}} \leq t_{\text{fit-}\Pi} \leq \frac{(1+\epsilon)rn \log n}{2}$.*

The main ingredient of the proof of Theorem 2 is the following Lemma.

Lemma 28. *Let $A \subset \text{GOOD}$ be such that $A = O(|\text{BAD}|)$. For $i \in [\ell]$ let $\bar{V}_i \subset V_i \cup A$ be such that $\bar{V}_i \subset \text{GOOD}$ and $|\bar{V}_i| = \frac{n}{\ell} - b$ where $b = O(|\text{BAD}|)$. Then w.h.p. there is a Π -colored cycle $C = (v_1, v_2, \dots, v_h, v_1)$ such that*

$$(i) \ V(C) = \cup_{i \in [\ell]} \bar{V}_i.$$

$$(ii) \ E(C) \subseteq \{e_i = (a, b) : \exists j \in [\ell], a \in \bar{V}_j, b \in \bar{V}_{j+1} \text{ such that } i \leq \tau_1, c(e_i) = \Pi_j\}.$$

Lemma 28 states that if we remove a set of $O(|\text{BAD}|)$ vertices from each V_i and then relocate $O(|\text{BAD}|)$ vertices to form the sets \bar{V}_i , given the constraint that the new sets \bar{V}_i are all of the same size, then w.h.p. there is a Π -colored cycle C in $G_{\tau_1}^r$ that spans $\cup_{i \in [\ell]} \bar{V}_i$ and “respects” the new partitioning $\bar{V}_1, \dots, \bar{V}_\ell$. That is $E(C)$ consists of Π_j colored edges from V_j to V_{j+1} that lie in $G_{\tau_1}^r$, $j \in [\ell]$.

The proof of Lemma 28, which we omit, is similar to the proof of Theorem 1 with the extra advantage that we do not have to take care of any “bad” vertices. Here we handle vertices in A in the same way that we handled the vertices in GOOD that we shuffled in the

proof of Theorem 1. We use the lower bound of τ_1 given in Lemma 27 in place of the lower bound on $\tau_{fit-\Pi}$ given in Corollary 13.

In order to prove Theorem 2 we construct a large Π -colored cycle to which we attach spikes. Here by a spike we mean a 3-star or equivalently the graph on 4 vertices r_0, r_1, r_2, r_3 and edge set $(r_0, r_1), (r_2, r_1)$ and (r_1, r_3) . The *base* vertices of a spike r_2, r_3 will belong to the cycle while typically its *head* vertex r_0 will be a bad vertex.

Let $h = \min \{i \in \mathbb{Z}_{\geq 0} : i = -|BAD| \pmod{\ell}\}$ and $\Gamma(BAD) = BAD \cup N(BAD)$ where $N(BAD)$ is the neighborhood of BAD . Let S_B be a random subset of $GOOD \setminus \Gamma(BAD)$ of size h . Finally let $B = BAD \cup S_B$. We use S_B to ensure that ℓ divides $|B|$. From Lemma 17 and similar reasoning to that in the proof of Lemma 20, for every $v \in B$ we can find a vertex $v_1 \in GOOD$ that is adjacent to v in $G_{\tau_1}^r$. Let $\Pi_{k(v)}$ be the color of (v, v_1) . For every $v \in B$ we can find $v_2, v_3 \in GOOD$ such that $(v_2, v_1), (v_1, v_3) \in E(G_{2\ell}^r)$ and $(v_2, v_1), (v_1, v_3)$ have colors $\Pi_{k(v)-1}$ and $\Pi_{k(v)+1}$ respectively. In addition the selection of all the vertices above can be done such that all of them are distinct and do not lie in B .

We now construct the sets $\bar{V}_i, i \in [\ell]$ as follows. We begin our construction of \bar{V}_i by first removing from V_i all the vertices in $\{v, v_1 : v \in B\}$. Then for every $v \in B$ we move v_2 into $\bar{V}_{k(v)-1}$ and v_3 into $\bar{V}_{k(v)+2}$. After this, for $i \in [\ell]$, we choose a random set R_i of size $4\ell|B|$ from the current vertices in V_i , not including the vertices in $\{v_2, v_3 : v \in B\}$. Finally we redistribute $\cup_{i \in [\ell]} R_i$ such that all the resulting sets \bar{V}_i are of the same size. By applying Lemma 28 with $A = \{v_2, v_3 : v \in B\} \cup (\cup_{i \in [\ell]} R_i)$ we get a Π -colored cycle C that spans $\cup_{i \in \ell} \bar{V}_i$ and “respects” the partitioning $\bar{V}_1, \dots, \bar{V}_\ell$. C along with the edges that belong to the spikes allow us to claim that $G_{\tau_1}^r$ is Π -connected. To see this, we orient the edges of C so that an edge of color Π_i is followed by an edge of color Π_{i-1} , for $i \geq 1$. Then for $v \in B$ we first enter C by travelling along v, v_1, v_2 . If we wish to travel to $w \in B$ then we travel around C until we reach w_3 and finish our path with w_3, w_1, w .

7 Directed versions

Observe that now a pattern $\vec{\Pi}$ has also the notion of direction embedded in it. That is we are looking for an arc of color $\vec{\Pi}_1$ followed by an arc of color $\vec{\Pi}_2$ that is leaving its out vertex e.t.c. The main difference between the proofs of Theorems 4 and 5 and the proofs of Theorems 1 and 2 is in defining the demand of a pattern. This is because an in- and an out-arc of the same color are not exchangeable. To deal with this we may think of the direction of an arc as a second coordinate of its color. The idea is that a vertex “sees” an in-arc of color red as an arc of color (red,-) and an out-arc of color blue as an arc of color (blue,+). Thus instead of looking for a red in-arc and a blue out-arc it looks for arcs of colors (red,-) and (blue,+) respectively.

Definition 29. Let $\ell \in \mathbb{N}$ then

$$\overrightarrow{\mathcal{D}}(\ell) := \{S \subset [\ell] \times \{+, -\} : \{(i, +), (i+1, -)\} \cap S \neq \emptyset \text{ for all } i \in [\ell]\}.$$

Definition 30. Let $r \in \mathbb{N}$ and let $\vec{\Pi}$ be a directed $[r]$ -pattern. For $i \in [\ell]$ set $\vec{\Pi}(i, +) =$

$(\vec{\Pi}_i, +)$ and $\vec{\Pi}(i, -) = (\vec{\Pi}_{i-1}, -)$ The “demand” of $\vec{\Pi}$ is

$$d(\vec{\Pi}) := \min \left\{ |\{\vec{\Pi}(i, *) : (i, *) \in S\}| : S \in \mathcal{D}(\vec{\Pi}) \right\}.$$

Once again for a given $[r]$ -pattern $\vec{\Pi}$, if there exists a set $S \in \overline{\mathcal{D}(\vec{\Pi})}$ and a vertex $v \in V$ such that v is not incident to any arc that is assigned one of the at least $d(\vec{\Pi})$ (color,sign) elements of $\{\vec{\Pi}_i : i \in S\}$ then v does not fit $\vec{\Pi}$. Conversely if v does not fit $\vec{\Pi}$ then such a set $S \in \overline{\mathcal{D}(\vec{\Pi})}$ exists. Each vertex may see $2r$ distinct pairs of (color,direction) among the arcs adjacent to it. Thus in place of Lemma 12 we have the following Lemma

Lemma 31. *Let $r = O(1)$, $\vec{\Pi}$ being a directed $[r]$ -pattern and $\epsilon > 0$. Then, w.h.p.*

$$\frac{2r}{d(\vec{\Pi})} \cdot \frac{(1 - \epsilon)n \log n}{2} \leq \tau_{fit-\vec{\Pi}} \leq \frac{2r}{d(\vec{\Pi})} \cdot \frac{(1 + \epsilon)n \log n}{2}.$$

For the proofs of Theorems 4 and 5 in the definition of the BAD sets we do not have to impose an ordering on the endpoints of the arcs. Instead we can use the one given by their direction. The rest of the proof of Theorem 1 can be extended to the setting of Theorem 4.

For the proof of Theorem 5 we can use a similar construction to the one used in the proof Theorem 4. The idea is once again to construct a $\vec{\Pi}$ -colored cycle \vec{C} that spans $n - O(|BAD|)$ vertices in *GOOD*. Then we join the rest of the vertices to \vec{C} by an in- and an out-arc. As before we can ensure that those arcs exist w.h.p. Finally we can use a subpath of \vec{C} and the aforementioned arcs to $\vec{\Pi}$ -connect any two vertices.

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